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# Numerical Methods for First Order Delay Differential Equations


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## Abstract

A delay differential equation (DDE) also called the functional differential equation, and a neutral delay differential equation (NDDE) can provide us with realistic model of many phenomena arising in applied mathematics. like biology, medicine, control theory, climate models, population dynamics, electrical networks, and many others. The delay differential equations (DDEs) are used to constitute basic mathematical models for such real phenomena. Due to the similarities between delay differential equations (DDEs) and ordinary differential equations (ODEs), less attention was paid to the numerical solutions of delay differential equations (DDEs) in early stage of research contrary to the fact at that time as solving delay differential equations (DDEs) numerically is more complicated as compare to the numerical treatments of ordinary differential equations (ODEs).

In delay differential equations (DDEs) the unknown not only depend on the derivatives of independent variables (time) at present value but also contain an additional derivatives which depend on the solution at some previous time. The principal difficulty in studying delay differential equations (DDEs) lies in their special transcendental nature. Hence, therefore they are often solved using numerical methods, asymptotic solutions, approximations and graphical approaches. In the first part of this thesis we will discuss the major difference between ordinary differential equations (ODEs) and delay differential

equations (DDEs) as far as numerical solution are concerned, And also we will refer to the major obstacles (difficulties) in the numerical solutions of delay differential equations (DDEs) and neutral delay differential equations (NDDEs).

In the second part of this thesis we will discuss the numerical solutions for the first order delay differential equation (DDE) and a neutral delay differential equation (NDDE). And we will use the matlab for solving the delay differential equations (DDEs) with constant Delay $y(t-\tau)$, with dde23 solver which is much like solving ordinary differential equations (ODEs) with ode23 solver, and solving delay differential equations (DDEs) with time_state dependent delay $y(t-\tau(t))$, $y(t-\tau(t,y(t)))$ respectively, with ddesd solver which was developed by L.F. Shampine (2005).[18,19] . Finally, we used ddeNsd solver to solve neutral delay differential equations (NDDEs), with general delays (with constant delays , time dependent delay, and state dependent delay ) which was developed by L.F. Shampine (2008). [17]

**ملخص**

المعادلات التفاضلية المتأخرة و المعادلات التفاضلية المتأخرة المحايدة تزودنا بالعديد من الظواهر الواقعية التي تبرز من خلال الرياضيات التطبيقية، مثل علم الأحياء، الطب، التعداد السكاني ، الشبكات الكهربائية ، غيرها الكثير من التطبيقات. تستعمل المعادلات التفاضلية المتأخرة لتشكيل نماذج رياضية أساسية لهذه الظواهر الحقيقية. نظرا لأوجه التشابه بين المعادلات التفاضلية المتأخرة والمعادلات التفاضلية العادية، كان الاهتمام لإيجاد الحلول العددية للمعادلات التفاضلية المتأخرة قليل في المرحلة المبكرة من البحث على عكس الحقيقة في ذلك الوقت كما أن الحل العددي لمعادلات تفاضلية متأخرة أكثر تعقيدا بالمقارنة للعلاجات العددية للمعادلات التفاضلية العادية .

المجهول في المعادلات التفاضلية المتأخرة لا يعتمد على مشتقات من المتغيرات المستقلة في القيمة الحالية فقط ولكن أيضا يحتوي على المشتقات الإضافية التي تعتمد على حل في وقت سابق . الصعوبة الرئيسية في دراسة المعادلات التفاضلية المتأخرة تكمن في وجود التأخير في الحل .وبالتالي فهي غالبا تحل باستخدام الطرق العددية ، والحلول المتقاربة ، وتقريب نماذج رسومية.

في الجزء الأول من هذه الأطروحة سنناقش الفرق الرئيسي في الحل العددي بين المعادلات التفاضلية العادية و المعادلات التفاضلية المتأخرة ، و كذلك سنشير إلى العقبات الرئيسية (الصعوبات) في الحلول العددية للمعادلات التفاضلية المتأخرة و المعادلات التفاضلية المحايدة.

في الجزء الثاني من هذه الأطروحة سنناقش الحلول العددية لمعادلات تفاضلية متأخرة و معادلات تفاضلية محايدة ، واستخدمنا في هذه الأطروحة برنامج الماتلاب لإيجاد الحل العددي لمعادلات تفاضلية متأخرة إذا كانت قيمة التأخير مقدارا ثابتا لإيجاد الحل العددي باستخدام الرمز (dde23) ،وكذلك لمعادلات تفاضلية متأخرة بتأخير عام تعتمد على المتغير(t) أو تعتمد على المتغير (t) و ((y(t)) باستخدام الرمز (ddesd) الذي طوره (L.F.shampine) عام(٢٠٠٥) واستخدمنا رمز (dde Nsd) لحل معادلات تفاضلية متأخرة محايدة والتي تعتمد تأخير ثابت أو متغير يعتمد على كل من (t) ((y(t)) والذي طور أيضا من قبل (L.F.shampine) (٢٠٠٥).

**Contents**

**Introduction**

Numerical mathematics is the branch of mathematics that proposes, develops, analyzes and applies methods from scientific computing to several fields including analysis, linear algebra, geometry, approximation theory, functional equations, optimization and differential equations.

Other disciplines such as physics, the natural and biological sciences, engineering, and economics and the financial sciences frequently give rise to problems that need scientific computing for their solutions. As such, numerical mathematics is the cross road of several disciplines of great relevance in modern applied sciences, and can become a crucial tool for their qualitative and quantitative analysis. This role is also emphasized by the continual development of computers and algorithms, which make it possible nowadays, using scientific computing, to tackle problems of such a large size that real-life phenomena can be simulated providing accurate responses at affordable computational cost. see[5,6,20,21]

The corresponding spread of numerical software represents an enrichment for the scientific community. However, the user has to make the correct choice of the method (or the algorithm) which best suits the problem at hand. As a matter of fact, no black-box methods or algorithms exist that can effectively and accurately solve all kinds of problems.

Delay differential equations (DDEs) are of sufficient importance in modeling real-life phenomena to merit the attention of numerical analysts. Delay differential equations (DDEs) arise in many areas of mathematical modeling, for example population dynamics (taking into account the gestation times),infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling, for example, the body's reaction to CO2, etc. in circulating blood) and chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft (with respectively large and short lags),and more general control problems. For more information. see [4,6,21]

The fact that many phenomena frequently modeled by ordinary differential equations (ODEs) can be better modeled by Delay differential equations (DDEs) has not escaped the attention of the numerical analysis community, see [6,14,21] .

Our objective in this research is to assess the issues (largely issues of numerical analysis, but some more subjective) to be addressed to understand how the numerical solution of delay differential equations DDEs can be settled. In the process, we shall refer to many of the authors who have made contributions in this area.

In 1966, descriptive L.E. El'sgol'ts and S.B. Norkin [21,22] ,in the last 15-20 years the area of application of differential equation with deviating argument has greatly expanded . There are a lot of the scientists who advanced researches in domain of the numerical analysis, in particular numerical analysis for delay differential equations. C.T.H Baker , C.A.H Paul and D.R

Wille [4,5,6] described the efficient implementation of numerical software for solving delay differential equations, and described several strategies that have been developed over the past 25 years for improving the efficiency of delay differential equation solvers. In 1996 Hiroshi Hayashi developed an analysis of a numerical method for solving retarded and neutral delay differential equation with vanishing delays, the recently developed techniques include continuous Runge Kutta (CRK) methods, and determine convergence properties of the numerical solutions. JAN M . Hevernanand Robert M . Corless [12] explored the use of a computer algebra system to solve some very simple linear delay differential equations (DDEs), some of these DDEs are useful by themselves, and may also be of use as test problems for more general methods, they give detailed descriptions of the classical method of steps, the Laplace Transform method and a novel least squares method, followed by some discussion on the limitions and successes of each.Fudziah Ismail, Raed Ali Al-Khasawneh,Aung San Lwin & Mohamed Suleiman [8] , Diagonally Implicit Runge-Kutta methods of different orders are used for the treatment of delay differential equations, that the delay argument is approximated using an appropriate Hermite Interpolation. The numerical results based on these methods are compared and the Q-stability region of the methods are presented.

L.F. Shampine and S.Thompson in [16] show how to find the numerical solution of the delay differential equation with Retarded and neutral type in

Matlab in three codes. First: dde23 solver, which is a code to solve delay differential equations(DDEs)with constant delays $y(t-\tau)$ [15,16,18]. Second: ddesd solver, used to solve delay differential equations (DDEs) with general delays (time dependent delay $y(t-\tau(t))$, the state dependent delay $y(t-\tau(t,y(t)))$ [17,19]. Finally, ddeNsd code, shows the numerical solution of neutral delay differential equations (NDDEs) with general delays (with constant delays $y'(t-\tau)$, time dependent delay $y'(t-\tau(t))$, the state dependent delay $y'(t-\tau(t,y(t)))$) [17].

This research is divided into three chapters.

*Chapter one:* contains the main concepts, definitions, and theorems that are necessary for the thesis, set out the major differences between delay differential equations DDEs and ordinary differential equations ODEs as far as numerical solutions are concerned, also we have pointed out the major obstacles (difficulties) for solving delay differential equations (DDEs).

*Chapter two*: contains some numerical methods for solving delay differential equations DDEs: $y'(t)=f(t,y(t),y(t-\tau(t)))$, for $t\in[t_0,t_f]$ with history function $\phi(t)$,and general delays (with constant delays $y(t-\tau)$, time dependent delay $y(t-\tau(t))$, the state dependent delay $y(t-\tau(t,y(t)))$)) such as bellman's method of steps ,Euler's method and Runge Kutta methods, and also contains some delay differential equations DDEs with constant delays $y(t-\tau)$, which is solved by dde23 solvers in matlab, and with general delays which is solve by ddesd solvers in matlab.

*Chapter three:* contains approximate solution of a neutral DDE with the solution of a retarded DDE and exploit the fact that retarded DDEs are much easier to solve numerically. After demonstrating the validity of the approach, we used Matlabprogram, ddeNsd, that solves DDEs of neutral type. The new approach and asimple user interface make it easy to solve a wide range of test problems from the literature to moderate accuracy.

# Chapter One

# Preliminaries

## 1.0 Introduction

This chapter contains the main concepts, and necessary definitions, and theorem in this research. Section 1.1 introduces the definition of DDEs, accompanied with some examples. Section 1.2 set out the major differences between the ODEs and the DDEs as far as their numerical solution are concerned. Section 1.3 contains the main obstacles in the numerical solutions of DDEs.

## 1.1 Definitions and Examples

A differential equation with deviating argument is a differential equation in which the unknown function appears with various values of the argument. The differential equations with deviating arguments are classified according to the following definitions:

**Definition 1.1.1**

A differential equation with retarded argument is a differential equation with deviating argument in which the highest order derivative of the unknown function appears for just one value of the argument, and this argument is not less than all arguments of the unknown function and its derivative appearing in the equation.

**Example 1.1.1**

Consider the equations

$$y'(t) = f(t, y(t), y(t-\tau)), t \geq t_0 \qquad (1.1.1)$$

$$y'(t) = f(t, y(t), y(t-\tau(t, y(t)))), t \geq t_0 \qquad (1.1.2)$$

$$y''(t) = f(t, y(t), y(t-\tau(t)), y'(t-\tau(t))), t \geq t_0 \qquad (1.1.3)$$

where $\tau$ is a constant delay, and $\tau(t)$ are a variable delay for $t \geq t_0$.

Equations (1.1.1),(1.1.2)and(1.1.3) are differential equations with retarded argument if $\tau > 0, \tau(t) > 0$ for $t \geq t_0$ .

**Example 1.1.2**

Consider the differential equation:

$$y'(t) = f(t, y(t), y(t-\tau_1(t)), ..., y(t-\tau_k(t))), \qquad (1.1.4)$$

For all $\tau_j(t) \geq 0$, $j = 1,2,...,k$

Where

$t \in I = [t_0, t_0 + h]$, $h > 0$, $f \in C[I \times \Re^m \times \Re^m \times \Re^m \times ... \times \Re^m, \Re^m]$

and

$y(t) = \phi(t), t \in I_0 = [t_0 - h, t_0]$

where $\phi(t)$ is an initial function (or called a history function) of equation (1.1.4) , or a solution at a time previous than $t \geq t_0$ .This kind of equation is called Delay Differential Equation (DDEs) with several delays.

A first order delay differential equation (DDE) with one constant delay term $(t-\tau)$, and a history function $\phi(t)$ can be written as:

$$\left.\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), t \geq t_0 \\ y(t) &= \phi(t), t \leq t_0 \end{aligned}\right\} \qquad (1.1.5)$$

And if it has several delay term as:

$$\left.\begin{aligned} y'(t) &= f(t, y, y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_n)), t \geq t_0 \\ y(t) &= \phi(t), t \leq t_0 \end{aligned}\right\} \qquad (1.1.6)$$

Where $\phi(t)$ is the initial function, $\tau(t, y(t))$ is called the delay argument, the value of $y(t-\tau(t))$ is the solution of the delay term or commonly referred to as the delay term only. If the delay is a constant then it is called constant delay $y(t-\tau)$, if it is function of time $t$, then it is called time dependent delay $y(t-\tau(t))$, if it is a function of time $t$ and $y(t)$ then it is called the state dependent delay $y(t-\tau(t, y(t)))$.

**Definition 1.1.7**

A neutral delay differential equation (NDDE) is a differential equation in which the highest order derivative of the unknown function appears in the equation both with and without delays,(retarded arguments).

**Example 1.1.3**

$$y'(t) = f(t, y(t), y'(t-\tau)), t \geq t_0 \qquad (1.1.7)$$

$$y'(t) = f(t, y(t), y'(t-\tau(t))), t \geq t_0 \qquad (1.1.8)$$

$$y'(t) = f(t, y(t), y'(t-\tau(t, y(t)))), t \geq t_0 \qquad (1.1.9)$$

$$y''(t) = f(t, y(t), y(t-\tau(t)), y'(t), y''(t-\tau(t))), t \geq t_0 \qquad (1.1.10)$$

Where equation (1.1.7) is a first order Neutral Delay Differential Equation NDDE with constant delay, equation (1.1.8) is a first order Neutral Delay Differential Equation NDDE with time dependent delay, equation (1.1.9) is a first order Neutral Delay Differential Equation NDDE with state dependent delay, and equation (1.1.10) is a second order Neutral Delay Differential Equation (NDDE) with time dependent delay. Assume that the existence, uniqueness, and stability of solution to the problem under consideration. For example sufficient conditions for the existence and uniqueness of solutions to the DDE(1.1.2) and NDDE (1.1.9) are:

- $f$ is continuous with respect to $t, y(t)$ and $y(t, \tau(t, y(t)))$, also $y(t)$ is continuous,
- $f$ satisfies a Lipschitz condition in the last two argument,
- $\phi(t)$ is continuous,
- $f$ is bounded.

A function $f$ that satisfies an inequality

$$\|f(t, y_2, z_2) - f(t, y_1, z_1)\| \leq L_1 \|y_2 - y_1\| + L_2 \|z_2 - z_1\| \tag{1.1.11}$$

for all $(t, y_2, z_2), (t, y_1, z_1)$ in a region $D$ is said to satisfiy a Lipschitz condition in $D$. We use a symbol $L_3$ for a Lipschitz constant corresponding to the third argument if a problem under consideration is an NDDE. We use symbols $L_y$ and $L_{y'}$ for Lipschitz constant for the function $y(t)$ and $y'(t)$ in inequalities

$$\|y(t_2) - y(t_1)\| \leq L_y \|t_2 - t_1\| \tag{1.1.12}$$

$$\|y'(t_2)-y'(t_1)\| \le L_{y'}\|t_2-t_1\|. \tag{1.1.13}$$

And $L_\tau$ for Lipschitz constant for the function $\tau$ in an inequality

$$\|\tau(t,y_2)-\tau(t,y_1)\| \le L_\tau\|y_2-y_1\|. \tag{1.1.14}$$

Let $y(t)$ be a solution of differential equation and $z_i(t)$ be a continuous approximation to $y(t)$ on $[t_i,t_{i+1}]$ associated with a method. The method is said to be convergent if

$$\max_{0\le i\le N}\max_{t_i\le t\le t_{i+1}}\|z_i(t)-y(t)\| \to 0 \text{ as } H=\max_i h_i \to 0 \text{ and } N\to\infty,$$

Where

$$h_i = t_{i+1}-t_i \text{ and } \sum_{i=0}^{N} h_i = t_N - t_0.$$

**Definition 1.1.8**

We define a solution (local) of the DDE (1.1.2) as the solution of

$$y_n'(t) = f(t, y_n(t), y_n(t-\tau(t,y_n(t)))) \quad \text{for} \quad t_n \le t \le t_{n+1}.$$

$y_n(t)=x(t)$, $t\le t_n$ Where $x(t)$ is a continuous approximation to $y(t)$ on $[t_0,t_n]$ associated with a method and $x(t)=\phi(t)$ for $t\le t_0$.

**Definition 1.1.9**

We define a solution (local) for NDDE (1.1.9) as the solution of

$$y_n'(t) = f(t, y_n(t), y_n'(t-\tau(t,y_n(t)))) \quad \text{for} \quad t_n \le t \le t_{n+1}.$$

$y_n(t)=x(t)$, $y_n'(t)=x'(t)$ $t\le t_n$ Where $x(t)$ is a continuous approximation to $y(t)$ on $[t_0,t_n]$ associated with a method and $x(t)=\phi(t), x'(t)=\phi'(t)$ for $t\le t_0$.