Conformable Fractional Differintegral Method For Solving Fractional Equations

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Abstract: The standard approaches to the problem of conformable fractional calculus has been studied extensively. Many researchers have shown that the obtained conditions for the theorem describing the general solution of; $y^{\alpha} = a(x)y + b(x)$ are generally weaker than those derived by using the classical norm-type expansion and compression theorem. In this paper, we propose conformable method for the fractional differential transform and established the prove for basic properties of differintegrals. Some solved examples have been reported to illustrate the possible application of the obtained results.

Index Terms: Conformable Fractional Derivatives, Fractional Calculus, α- differintegrals, the integrating factor, Caputo, Riemann– Liouville, positive solution

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1 INTRODUCTION

The fractional Calculus (FC) is a branch of mathematics that investigate the properties of integrals and derivatives of non-integer orders called differintegrals which represents either fractional (FD) derivatives or fractional integrals (FI). The following definitions are very important in this study;

  I.   conformable fractional derivative;[1]

$$f^{\alpha}(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon} \; , \; .$$

$$\forall x > 0 \;,\; 0 < \alpha < 1.[1]$$

  II. Riemann improper integral;[8]

$$D_{\alpha}^{-n} f(x) = \frac{1}{(n-1)!} \int_{a}^{x} (x-t)^{n-1} f(t) dt.$$

These definitions are a natural extension of the usual derivatives and integral, and it achieves the general characteristics of the fractional integration. Lately, scholars have given new definitions of fractional calculus which is conspicuously compatible with the known definitions of fractional derivative and integral [1]. Unlike previous definitions, this definition gratifies formularies the quotient of two functions with application to derivative of product. For more results on fractional derivatives and fractional differential equations (FDE), see the references therein: [1], [2], [3], [4], [5], [6].

2 PRELIMINARIES

In this section, we define some preliminary results which would help researchers to understand the basic concept behind this work.

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2.1 Fractional Integral:

The fractional Integral is an extension of fractional derivatives; there are some accepted and common definitions in many researches as we stated a number of important ones above. The conformable fractional Integral is defined as follows. Definition 1.[2]

Suppose f is differentiable on $[a, \infty), a > 0$. Then,

$$I_{\alpha}^{a}(f(x)) = I_{\alpha}^{1}(x^{\alpha-1}f) = \int_{a}^{x} \frac{f(t)}{t^{1-\alpha}} dt,$$

where $I_{1}^{a}(x^{\alpha-1}f)$ is the general form of Riemann improper integral, and $\alpha \in (0,1)$.

Theorem 1

Let $\alpha \in (0,1)$ and f, g be α-differentiable functions at a point $x > 0$, and if f is differentiable, then:

$$D_{\alpha}(f(x)) = x^{1-\alpha} \frac{d}{dx} f(x)$$

Proof:

By applying the fractional derivative definition we get;

$$D_{\alpha}(f(x)) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

Then,

Suppose that, $h = \varepsilon x^{1-\alpha}$, hence $h \to 0$ when $\varepsilon \to 0$

and $\varepsilon = x^{\alpha-1} h$,

$$D_{\alpha}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x^{\alpha-1} h}$$

$$D_{\alpha}(f(x)) = x^{1-\alpha} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Hence

$$D_{\alpha}(f(x)) = x^{1-\alpha} \times \frac{d}{dx}(f(x)) \quad \square$$

Theorem 2

Supposse f is continuous on $[a, \infty)$, $a > 0$. Then

$$D_{\alpha}(I_{a}^{a}(f(x))) = f(x), \text{for } x \geq a .$$

Proof:

Since f is smooth, by difinition1; and, $I_{a}^{a}(f(x))$ is differentiable. Then, by applying Theorem1, we get;

$$D_{\alpha}(I_{a}^{a}(f(x))) = x^{1-\alpha}\frac{d}{dt}I_{a}^{a}(f(x))$$

$$= x^{1-\alpha}\frac{d}{dx}\int_{a}^{x}\frac{f(t)}{t^{1-\alpha}}dt$$

$$= x^{1-\alpha}\frac{f(x)}{x^{1-\alpha}}$$

$$= f(x) \quad \square$$

Theorem 3

Let f be a differentiable function on (a, ∞), with f (a)=0, then

$$I_{a}^{a}(D_{\alpha}(f(x))) = f(x), \quad \text{for } x \geq a > 0.$$

Proof:

Let f be a differentiable function, then by the definition 1,

$$I_{a}^{a}(D_{\alpha}(f(x))) = I_{a}^{a}(\lim_{\varepsilon \to 0}\frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon})$$

$$\forall x > 0, \alpha \in (0,1).$$

Let $\quad h = \varepsilon x^{1-\alpha}$; so clearly as $\varepsilon \to 0$,

then $\quad h \to 0$ and $\varepsilon = x^{\alpha-1} \times h$.

Therefore,

$$I_{a}^{a}(D_{\alpha}(f(x))) = I_{a}^{a}(\lim_{h \to 0}\frac{f(x + h) - f(x)}{x^{\alpha-1} \times h})$$

$$I_{a}^{a}(D_{\alpha}(f(x))) = I_{a}^{a}(x^{1-\alpha} \times \lim_{h \to 0}\frac{f(x + h) - f(x)}{h})$$

$$= I_{a}^{a}(x^{1-\alpha} \times \frac{df}{dx})$$

Using Definition 1

$$I_{a}^{a}(D_{\alpha}(f(x))) = \int_{a}^{x}t^{1-\alpha}\frac{1}{t^{1-\alpha}}\frac{df}{dt}dt$$

$$I_{a}^{a}(D_{\alpha}(f(x))) = \int_{a}^{x}\frac{df}{dt}dt$$

$$= f(x) - f(a) .$$

But $f(a) = 0$ that implies,

$$I_{a}^{a}(D_{\alpha}(f(x))) = f(x) \quad \square$$

Hence,
The relation between fractional derivative and fractional integration is satisfies.

$$I_{a}^{a}(D_{\alpha}(f(x))) = D_{\alpha}(I_{a}^{a}(f(x))) = f(x), \text{for } x \geq a > 0.$$

From the above results, we derived the proposed scheme as follows.

3   METHOD FORMULATION

The derived formula for the solution of FDE with variable coefficients is defined as follows. The following Theorem is very useful in the derivation process.

Theorem 4 (Variable Coefficients)

Suppose $y = f(x)$ is differentiable and $a(x)$ and $b(x)$ are continuous functions, then the FDE,

$$y^{\alpha} = a(x)y + b(x) .$$

Has infinitely many solutions given by

$$y(x) = ce^{A(x)} + e^{A(x)}I_{\alpha}\left(e^{-A(x)} \times b(x)\right) \qquad (1)$$

Where $A(x) = I_{\alpha}a(x)$, $\alpha \in (0,1)$ and $c \in \Re$.

Remarks:

a. The expression in Eq. (1) is called the solution of fractional differential equation.

b. The function $\mu(x) = e^{-A(x)}$ is called the integrating factor of the equation.

Proof:

Writing the fractional differential equation with y on one side only, as

$$y^{\alpha} - a(x)y = b(x) .$$

Then multiply the above by $\mu(x)$, we need to solve for $\mu(x)$ such that, $\qquad \mu^{\alpha}(x) = -a(x)\mu(x)$

Suppose there is a function $\mu(x)$ satisfying the equation (4), then

$$\mu(x)y^{\alpha} + \mu^{\alpha}(x)y = \mu(x)b(x) \qquad (5)$$

By definition 1, we have

$$(\mu(x)y)^{\alpha} = \mu(x)b(x) \qquad (6)$$

Using Theorem 1 to integrating both sides of equation (6), then

$$\mu(x)y = I_{\alpha}(\mu(x)b(x)) + c_{1} \quad c_{1} \in \Re \qquad (7)$$

And so,

$$y = \frac{1}{\mu(x)}I_{\alpha}(\mu(x)b(x)) + \frac{c_{1}}{\mu(x)} \qquad (8)$$

To find $\mu(x)$, we go back to equation (4), then

$$\mu^{\alpha}(x) = -a(x)\mu(x) .$$

Take,

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$$\mu(x) = c_1 e^{-A(x)}, \quad \text{where} \quad A(x) = I_\alpha(a(x)).$$

Then,

$$\mu^\alpha(x) = -c_1 e^{-A(x)}.A^\alpha(x).$$

But, by Theorem 1 we get $A^\alpha(x) = a(x)$, then

$$\mu^\alpha(x) = -c_1 e^{-A(x)} a(x)$$

$$\mu^\alpha(x) = -a(x).\mu(x)$$

Thus $\mu(x)$ satisfies equation (4), back to equation (8)

$$y(x) = 1/c_2\, e^{A(x)} I_\alpha\big(c_2 e^{-A(x)} b(x)\big) + c_1/c_2\, e^{-A(x)}$$

Let $c_1/c_2 = c_3$

$$y(x) = e^{A(x)}\big(I_\alpha e^{-A(x)} b(x)\big) + c_3 e^{A(x)}.$$

4 RESULTS AND DISCUSSIONS

In this section, we present the solutions of solved examples to support our theoretical analysis. We begin with a given remark.

Remark: The relation between fractional derivatives and fractional integral are solved using example 1.

Example 1

If $f(x) = \sin x$, then show that

$$I_\alpha^\alpha(D_\alpha(x)) = \sin x, \text{ for } x \geq \pi.$$

Solution:

Since f (x) is differentiable function, then by applying definition 1 on

$$D_\alpha(\sin x)$$

we get

$$I_\alpha^\pi\big(D_\alpha(\sin(x))\big) = I_\alpha^\pi\big(x^{1-\alpha} d/dx \sin(x)\big)$$

$$I_\alpha^\pi\big(D_\alpha(\sin(x))\big) = I_\alpha^\pi(x^{1-\alpha}\cos(x))$$

Applying definition 1.1 on $I_\alpha^\pi(x^{1-\alpha}\cos(x))$, we have

$$I_\alpha^\pi\big(D_\alpha(\sin(x))\big) = \int t^{1-\alpha}\cos(t)/t^{1-\alpha} d(t)$$
$$= \sin(x) = \sin(\pi)$$

Hence

$$I_\alpha^\pi\big(D_\alpha(\sin(x))\big) = \sin(x).$$

Problem 2.

Consider the following initial fractional equation.

$$y^{1/2} + y = x^2 + 2x^{3/2}, y(0) = 0$$

Solution

Rewrite the equation with y on only one side,

$$y^{\left(\frac{1}{2}\right)} = -y + x^2 + 2x^{\left(\frac{1}{2}\right)}$$

Using Theorem 4 that given general solution

$$y^\alpha = a(x)y + b(x),$$

where

$$a(x) = -1, \quad b(x) = x^2 + 2\sqrt{x^3}$$

As in the above theorem the integrating factor;

$$\mu(x) = ce^{-I(1/2)(-1)}$$

$$= ce^{2\sqrt{x}}$$

which implies

$$ce^{2\sqrt{x}}y^{1/2} + ce^{2\sqrt{x}}y = ce^{2\sqrt{x}}(x^2 + 2\sqrt{x^3})$$

But

$$(ce^{2\sqrt{x}}y)^{1/2} = ce^{2\sqrt{x}}y^{1/2} + ce^{2\sqrt{x}}y$$

So,

$$c(e^{2\sqrt{x}}y)^{1/2} = ce^{2\sqrt{x}}(x^2 + 2\sqrt{x^3})$$

By Theorem 4

$$I_{1/2}((e^{2\sqrt{x}}y)^{1/2}) = e^{2\sqrt{x}}y$$

Since, $e^{(0)}y(0) = 0$

From Theorem 1

$$ye^{2\sqrt{x}} = \int \frac{\quad}{\quad} d(t),$$

$$\int (t^{3/2}e^{2\sqrt{t}} + 2te^{2\sqrt{t}})d(t),$$

Then integration by parts we get.
Let $u = \sqrt{t} \to 2udu = dt$. If

$$ye^{2\sqrt{t}} = 2\left[\frac{1}{2}u^4 e^{2u} - -\frac{4}{4}u^3 e^{2u}x + -u\,e\sqrt{x} - \frac{24}{16}u\,e^{2u} + \frac{24}{32}e^{2u}\right]_0^{\sqrt{t}}$$

Also,

$$t = 0 \to u = 0$$

$$+ 4\left[\frac{1}{2}u^3 e^{2u} - -\frac{3}{4}u^2 e^{2u} + \frac{6}{8}u\,e^{2u} - \frac{6}{16}e^{2u}\right]_0^{\sqrt{t}}$$

Substituting u and solving the equation gives:

$$ye^{2\sqrt{t}} = \left[\int_0^{\sqrt{t}} e^{2u}(u^3 + 2u^2)\times 2udu\right.$$
$$= \left[(x^2 e^{2\sqrt{t}} - 2x e^{2\sqrt{t}} + 3xe^{2\sqrt{t}} - 3\sqrt{x}e^{2\sqrt{t}} + \frac{3}{2}e^{2\sqrt{t}}) - \frac{3}{2}\right.$$
$$= 2\int_0^{\sqrt{t}} u^4 e^{2u} + 2u^3 e^{2u})du$$
$$\left. + \left[(2x^2 e^{2\sqrt{t}} - 3xe^{2\sqrt{t}} + 3\sqrt{x}e^{2\sqrt{t}} - \frac{3}{2}e^{2\sqrt{t}}) - \frac{3}{2}\right]\right.$$
$$= 2\int_0^{\sqrt{t}} u^4 e^{2u}du + 4\int_0^{\sqrt{t}} u^3 e^{2u}du$$

$$ye^{2\sqrt{t}} = x^2 e^{2\sqrt{t}} - 3, \text{ so}$$

$$y = \frac{x^2 e^{2\sqrt{t}} - 3}{e^{2\sqrt{t}}}$$

$$y = x^2 - 3e^{-2\sqrt{t}} \quad \square$$

Example 3.

Consider the following initial fractional equation

$$y^{\left(\frac{1}{2}\right)} - y = 0, \quad y(0) = 0.$$

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Solution:

Rewrite the equation with y on only one side,

$$y^{\left(\frac{1}{2}\right)} = y$$

Given the general solution $y^{\alpha} = a(x)y + b(x)$ then by Theorem 4, we have,

$$a(x) = 1 \quad , \quad b(x) = 0$$

and the integrating factor;

$$\mu(x) = ce^{-I_{\frac{1}{2}}(1)}$$

We want to found the value of,

$$I_{\frac{1}{2}}(1) = \int_0^x \frac{1}{t^{1-0.5}}dt,$$

$$I_{\frac{1}{2}}(1) = \int_0^x t^{-0.5}dt \quad \rightarrow \quad I_{\frac{1}{2}}(1) = 2\sqrt{x}$$

then, $\mu(x) = ce^{-2\sqrt{x}}$

Hence, $ce^{-2\sqrt{x}}y^{\frac{1}{2}} - ce^{-2\sqrt{x}}y = 0$

$$\left(e^{-2\sqrt{x}}y\right)^{\frac{1}{2}} = 0$$

By Theorem 4

$$I_{\frac{1}{2}}(e^{-2\sqrt{x}}y)^{\frac{1}{2}} = I_{\frac{1}{2}}(0);$$

$$e^{-2\sqrt{x}}y = c$$

$$\Rightarrow y = ce^{2\sqrt{x}} \quad \square$$

5 CONCLUSIONS

In this study, some properties of fractional derivative and integrals were established, and the fractional integral for constant function is zero was proved. We also obtained a relationship between the fractional calculus and calculus that satisfies, and a general solution was obtained to fractional differential equations $y^{\alpha} = a(x)y + b(x)$.

6 ACKNOWLEDGMENT

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