Deanship of Graduate Studies

Al – Quds University

Oscillation of Solutions of Third Order Linear Neutral Delay Differential Equations

Ayoub Hasan Ahmad Saleh

M. Sc. Thesis

Jerusalem – Palestine

1436/2015

Deanship of Graduate Studies

Al – Quds University

Oscillation of Solutions of Third Order Linear Neutral Delay Differential Equations

Ayoub Hasan Ahmad Saleh

M. Sc. Thesis

Jerusalem – Palestine

1436/2015

Oscillation of Solutions of Third Order

Linear Neutral Delay Differential Equations

Prepared by :

Ayoub Hasan Ahmad Saleh

B. Sc. Mathematics, Birzeit University,

Palestine

Supervisor : Dr. Taha Abu Kaff

Thesis Submitted in Partial Fuifillment of the Requirements for the Degree of Master of Mathematics at Al – Quds University

1436/2015

Al-Quds University

Deanship of Graduate Studies

Graduate Studies/Mathematics

Thesis Approval

Oscillation of Solution of Third Order Linear Neutral Delay Differential Equation

Prepared By : Ayoub Hasan Ahmad Saleh

Registration No: 20913396

Supervisor: Dr. Taha Abu Kaff

Master thesis submitted and accepted, Date: 10/5/2015.

The names and signatures of the examining committee members are as follows:

1) Dr. Taha Abu Kaff Head of committee: signature………………

2) Dr. Jamil Jamal Internal Examiner: signature………………

3) Dr. Ali Zain External Examiner: signature………………

Jerusalem-Palestine

1436/2015

Dedication

To my mother,

my father,

my wife,

my daughter,

my sons,

my brothers,

and

my sisters,

I dedicate this work.

Declaration

The work provided in this thesis , unless otherwise references , is the researcher's own work , and has not been submitted elsewhere for any other degree or qualification .

Student's Name : Ayoub Hasan Ahmad Saleh

Signature : ....

Date: 10/ 5 /2015

Acknowledgment

I gratefully extended my thanks to those people who helped in completing this work, specially and personally to my supervisor, Dr. Taha Abu Kaff for his helps and advices throughout the period of the study.

I would like to thank Dr. Khalid Salah from Al-Quds University for his Valuable help.

Also, my thanks to the other members of the department of mathematics at Al-Quds University.

Table of Contents

Index of Special Notations

$\mathbb{N}$ set of natural numbers

$\mathbb{R}$ set of real numbers

$\mathbb{R}^+$ set of positive real numbers

$f(t) \in C(D) = \{f : D \to \mathbb{R} :\ f$ is continuouse function $\}$

$f(t) \in C^1(D) = \{f : D \to \mathbb{R} :\ f$ is continuously differentiable function $\}$

$f(t)$ $f : [\,0\,,\infty) \to \mathbb{R}$ function of time t .

$y''(t)$ the usual second derivative $\dfrac{d^2y}{dt^2}$.

sup S the least upper bound , or the supremum of the set S .

$\int_a^b f(t)dt$ usual definite integral .

■ ends of proofs .

The form (A . B . C) A is the number of chapter .

B is the number of section .

C is the serial number .

Abstract

It is well-known that there are many types of differential equations and each type has its own applications and solution, one of these types known as neutral delay linear differential equations. This type of equations has solutions divided into three shapes: oscillatory , almost oscillatory , and nonoscillatory solutions. And to decide whether the solution is oscillatory or nonoscillatory, we have some necessary and sufficient conditions that must satisfied .

In this thesis , we study the oscillation of nontrivial real valued solutions $y(t)$ to the third order linear neutral delay differential equations of the form

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\big(y(t)+p(t)y(t-\tau)\big)\right)\right)+f(t)y(t-\sigma)=0 \qquad (1N1-A)$$

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\big(y(t)+p(t)y(\tau(t))\big)\right)\right)+f(t)y(\sigma(t))=0 \qquad (1N1-B)$$

$$\frac{d}{dt}\left(r(t)\,\frac{d^2}{dt^2}\left(\big(y(t)+p(t)y(\tau(t))\big)\right)\right)+f(t)y(\sigma(t))=0 \qquad (1N2-B)$$

$$\frac{d^2}{dt^2}\left(r(t)\,\frac{d}{dt}\big(y(t)+p(t)y(t-\tau)\big)\right)+f(t)y(t-\sigma)=0 \qquad (2N\,1-A)$$

$$\frac{d^2}{dt^2}\left(r(t)\,\frac{d}{dt}\left(y(t)+p(t)y\big(\tau(t)\big)\right)\right)+f(t)y\big(\sigma(t)\big)=0 \qquad (2N1-B)$$

where

$p(t)$, $f(t) \in C\,([t_0\,,\infty),\mathbb{R}\,)$, $f(t) \geq 0$, $r_1(t)$, $r_2(t)$, $r(t) \in C^1\,([t_0\,,\infty),\mathbb{R}^+\,)$

and τ , $\sigma \in [0\,,t\,)$.

The purpose of this thesis is to examine sufficient conditions established so that every solution to equations $(1N1-A)$, $(1N1-B)$, $(2N1-A)$, $(2N1-B)$, $(1N2-A)$, $(1N2-B)$ is either oscillatory or converge to zero . In particular , we extend the results that obtained in K. V. V. Seshagiri Rao to the equation

$$\frac{d}{dt}\left(r(t)\,\frac{d^2}{dt^2}\left(\big(y(t)+p(t)y(t-\tau)\big)\right)\right)+f(t)y(t-\sigma)=0 \qquad (1N2-A)$$

and follow the similar steps that used specially in Tongxing Li in studying equation $(1N1-B)$ to examine oscillatory properties that presented for the equation $(1N1-A)$ when the function $r_1(t)=1$. These criteria improve and complement those results in the literature . Moreover , we give the proof of the comparison lemma that appear in Seshagiri Rao , and some illustrating examples .

Introduction

Many important and significant problems in engineering, physical sciences, and social sciences when formulated in mathematical terms, require the determination of a function satisfying an equation that has one or more derivatives of an unknown function, which may be a function of time t . Such equations are called differential equations . Newton's second law of motion

$$m\frac{d^2u(t)}{dt^2} = F\left(t, u(t), \frac{du(t)}{dt}\right) \qquad (1)$$

for the position $u(t)$ of the particle acted on a force F is a good example, the position $u(t)$, and the velocity $\dfrac{d\,u(t)}{dt}$.

If the differential equation of the form

$$F\left(t\,, y(t)\,, y'(t)\,, y''(t), \ \ . \ . \ . \ \ , \ \ y^{(n)}(t)\right) = 0 \qquad (2)$$

Or $$y^{(n)}(t) = f\left(t\,, y(t)\,, y'(t)\,, y''(t), \ \ . \ . \ . \ \ , \ \ y^{(n-1)}(t)\right) \qquad (3)$$

$n \in \mathbb{N}$ is called nth order differential equation, where only the function y(t) and its derivatives are used in determining if the differential equation is linear, see [31] .

A solution of equation (3) on the interval $(\alpha\,, \beta)$ is a function $\emptyset$ such that $\emptyset'(t)\,, \emptyset''(t), \ . \ . \ . \ , \ \emptyset^{(n-1)}(t)$ exist and satisfy

$\emptyset^{(n)}(t) = f\left(t\,, \emptyset(t)\,, \emptyset'(t)\,, \emptyset''(t), \ . \ . \ . \ , \ \emptyset^{(n-1)}(t)\right)$ for every $t \in (\alpha\,, \beta)$.

We will assume that the function f is real valued function and we are interested in obtaining real valued solutions $y = \emptyset(t)$ in our work. Now, if the solution $y = \emptyset(t)$ has arbitrary large zeros on interval $(t_x\,, \infty)$ then this solution is said to be oscillatory; Otherwise, it is said to be nonoscillatory. If all solutions are oscillatory or converge to zero asymptotically then the differential equation is said to be almost oscillatory.

A type of differential equation in which the derivative of the unknown function at certain time (t) is given in terms of the values of the function at previous times $(t-\tau)$ are called delay equations . It is also called time-delay-systems , equation with deviating argument . The simplest constant delay has the form

$$y'(t) = f\big(t\,, y(t)\,, y(t-\tau_1)\,, y(t-\tau_2),\ \ .\ .\ .\ ,\ \ y(t-\tau_n)\big) \qquad (4)$$

where the delays (lags) τ_j's are positive constants. More generally , state dependent delays may depend on the solution that is $\tau_j = \tau_j\big(t\,, y(t)\big)$.

Oscillation problems for first order ordinary differential equation with deviating arguments are interesting from a theoretical as well as the practical point of view. In fact , Bernoulli (1728), while studying the problem of sound in a tube with finite size , investigated the properties of solutions of first order ordinary differential equation with deviating argument , this was the first work in this area . Myskis investigated several oscillation problems of this type of equations, see [7].

Now a differential equation in which the highest order derivative of the unknown function appears both with and without delays is called Neutral Delay Differential Equation (NDDE). Concerning existence , uniqueness and continuous dependence for (NDDE), we refer to Driver [25 , 26] .

There has been great interest in studying the oscillatory behavior of differential equations , since there are many types of them not easy to solve and has many applications , see [7, 9 , 10 , 16 , 18] and the references cited therein . In fact oscillation theory of neutral delay differential equations has grown rapidly and has many interesting applications from the real world in many fields . Delay differential equations are important class of dynamical systems , so they often arise in either natural or technological control problems. In these systems, a controller monitors the state of the system, and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action. They also have applications to electric networks containing lossless transmission lines. Such networks appear in high speed computers where lossless transmission lines are used to interconnect switching circuits. They also occur in problems dealing with vibrating masses attached to elastic bar and in some variational problems , see [12 , 22 , 28] . In addition to that, they now occupy a place of central importance in the biological applications since they give a better description of fluctuations, in population dynamics and epidemiology, see [1 , 8 , 21 , 23] for more applications . Lastly , they appear in dynamical economics as a delay differential equation models of cyclic economic behavior , and it is now known that a broad spectrum of dynamic behaviors can be found in nonlinear delay differential equations , see Saari [5], Mackey [17], Franke [27], and the references cited therein . In the last forty years , there has been many researches that study the oscillatory behavior of linear neutral delay differential equations of the form:

$$\frac{d^n}{dt^n}\left(y(t)+p(t)y(t-\tau)\right)+f(t)y(t-\sigma)=0 \qquad (5)$$

where $n \in \mathbb{N}$, $t - \tau \leq t$, $t - \sigma \leq t$, see [6] . For n = 1, equation (5) has been studied by Ladas and Sficas [6], Grammatikopoulos, Grove Ladas [19], and Zhang , Wang[32] , in these papers they established conditions for the oscillation of all solutions of first order linear neutral delay differential equations. For n=2 equation (5) has been studied by M. K. Grammatikopoulos , Grove Ladas and A. Meimaridou[20] and Philo[4], Dzurina , and Stavroulakis [11], Agarwal Shieh and Yeh [29] , Seshagiri Rao and Sai Kumar [14] in these papers they established conditions for the oscillation of all solutions of second order linear neutral delay differential equations. For n=3 , third order linear neutral delay differential equations have received less attention compared to first and second order . This research concerned with third order linear neutral delay differential equations of the form:

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\left(y(t)+p(t)y(t-\tau)\right)\right)\right)+f(t)y(t-\sigma)=0 \qquad (1N1-A)$$

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\left(y(t)+p(t)y(\tau(t))\right)\right)\right)+f(t)y(\sigma(t))=0 \qquad (1N1-B)$$

Some researches study equations $(1N1-A)$ and $(1N1-B)$ when $\mathrm{p(t)}=0$ and $r_1(t)=1$, for that we mention the works of Cemil [3] , Erbe [15] , and Paul [24]. And for $p(t)\neq 0$ equation $(1N1-A)$ was studied by Seshagiri Rao and Sai Kumar [13], and references therein. For $p(t)\neq 0$. Equation $(1N1-B)$ was studied by Tongxing Li , Chenghui Zhang , and Guojing Xing , see [30] and references cited therein .

From now , by solution of equations $(1N1-A)$ and $(1N1-B)$ we mean a nontrivial function $y(t)\in C\left(\left[t_y\,,\infty\right)\right)$, where $t_y\geq t_0$ which satisfies $(1N1-A)$ and $(1N1-B)$ on $\left[t_y\,,\infty\right)$. We consider only those solutions of $y(t)$ of $(1N1-A)$ and $(1N1-B)$ which satisfy $sup\{|y(t)|: t\geq T\}>0$ for all $T\geq t_y$ of $(1N1-A)$ and $(1N1-B)$ possesses such solution . Also when we write a functional inequality , it will be assumed to hold for sufficiently large t in our subsequent discussion .

The purpose of this research is to examine oscillatory behavior of third order linear neutral delay differential equations $(1N1-A)$ and $(1N1-B)$ and to establish some sufficient conditions which ensure that any solution of this type of equations are oscillate or converge to zero . We give examples to illustrate the main results .

This thesis consists of four chapters:

Chapter One: is devoted to introduce basic definitions lemmas and theorems needed in our proofs .

Chapter Two: is devoted to discuss the conditions that guarantee oscillation of every solution of third order linear neutral delay differential equation of the form

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\left(\left(y(t)+p(t)y(t-\tau)\right)\right)\right)+f(t)y(t-\sigma)=0 \quad (1N2-A)$$

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\left(\left(y(t)+p(t)y(\tau(t))\right)\right)\right)+f(t)y(\sigma(t))=0 \quad (1N2-B)$$

Chapter Three: is devoted to discuss the main result and the conditions that guarantee oscillation of every solution of third order linear neutral delay differential equation of the form

$$\frac{d^2}{dt^2}\left(r(t)\ \frac{d}{dt}\ (\ y(t)+p(t)y(t-\tau)\right)+f(t)y(t-\sigma)=0 \qquad (2N1-A)$$

$$\frac{d^2}{dt^2}\left(r(t)\ \frac{d}{dt}\ (\ y(t)+p(t)y(\tau(t))\right)+f(t)y\big(\sigma(t)\big)=0 \qquad (2N1-B)$$

Chapter Four: is devoted to answer the problem that appear in the summary of Seshagiri Rao and Sai Kumar[13], and to give conditions guarantee oscillatory of equation $(1N1-A)$ and generalize it to the equations of the form $(1N1-B)$.

Chapter One

Preliminaries

This chapter contains some basic inequalities definitions and results which are essential for the proofs and studying the main results of this research.

1.1 Some Basic Lemmas and Theorems

Lemma 1.1.1 [13]

Let $u, v, x \in \mathbb{R}$ then $v\,x - u\,x^2 \leq \frac{1}{4}\,\frac{v^2}{u}$, $u > 0$.

Proof :

It is clear that if u, v, x in $\mathbb{R}$ then $(v - 2\,u\,x)^2 \geq 0$

which implies $v^2 - 4v\,u\,x + 4\,u^2x^2 \geq 0$,

so $v^2 \geq 4v\,u\,x - 4\,u^2x^2$,

dividing by $4u$, we get

$v\,x - u\,x^2 \leq \frac{1}{4}\frac{v^2}{u}$. The proof is complete . ■

Lemma 1.1.2: [13]

Suppose $x(t)$ is twice continuously differentiable real valued function on the Interval $[t_0, \infty)$ such that $x(t) > 0$, $x'(t) \geq 0$, $x''(t) \leq 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$ and $\sigma > 0$, then for each K with $0 < K < 1$, there exists $t_2 \geq t_1$

such that

$$\frac{x(t-\sigma)}{x(t)} \geq K \frac{(t-\sigma)}{t} \quad , \qquad t \geq t_2 \tag{1.1.1}$$

Proof:

Let $x : [\, t - \sigma \,, t \,] \rightarrow R \,, \sigma > 0$. Given the function $x(t)$ is continuous on the interval $[\, t - \sigma \,, t \,] \subset [t_0 \,, \infty)$ and differentiable on the interval $(\, t - \sigma \,, t \,)$ so $x(t)$ satisfies Lagrange's Mean Value Theorem. And so, there exists $\xi \,\epsilon \left(\, t - \sigma \,, t \,\right)$ such that

$$\frac{x(t) - x(t-\sigma)}{t - (\, t - \sigma)} = x'(\,\xi\,)\,. \tag{1.1.2}$$

But $x'(t) > 0$ and $x''(t) \leq 0$ for each $t_1 > t_0$, so for $t > \xi > t - \sigma$, we have $0 \leq x'(t) \leq x'(\,\xi\,) \leq x'(t-\sigma)$.

Using $x'(\xi) < x'(t-\sigma)$ and equation (1.2.2), we have

$$\frac{x(t) - x(t-\sigma)}{t - (t-\sigma)} \leq x'(t-\sigma)\,,$$

so

$$x(t) - x(t-\sigma) \leq x'(t-\sigma)\big(t - (t-\sigma)\big)\,,$$

so

$$x(t) \leq x(t-\sigma) + x'(t-\sigma)\big(t - (t-\sigma)\big)\,,$$

hence

$$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{x'(t-\sigma)}{x(t-\sigma)}\big(t-(t-\sigma)\big)\,. \qquad (1.1.3\,)$$

Once again, applying Lagrange's Mean Value Theorem on $x(t)$ on the interval $[t_1, t-\sigma] \subset [t_0, \infty)$ for $t-\sigma > t_1 > t_0$, so there exists $\eta \in (t_1, t-\sigma)$ where

$$\frac{x(t-\sigma)-x(t_1)}{(t-\sigma)-t_1} = x'(\eta)\,. \qquad (1.1.4)$$

But $x'(t)$ is non increasing, therefore

$$t-\sigma > \eta > t_1 \;\; => \;\; 0 < x'(t-\sigma) \leq x'(\eta) \leq x'(t_1)\,.$$

Using $x'(\eta) \geq x'(t-\sigma)$ and equation (1.2.4), this implies

$$\frac{x(t-\sigma)-x(t_1)}{(t-\sigma)-t_1} \geq x'(t-\sigma)\,,$$

hence

$$x(t-\sigma)-x(t_1) \geq x'(t-\sigma)(\,(t-\sigma)-t_1)\,. \qquad (1.1.5)$$

Since $x(t_1) > 0$, we get

$$x(t-\sigma) \geq x'(t-\sigma)\big((t-\sigma)-t_1\big)$$

and since $x'(t-\sigma) > 0$, we have

$$\frac{x(t-\sigma)}{x'(t-\sigma)} \;\geq\; (t-\sigma)-t_1$$

$$\frac{x(t-\sigma)}{x'(t-\sigma)} \;\geq\; \left(1-\frac{t_1}{(t-\sigma)}\right)(t-\sigma).$$

Take any $t_2 \leq t$ such that $t_1 \leq t_2-\sigma \leq t-\sigma$, this implies

$$1 \geq \frac{t_1}{t_2 - \sigma} \geq \frac{t_1}{t - \sigma} > 0$$

hence

$$0 < 1 - \frac{t_1}{t_2 - \sigma} < 1 \,.$$

So if given constant $K \epsilon (0, 1)$, then we can find $t_2 - \sigma \geq t_1, \sigma > 0$, or $t_2 \geq t_1 + \sigma$ such that

$$K = 1 - \frac{t_1}{(t_2 - \sigma)} \quad \text{and} \quad \frac{x(t-\sigma)}{x'(t-\sigma)} \geq \mathrm{K}\,(t - \sigma) \quad \text{for} \quad t \geq t_2 \;.$$

From (1.1.3) and for every $t \geq t_2$, we have

$$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{x'(t-\sigma)}{x(t-\sigma)} \left(t - (t - \sigma)\right),$$

or

$$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{1}{\frac{x(t-\sigma)}{x'(t-\sigma)}} \left(t - (t - \sigma)\right)$$

$$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{1}{K(t-\sigma)} \left(t - (t - \sigma)\right)$$

$$\frac{x(t)}{x(t-\sigma)} \leq 1 + \frac{t}{K(t-\sigma)} - \frac{(t-\sigma)}{K(t-\sigma)} = 1 + \frac{t}{K(t-\sigma)} - \frac{1}{K} = \frac{t}{K(t-\sigma)} - \left(\frac{1}{K} - 1\right),$$

or

$$\frac{x(t)}{x(t-\sigma)} \leq \frac{t}{K(t-\sigma)} - \left(\frac{1-K}{K}\right)$$

But $0 < K < 1$, so $\dfrac{1-K}{K} > 0$, hence

$$\frac{x(t)}{x(t-\sigma)} \leq \frac{t}{K\,(t-\sigma)} \quad ,$$

but $x(t)$ and $x(t-\sigma)$ are positives , hence

$$\frac{x(t-\sigma)}{x(t)} \geq K\,\frac{(t-\sigma)}{t} \quad .$$

And this completes the proof . ■

Lemma 1. 1. 3 : Error! Reference source not found.

Assume that $x(t) > 0$, $x'(t) \geq 0$, $x''(t) \leq 0$ on $(t_0\,,\infty)$, then for each $k \in (0\,,1)$ there exists a $t_k \geq t_0$ such that

$$\frac{x(\sigma(t))}{x(t)} \geq k\,\frac{\sigma(t)}{t} \ \text{ for } t \geq t_k \ , \ \sigma(t) \leq t.$$

Proof:

The proof is the same as the proof of lemma 1.2.2 . See lemma 3 in [1] . ■

Lemma 1. 1. 4 :[13]

Let $z(t) = y(t) + p(t)\,y(t-\tau)$ where $z(t)$ is three contiuously differentiable real valued function on $[t_0\,,\infty)$ and suppose that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ and $z'''(t) < 0$ on $[\,t_1\,,\infty\,)$ for some $t_1 \geq t_0$. Then there exists $t_2 \geq t_1$ such that $z(t) \geq \frac{1}{2}\,Mt\,z'(t)$, $t \geq t_2$, for each M ; $0 < M < 1$.

Proof:

Define a function $H(t)$ for $t \geq t_2 \geq t_1$ as

$$H(t) = (t - t_2)z(t) - \frac{M(t-t_2)^2}{2} z'(t). \qquad (1.1.6\,)$$

It is clear that $H(t_2) = 0$ and $H(t)$ is a differentiable function , so

$$H'(t) = (t - t_2)z'(t) + z(t) - \left\{\frac{M}{2}.2(t-t_2)z'(t) + \frac{M}{2}(t-t_2)^2 z''(t)\right\}.$$

$$H'(t) = z(t) + (t-t_2)(1-M)z'(t) - \frac{M}{2}(t-t_2)^2 z''(t). \qquad (1.1.7)$$

Now we need to prove that $H'(t) > 0$ and $H(t) > 0$

Using Taylor′s Theorem and $z''(t)$ is nonincreasing function , we have

$$z(t) \geq z(t_2) + (t-t_2)\, z'(t_2) + \frac{(t-t_2)^2}{2} z''(t) \ ,$$

and substituting it in equation (1.1.7) , we get

$$H'(t) \geq z(t_2) + (t-t_2)\, z'(t_2) + \frac{(t-t_2)^2}{2} z''(t) + (t-t_2)(1-M)z'(t) - \frac{M}{2}(t-t_2)^2 z''(t)$$

$$H'(t) \geq z(t_2) + (t-t_2)\, z'(t_2) + (t-t_2)(1-M)z'(t) + (\frac{1-M}{2})(t-t_2)^2 z''(t)\,. \quad (1.1.8)$$

All terms on the right side in (1.2.8) are positive terms , hence $H'(t) > 0$.

So for

$$t > t_2 \implies H(t) > H(t_2) = 0 \ ,$$

hence

$$H(t) > 0 \ \text{ for all } \ t \in [t_0\,, \infty) \ .$$

So

$$(t - t_2)\, z(t) - \frac{M(t-t_2)^2}{2}\, z'(t) > 0 \quad \text{for all} \quad t > t_2 \ .$$

Hence

$$(t - t_2) z(t) \ > \ \frac{M(t-t_2)^2}{2}\, z'(t) \ .$$

So

$$\frac{z(t)}{z'(t)} > \frac{M(t-t_2)}{2} \ ,$$

which gives

$$\frac{z(t)}{z'(t)} \geq \frac{Mt}{2} - \frac{Mt_2}{2} \ ,$$

since $0 < M < 1$ and $t_2 > t_0 > 0$, we have

$$\frac{z\,(t)}{z\,'(t)} \geq \frac{Mt}{2} \ ,$$

hence

$$z\,(t) \geq \frac{1}{2}\, Mt\, z'(t)\,, \ \text{for all} \ t \geq t_2 \ .$$

This completes the proof . ■

1.2 Some Basic Definitions:

Here are some basic definitions that needed later in our literature .

Definition 1.2.1:[1]

A delay differential equation is an ordinary differential equation where the derivative at any time t depends on the solution at prior times, where the time delays (lags) are positive quantities.

For example :

$$y'(t) = f\left(t, y(t), y(t-\tau_1), y(t-\tau_2), \dots, y(t-\tau_k)\right) \quad , \tau_j > 0 \;\; \forall \; j = 1, 2, 3, \; . \; . \; . \; k$$

is the general form of the simplest constant delay equations.

Definition 1.2.2: [13]

A solution of a delay differential equation is called oscillatory if it has arbitrarily large zeros on the interval $[t_y, \infty)$; otherwise it is called nonoscillatory.

This definition means that a solution $y(t)$ of a delay differential equation is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \longrightarrow \infty$ as $i \longrightarrow \infty$ and $y(t_i) = 0$ for all $(i = 1, 2, 3, \; . \; . \; .)$, and nonoscillatory if and only if $y(t) \neq 0$ for all large t.

Definition 1.2.3 : [30]

A solution of a delay differential equation is called almost oscillatory if it has arbitrarily large zeros or converge to zero asymptotically on the interval $[t_y, \infty)$.

Definition 1.2.4 :[13]

A delay differential equation is said to be oscillatory if all its solutions are oscillate and nonoscillatory if at least one of its solutions is nonoscillate .

Definition 1.2.5 : [26]

A differential equation in which the highest order derivatives of the unknown function appears both with and without delays is called neutral delay differential equation .

For example , the following differential equations :

$$\big(y(t) + p(t)y(\tau(t))\big)' + f(t)y\big(\sigma(t)\big) = 0\,, \tag{1.2.1}$$

$$\Big(a(t)\ \big(y(t) + p(t)y(\tau(t))\big)'\Big)' + f(t)y\big(\sigma(t)\big) = 0\,, \tag{1.2.2}$$

$$\Big[a(t)\Big(b(t)\ \big(y(t) + p(t)y(\tau(t))\big)'\Big)'\Big]' + f(t)y\big(\sigma(t)\big) = 0\,, \tag{1.2.3}$$

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\ (\,y(t) + p(t)y(\tau(t))\right) + f(t)y\big(\sigma(t)\big) = 0\,, \tag{1.2.4}$$

where $0 < p(t) < 1$ and $0 \le \tau(t) \le t$, $0 \le \sigma(t) \le t$, are neutral delay differential equations .

Chapter Two

Oscillation of Solution of the Equation of the Form

$(\mathbf{1N\ 2-A})$ and $(\mathbf{1\ N\ 2-B})$

2.1 Introduction :

We shall consider in this chapter the two forms of a third order linear neutral delay differential equations

$$(1N2-A): \qquad \frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\ (y(t)+p(t)y(t-\tau)\right)+f(t)y(t-\sigma)=0$$

and

$$(1\ N\ 2-B): \qquad \frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\ (y(t)+p(t)y(\tau(t))\right)+f(t)y\big(\sigma(t)\big)=0$$

K . V. V. Seshagiri Rao [13] has discussed the first form $(1N2-A)$ and established sufficient conditions for oscillation of solutions of this type of linear neutral delay differential equations when

$p(t)$, $f(t)\ \in C\ ([t_0\,,\infty),\mathbb{R}\,)$ and $f(t)\ \geq 0$, $r(t)\in C^1\left([t_0\,,\infty),(0\,,\infty)\right)$

$$r'(t)\geq 0 \text{ and } \int_{t_0}^{\infty}\frac{1}{r(s)}\,ds=\infty\,.$$

2.2 Oscillation Conditions of the Solution of the Equation $(1N2-A)$

Theorem 2.2.1: [13]

Assume that :

(H_1): $r(t) \in C^1([\,t_0\,,\infty),(\,0\,,\infty\,))$, $r'(t) \geq 0$ for $t \geq t_0$.

(H_2): $p(t) \in C([\,t_0\,,\infty), R)$, where $0 \leq p(t) \leq p < 1$ and p is constant .

(H_3): $f(t) \in C([\,t_0\,,\infty),[\,0\,,\infty))$

(H_4): There exists a positive decreasing function $q(t)$ such that $f(t) \geq q(t)$

for $t \in [t_0\,,\infty)$. (or $q(t) > 0$, $q'(t) < 0\,, q(t) \leq f(t)$ $\forall\, t \geq t_0$)

(H_5): $\displaystyle\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{r(u)}\int_{u}^{\infty} f(s)ds\right] du\, dv = \infty$,

(H_6): $\displaystyle\lim_{t\to\infty} \sup \int_{t_1}^{t} [\,2\, q(s)\big(1-p(s-\sigma)\big)K\, M\, (s-\sigma)^2 - \frac{r(s)}{s}]\, ds = \infty$,

for some K , $M \in (\,0\,,1)$ for sufficiently large $t_1 \geq t_0$.

(H_7): $\displaystyle\lim_{t\to\infty} \sup \int_{t_2}^{t}\left[2\, q(s)\big(1-p(s-\sigma)\big)K\, M\, \frac{(s-\sigma)^2}{S} - \frac{1}{R(s)r(s)}\right] ds = \infty$,

hold for some K , $M \in (\,0\,,1)$ for sufficiently large $t_2 \geq t_0$, where

$$R(t) = \int_{t}^{\infty}\frac{1}{r(s)}\, ds\,,$$

then equation $(1N2-A)$ is almost oscillatory.

Proof :

Let $z(t) = y(t) + p(t)\, y(t-\tau)$. (2.2.1)

Suppose that equation $(1N2 - A)$ has a nonoscillatory solution $y(t)$.

Without loss of generality suppose that $y(t)$ is positive solution of equation $(1N2 - A)$, that is $y(t) > 0$. Then there exist three possible cases for $z(t)$

(I) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $z'''(t) \leq 0$,

(II) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$,

(III) $z(t) > 0$, $z'(t) > 0$, $z''(t) < 0$, $(r(t)(z''(t)))' \leq 0$,

for $t \geq t_1 \geq t_0$

Case I: $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, $z'''(t) \leq 0$,

Since $z(t)$ is a positive decreasing function , there exists finite limit $\lim\limits_{t\to\infty} z(t) = k$. We shall prove that $k = 0$.

Assume that $k > 0$. Then for any $\varepsilon > 0$, there is $t_2 \geq t_1$ such that

$$k + \varepsilon > z(t) > k \,, \text{ for } t \geq t_2 \,. \tag{2.2.2}$$

We have $k > 0$ and $1 > p > 0$, this implies $k > k\,p > 0$,

so

$$k - k\,p > 0 \,,$$

hence

$$\frac{k(1-p)}{p} > 0\,,$$

so we can choose $0 < \varepsilon < \dfrac{k(1-p)}{p}$ and adding k, we get

$$k < k + \varepsilon < k + \frac{k(1-p)}{p} \;\Rightarrow\; p\,k < p(k+\varepsilon) < k$$

From equation (2.2.1), we have

$$y(t) = z(t) - p(t)y(t-\tau), \tag{2.2.3}$$

but $z(t) > k > 0$ which implies $y(t) > k - p(t)y(t-\tau)$.

Since $z(t) > y(t) > 0 \implies z(t-\tau) > y(t-\tau)$,

hence

$$k - p(t)y(t-\tau) > k - p(t)z(t-\tau)\,,$$

which implies from equation (2.2.3)

$$y(t) > k - p(t)z(t-\tau)\,,$$

but

$$k + \varepsilon > z(t) > z(t-\tau)\,,$$

so

$$p(k+\varepsilon) > pz(t-\tau)\,,$$

hence

$$y(t)\ > k - p(k+\varepsilon) \Longrightarrow\ y(t) > \frac{k - p(k+\ \varepsilon)}{(k+\varepsilon)}\ (k+\varepsilon)\,.$$

Letting

$$m = \frac{k - p(k+\varepsilon)}{(k+\varepsilon)} \quad ,$$

this implies

$$y(t) > m\ (k+\varepsilon)$$

and using equation (2.2.2) , we get

$$y(t) > m\, z(t)\ . \tag{2.2.4}$$

Now from equation $(1N2 - A)$, we have

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\ (\,y(t) + p(t)y(t-\tau)) \right) = -f(t)y(t-\sigma)\ ,$$

so

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2} z(t) \right) =\ -f(t)y(t-\sigma) \quad ,$$

but from equation (2.2.4) , we have $y(t) > m\, z(t)$ and $f(t) > 0$, hence

$$f(t)\, y(t-\sigma) \geq f(t)m\, z(t-\sigma) \quad ,$$

hence

$$-f(t)\, y(t-\sigma) \leq -f(t)\, m\, z(t-\sigma)\ ,$$

hence

$$(r(t)\, z''(t))' \leq\ -f(t)\, m\, z(t-\sigma) \quad ,$$

$$-\big(r(t)\, z''(t)\big)' \geq f(t)\, m\, z(t-\sigma) \quad .$$

Integrating the last inequality from t to ∞ , we get

$$-\int_t^{\infty} \big(r(s)\, z''(s)\big)' ds \geq m \int_t^{\infty} f(s)\, z(s-\sigma)\, ds$$

$$-r(s)\, z''(s)\,]_t^{\infty} \geq m \int_t^{\infty} f(s)\, z(s-\sigma)\, ds$$

$$-\left(\lim_{\lambda \to \infty} r(\lambda) z''(\lambda) - r(t) z''(t) \right) \geq m \int_t^{\infty} f(s)\, z(s-\sigma)\, ds \quad ,$$

$$r(t)\, z''(t) \geq \lim_{\lambda \to \infty} r(\lambda) z''(\lambda) + m \int_t^{\infty} f(s)\, z(s-\sigma)\, ds \quad .$$

But $z''(t)$ positive decreasing function and $0 < r(t) < \infty$, so $\lim\limits_{\lambda \to \infty} r(\lambda) z''(\lambda) > 0$,

hence

$$r(t)\, z''(t) \geq m \int_t^{\infty} f(s)\, z(s-\sigma)\, ds \quad .$$

Using the fact that $z(t-\sigma) \geq k$, we obtain

$$r(t)\, z''(t) \geq m \int_t^{\infty} f(s)\, k\, ds \quad ,$$

hence

$$r(t)\, z''(t) \geq m\, k \int_t^{\infty} f(s)\, ds \quad ,$$

dividing both sides by $r(t) > 0$, we get

$$z''(t) \geq m\, k \left(\frac{1}{r(t)} \int_t^{\infty} f(s)\, ds \right) ,$$

and integrating the last inequality from t to ∞ , we get

$$\int_t^\infty z''(s)\,ds \;\geq\; m\,k \int_t^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du \;,$$

$$z'(s) \Big]_t^\infty \;\geq\; m\,k \int_t^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du \;,$$

$$\lim_{\lambda\to\infty} z'(\lambda) - z'(t) \geq\; m\,k \int_t^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du \;,$$

$$-z'(t) \geq -\lim_{\lambda\to\infty} z'(\lambda) +\; m\,k \int_t^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du \;,$$

but $z'(t)$ negative increasing function, so $-\lim_{\lambda\to\infty} z'(\lambda) > 0$ and we obtain

$$-z'(t) \geq\; m\,k \int_t^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du \;,$$

Integrating this inequality from t_1 to ∞ , we get

$$-\int_{t_1}^\infty z'(t)\,dt \geq\; m\,k \int_{t_1}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du\,dv \;,$$

$$-\,z(t)]_{t_1}^\infty \;\geq\; m\,k \int_{t_1}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du\,dv \;,$$

$$-\left(\lim_{\lambda\to\infty} z(\lambda) - z(t_1)\right) \geq\; m\,k \int_{t_1}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du\,dv \;,$$

$$z(t_1) \;\geq\; \lim_{\lambda\to\infty} z(\lambda) +\; m\,k \int_{t_1}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty f(s)\,ds\right] du\,dv \;,$$

but $z(t)$ positive decreasing function , so

$$z(t_1) \geq m\,k \int_{t_1}^{\infty} \int_{v}^{\infty} \left[\frac{1}{r(u)} \int_{u}^{\infty} f(s)\,ds \right] du\,dv \ ,$$

dividing by $m\,k > 0$, we obtain

$$\int_{t_1}^{\infty} \int_{v}^{\infty} \left[\frac{1}{r(u)} \int_{u}^{\infty} f(s)\,ds \right] du\,dv \ \leq \ \frac{z(t_1)}{mk} \ < \infty \ .$$

This contradicts (H_5) , so we must have $k = 0$.

But $0 \leq y(t) \leq z(t)$, so

$$\lim_{t\to\infty} 0 \leq \lim_{t\to\infty} y(t) \leq \lim_{t\to\infty} z(t) = 0 \ \Longrightarrow \ \lim_{t\to\infty} y(t) = 0 \ .$$

Now consider

Case II: $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) < 0$.

Let

$$z(t) = y(t) + p(t)y(t-\tau) \ > 0 \ ,$$

we obtain

$$y(t) = z(t) - p(t)y(t-\tau) \ ,$$

so

$$y(t-\sigma) = z(t-\sigma) - p(t-\sigma)y\big((t-\sigma)-\tau\big) \ . \tag{2.2.5}$$

But $z(t)$ is increasing and $z(t) > y(t)$ so

$$z(t-\sigma) \geq z\big((t-\sigma)-\tau\big) \geq y\big((t-\sigma)-\tau\big) \ ,$$

hence equation (2.2.5) gives

$$y(t-\sigma) \geq z(t-\sigma) - p(t-\sigma)z(t-\sigma) \ ,$$

which implies

$$y(t-\sigma) \geq \big(1-p(t-\sigma)\big)\, z(t-\sigma) \ . \qquad (2.2.6)$$

Now again from equation $(1N2-A)$, we have

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2}\ (\,y(t)+p(t)y(t-\tau))\right) = -f(t)y(t-\sigma) \ ,$$

so

$$\frac{d}{dt}\left(r(t)\ \frac{d^2}{dt^2} z(t)\right) = \ -f(t)y(t-\sigma) \ .$$

But from (H_4) , $\exists\, q(t){:}\, q(t) > 0$, $q'(t) < 0$, $q(t) \leq f(t)$, $\forall\, t \geq t_0$, so

$$q(t)\, y(t-\sigma) \leq \ f(t)\, y(t-\sigma) \ ,$$

$$-\,f(t)\, y(t-\sigma) \leq \ -q(t)\, y(t-\sigma) \leq 0 \ ,$$

hence

$$(\, r(t)\, z''(t)\,)' \leq \ -q(t)\, y(t-\sigma)$$

and using (2.2.6) we have

$$(\, r(t)\, z''(t)\,)' \leq \ -q(t)\big(1-p(t-\sigma)\big)\, z(t-\sigma) \ , \qquad (2.2.7)$$

so

$$(r(t)\, z''(t))' \leq 0 \ .$$

Define the function $\varphi(t) = t\, \dfrac{r(t)\, z''(t)}{z'(t)}$, $t \geq t_1$. It is clear that $\varphi(t) > 0$

$$\varphi'(t) = \frac{r(t)\, z''(t)}{z'(t)} + t \left(\frac{r(t)\, z''(t)}{z'(t)} \right)'$$

$$\varphi'(t) = \frac{r(t)\, z''(t)}{z'(t)} + t \left(\frac{z'(t)[\, r(t)\, z''(t)]' - r(t)\, z''(t) z''(t)}{(z'(t))^2} \right)$$

$$\varphi'(t) = \frac{r(t)\, z''(t)}{z'(t)} + t \ \frac{z'(t)[\, r(t)\, z''(t)]'}{(z'(t))^2} - t \ \frac{r(t)\, z''(t)^2}{(z'(t))^2}$$

$$\varphi'(t) = \frac{1}{t} \left(t\, \frac{r(t)\, z''(t)}{z'(t)} \right) + t\, \frac{[\, r(t)\, z''(t)]'}{z'(t)} - \left(t \ \frac{r(t)\, z''(t)}{z'(t)} \right) \frac{z''(t)}{z'(t)}$$

$$\varphi'(t) = \frac{1}{t}\, \varphi(t) + t\, \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \varphi(t)\, \frac{z''(t)}{z'(t)} \ ,$$

$$\varphi'(t) = \frac{\varphi(t)}{t} + t\, \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \varphi(t)\, \frac{z''(t)}{z'(t)}\, \frac{t\, r(t)}{t\, r(t)} \ ,$$

using the inequality (2.2.7) , we get

$$\varphi'(t) \leq \frac{\varphi(t)}{t} + t\, \frac{-q(t)\big(1 - p(t-\sigma)\big)\, z(t-\sigma)}{z'(t)} - \varphi(t) \left(t\, \frac{r(t)\, z''(t)}{z'(t)} \right) \frac{1}{t\, r(t)}$$

$$\varphi'(t) \leq \frac{\varphi(t)}{t} + \frac{-t\, q(t)\big(1 - p(t-\sigma)\big)\, z(t-\sigma)}{z'(t)} - \varphi(t).\varphi(t)\, \frac{1}{t\, r(t)}$$

$$\varphi'(t) \leq \frac{\varphi(t)}{t} - t\, q(t)\big(1 - p(t-\sigma)\big)\, \frac{z(t-\sigma)}{z'(t)} - \frac{\varphi(t)^2}{t\, r(t)} \ . \qquad (2.2.8)$$

Also from Lemma 1.1.2 , we have

$$\frac{x(t-\sigma)}{x(t)} \geq K \frac{(t-\sigma)}{t} \quad , \qquad t \geq t_2 \, .$$

Setting $x(t) = z'(t)$ and $x(t-\sigma) = z'(t-\sigma)$, we have

$$\frac{z'(t-\sigma)}{z'(t)} \geq \frac{K\,(t-\sigma)}{t} \quad , \quad t-\sigma \geq t_1 \, ,$$

multiplying both sides by $\dfrac{1}{z'(t-\sigma)} > 0$,

$$\Rightarrow \frac{1}{z'(t)} \geq \frac{K\,(t-\sigma)}{t} \frac{1}{z'(t-\sigma)} \quad \text{for} \quad t-\sigma \geq t_1 \geq t_2 \, .$$

Multiplying by $z(t-\sigma) > 0$

$$\Rightarrow \frac{z(t-\sigma)}{z'(t)} \geq \frac{K\,(t-\sigma)}{t} \frac{z(t-\sigma)}{z'(t-\sigma)} \, .$$

By Lemma 1.1.4 , we have

$$z\,(t) > \frac{1}{2} M\, t\, z'(t) \quad \text{for each} \quad M \in (0\,,1)\, ,$$

so

$$z\,(t-\sigma) > \frac{1}{2} M(t-\sigma) z'(t-\sigma)$$

$$\Rightarrow \frac{z(t-\sigma)}{z'(t)} \geq \frac{K\,(t-\sigma)}{t} \frac{\frac{1}{2} M(t-\sigma) z'(t-\sigma)}{z'(t-\sigma)}$$

$$\Rightarrow \frac{z(t-\sigma)}{z'(t)} \geq \frac{K\,(t-\sigma)}{t} \frac{M(t-\sigma)}{2}$$

$$\Rightarrow \frac{z(t-\sigma)}{z'(t)} \geq \frac{KM}{2} \frac{(t-\sigma)^2}{t} \, . \tag{2.2.9}$$

Substituting (2.2.9) in (2.2.8) , we get

$$\varphi'(t) \leq \frac{\varphi(t)}{t} - t\, q(t)\big(1 - p(t-\sigma)\big)\frac{KM}{2}\ \frac{(t-\sigma)^2}{t} - \frac{\varphi(t)^2}{t\, r(t)}\ .$$

Rearrange the last inequality we obtain

$$\varphi'(t) \leq\ -t\, q(t)\,\big(1 - p(t-\sigma)\big)\frac{KM}{2}\ \ \frac{(t-\sigma)^2}{t} + \left(\frac{1}{t}\varphi(t)\ -\ \frac{1}{t\, r(t)}\varphi(t)^2\right)$$

Using Lemma 1.1.1 with $x = \varphi(t)$, $u = \dfrac{1}{t\, r(t)}$, $v = \dfrac{1}{t}$, we have

$$\left(\frac{1}{t}\ \varphi(t) - \frac{1}{t\, r(t)}\varphi(t)^2\right) \leq \frac{1}{4}\ \frac{\left(\ \frac{1}{t}\ \right)^2}{\left(\frac{1}{t\, r(t)}\right)} = \frac{1}{4}\ \frac{r(t)}{t}\ \ ,$$

hence

$$\varphi'(t) \leq\ -t\, q(t)\big(1 - p(t-\sigma)\big)\frac{1}{2}KM\ \ \frac{(t-\sigma)^2}{t} + \frac{1}{4}\,\frac{r(t)}{t}\ \ .$$

Hence

$$q(t)\big(1 - p(t-\sigma)\big)\frac{1}{2}KM\,(t-\sigma)^2 - \frac{1}{4}\,\frac{r(t)}{t}\ \leq\ -\varphi'(t)$$

or

$$2\, q(t)\big(1 - p(t-\sigma)\big)\, KM\, .\, (t-\sigma)^2 - \frac{r(t)}{t} \leq\ -4\, \varphi'(t)\ . \tag{2.2.10}$$

Integrating the inequality (2.2.10) from t_2 to t , we obtain

$$\int_{t_2}^{t}\left[2\, q(s)\big(1 - p(s-\sigma)\big)\, KM\, .\, (s-\sigma)^2 - \frac{r(s)}{s}\right]ds \leq 4\,\varphi(t_2) - 4\,\varphi(t)\, ,\ \ \varphi(t) > 0.$$

$$\int_{t_2}^{t} \left[2\, q(s)\big(1 - p(s-\sigma)\big) KM \,.\, (s-\sigma)^2 - \frac{r(s)}{s} \right] ds \le 4\, \varphi(t_2) \, < \infty \quad ,$$

hence

$$\lim_{t\to\infty} \, sup \int_{t_2}^{t} \left[2\, q(s)\big(1 - p(s-\sigma)\big) KM \,.\, (s-\sigma)^2 - \frac{r(s)}{s} \right] ds \, < \infty \; .$$

This is a contradiction to (H_6) .

We now consider

Case $(\boldsymbol{III})$: $z(t) > 0$, $z'(t) > 0$, $z''(t) < 0$, $\big(r(t)\, z''(t)\big)' \le 0$.

Define the function $\emptyset$ by $\emptyset(t) = \dfrac{r(t)\, z''(t)}{z'(t)}$, $t \ge t_1$.

It is clear that $\emptyset(t) < 0$. And from the decreasing function $r(t)\, z''(t)$, we get

$$r(s)\, z''(s) \;\le\; r(t)\, z''(t) \;, \qquad \text{for} \qquad s \ge t \ge t_1 \,.$$

Dividing the above inequality by the positive function $r(s)$, we get

$$\frac{r(s)\, z''(s)}{r(s)} \le \frac{r(t)\, z''(t)}{r(s)} \quad ,$$

hence we obtain

$$z''(s) \;\le \frac{r(t)\, z''(t)}{r(s)}$$

Integrating the last inequality from t to l , we obtain

$$\int_{t}^{l} z''(s)\, ds \;\le\; \int_{t}^{l} \frac{r(t)\, z''(t)}{r(s)}\, ds \;,$$

which gives

$$z'(s)]_t^l \ \leq\ r(t)\, z''(t) \int_t^l \frac{1}{r(s)}\, ds \Rightarrow\ z'(l) -\ z'(t) \leq\ r(t)\, z''(t) \int_t^l \frac{1}{r(s)}\, ds$$

$$z'(l) \leq\ z'(t) +\ r(t)\, z''(t) \int_t^l \frac{1}{r(s)}\, ds\ ,$$

But (H_7) assumes $R(t) = \int_t^l \frac{1}{r(s)}\, ds$ as $l \to \infty$,

and since $z'(t)$ is a positive decreasing function , we have

$$0\ \leq\ z'(t) +\ r(t)\, z''(t)\, R(t)$$

$$-\, r(t)\, z''(t)\, R(t) \leq\ z'(t)\ ,$$

dividing the last inequality by $z'(t) > 0$, we have

$$-\ R(t)\, \frac{r(t) z''(t)}{z'(t)} \leq\ 1\ ,$$

but

$$\emptyset(t) =\ \frac{r(t)\, z''(t)}{z'(t)} \quad \Rightarrow -\ R(t)\, \emptyset(t) \leq\ 1\ . \qquad (2.2.11)$$

Differentiating the function $\emptyset(t)$

$$\emptyset'(t) = \left(\frac{r(t)\, z''(t)}{z'(t)}\right)'$$

$$\emptyset'(t) = \frac{z'(t)\big(r(t)\, z''(t)\big)' -\ r(t)\, z''(t) z''(t)}{(\ z'(t)\ )^2}$$

$$= \frac{z'(t)\big(r(t)\, z''(t)\big)'}{(\, z'(t)\,)^2} - \frac{r(t)\, z''(t) z''(t)}{(\, z'(t)\,)^2}$$

$$\emptyset'(t) = \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \left(\frac{r(t)\, z''(t)}{z'(t)}\right)\frac{z''(t)}{z'(t)} = \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \emptyset(t)\ \frac{z''(t)}{z'(t)}$$

$$\emptyset'(t) = \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \emptyset(t)\ \frac{z''(t)}{z'(t)}\ \frac{r(t)}{r(t)} = \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \emptyset(t)\left(\frac{r(t) z''(t)}{z'(t)}\right)\frac{1}{r(t)}$$

$$\emptyset'(t) = \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \emptyset(t)\emptyset(t)\frac{1}{r(t)} \ .$$

Thus ,

$$\emptyset'(t) = \frac{\big(r(t)\, z''(t)\big)'}{z'(t)} - \frac{\big(\emptyset(t)\big)^2}{r(t)} \le 0 \ .$$

By assumption $z'(t) > 0$, $z''(t) < 0$, so $\big(r(t)\, z''(t)\big)' \le 0$ and $r(t) > 0$, hence $\emptyset(t)$ is a negative decreasing function. But from (2.2.7) , in proving case (II) , we have

$$(\, r(t)\, z''(t)\,)' \le -q(t)\big(1 - p(t-\sigma)\big)\, z(t-\sigma),$$

$$\frac{z(t-\sigma)}{z'(t)} \ge \frac{KM}{2}\ \frac{(t-\sigma)^2}{t} \ ,$$

$$\emptyset'(t) \le \frac{-q(t)\big(1 - p(t-\sigma)\big)\, z(t-\sigma)}{z'(t)} - \frac{\big(\emptyset(t)\big)^2}{r(t)}$$

$$\emptyset'(t) \le -q(t)\big(1 - p(t-\sigma)\big)\frac{z(t-\sigma)}{z'(t)} - \frac{\big(\emptyset(t)\big)^2}{r(t)} \ .$$

By (2.2.9) , we have

$$\emptyset'(t) \le -q(t)\big(1-p(t-\sigma)\big)\frac{KM}{2}\ \frac{(t-\sigma)^2}{t} - \frac{\big(\emptyset(t)\big)^2}{r(t)} \quad ,$$

multiplying this inequality by the positive function $R(t)$, we have

$$R(t)\emptyset'(t) \le -q(t)\big(1-p(t-\sigma)\big)\frac{KM}{2}\ \frac{(t-\sigma)^2}{t}\ R(t) - \frac{\big(\emptyset(t)\big)^2}{r(t)}\ R(t)\ . \qquad (2.2.12)$$

Integrating (2.2.12) from $t_3 \ge t_1$ to t , we get

$$\int_{t_3}^{t} R\,(s)\emptyset'(s)ds \le \int_{t_3}^{t}\left[-q(s)\big(1-p(s-\sigma)\big)\frac{KM}{2}\ \frac{(s-\sigma)^2}{s}R(s) - \frac{\big(\emptyset(s)\big)^2}{r(s)}\ R(s)\right] ds\ ,$$

hence

$$\int_{t_3}^{t} R\,(s)\emptyset'(s)ds \le -\int_{t_3}^{t} q(s)\,\big(1-p(s-\sigma)\big)\frac{KM}{2}\ \frac{(s-\sigma)^2}{s}R(s)\ ds - \int_{t_3}^{t}\frac{\big(\emptyset(s)\big)^2}{r(s)}\ R(s)\ ds\ .$$

Using integration by parts with

$$u = R(s) \implies du = R'(s)\ ds\ = \frac{-1}{r(s)}\ ds\ ,\ \text{ since } R(s) = \int_{t}^{\infty}\frac{1}{r(s)}\ ds\ ,$$

and $dv = \emptyset'(s)\ ds \implies v = \emptyset(s)$,

hence

$$R(s)\ \emptyset(s)\ ]_{t_3}^{t} - \int_{t_3}^{t}\frac{-1}{r(s)}\emptyset(s)\ ds \le -\int_{t_3}^{t} q(s)\,\big(1-p(s-\sigma)\big)\frac{KM}{2}\ \frac{(s-\sigma)^2}{s}R(s)\ ds - \int_{t_3}^{t}\frac{\big(\emptyset(s)\big)^2}{r(s)}R(s)ds.$$

$$R(t)\,\emptyset(t) - R(t_3)\,\emptyset(t_3) - \int_{t_3}^{t} \frac{-1}{r(s)} \emptyset(s)\,ds$$

$$\leq - \int_{t_3}^{t} q(s)\left(1 - p(s-\sigma)\right) \frac{KM}{2}\, \frac{(s-\sigma)^2}{s} R(s)\,ds - \int_{t_3}^{t} \frac{\left(\emptyset(s)\right)^2}{r(s)} R(s)ds.$$

$$R(t)\,\emptyset(t) - R(t_3)\,\emptyset(t_3) + \int_{t_3}^{t} \frac{1}{r(s)} \emptyset(s)\,ds + \int_{t_3}^{t} q(s)\left(1 - p(s-\sigma)\right) \frac{KM}{2}\, \frac{(s-\sigma)^2}{s} R(s)\,ds$$

$$+ \int_{t_3}^{t} \frac{\left(\emptyset(s)\right)^2}{r(s)} R(s)ds \;\leq 0$$

$$R(t)\,\emptyset(t) - R(t_3)\,\emptyset(t_3) \;+ \int_{t_3}^{t} \frac{\left(\emptyset(s)\right)^2}{r(s)} R(s)ds + \int_{t_3}^{t} \frac{1}{r(s)} \emptyset(s)\,ds$$

$$+ \int_{t_3}^{t} q(s)\left(1 - p(s-\sigma)\right) \frac{KM}{2}\, \frac{(s-\sigma)^2}{s} R(s)\,ds \leq 0$$

$$R(t)\emptyset(t) - R(t_3)\,\emptyset(t_3) + \int_{t_3}^{t} \left[\frac{R(s)}{r(s)} \left(-\emptyset(s)\right)^2 - \frac{1}{r(s)}\left(-\emptyset(s)\right)\right] ds$$

$$+ \int_{t_3}^{t} q(s)\left(1 - p(s-\sigma)\right) \frac{KM}{2}\, \frac{(s-\sigma)^2}{s} R(s)ds \leq 0$$

Letting $u = \dfrac{R(s)}{r(s)}$, $v = \dfrac{1}{r(s)}$, $x = -\emptyset(s)$ and using Lemma 1.1.1 ,

we have

$$\frac{R(s)}{r(s)} \left(-\emptyset(s)\right)^2 - \frac{1}{r(s)}\left(-\emptyset(s)\right) \geq \; -\frac{1}{4}\, \frac{\left(\frac{1}{r(s)}\right)^2}{\frac{R(s)}{r(s)}} = -\frac{1}{4}\, \frac{1}{R(s)\,r(s)}$$

$$R(t)\,\emptyset(t) - R(t_3)\,\emptyset(t_3) + \int_{t_3}^{t} \left[-\frac{1}{4}\,\frac{1}{R(s)\,r(s)} \right] ds$$

$$+ \int_{t_3}^{t} q(s)\left(1 - p(s-\sigma)\right)\frac{KM}{2}\,\frac{(s-\sigma)^2}{s} R(s)\,ds \le 0$$

$$\int_{t_3}^{t} \left[q(s)\left(1 - p(s-\sigma)\right)\frac{KM}{2}\,\frac{(s-\sigma)^2}{s} R(s) - \frac{1}{4}\,\frac{1}{R(s)\,r(s)} \right] ds \le R(t_3)\,\emptyset(t_3) - R(t)\,\emptyset(t) \;.$$

But from (2.2.11) , we have $-R(t)\,\emptyset(t) \le 1$, so

$$\int_{t_3}^{t} \left[q(s)\left(1 - p(s-\sigma)\right)\frac{KM}{2}\,\frac{(s-\sigma)^2}{s} R(s) - \frac{1}{4}\,\frac{1}{R(s)\,r(s)} \right] ds \le \; R(t_3)\,\emptyset(t_3) + 1 \,.$$

Multiplying by 4 and take the *limit sup* as $t \to \infty$, we have

$$\lim_{t\to\infty} \sup \int_{t_3}^{t} \left[2q(s)\left(1 - p(s-\sigma)\right)KM\,\frac{(s-\sigma)^2}{s} R(s) - \frac{1}{R(s)\,r(s)} \right] ds \;\le \lim_{t\to\infty} \sup 4\left(R(t_3)\,\emptyset(t_3) + 1\right) < \infty$$

This is a contradiction to (H_7) . Therfore , all the solutions of the equation $(1N2 - A)$ are oscillatory . This completes the proof . ■

2. 3 Oscillation Conditions of the solution of the equation $(\mathbf{1N2} - \mathbf{B})$

Consider the delay differential equation :

$$\frac{d}{dt}\left(r(t)\,\frac{d^2}{dt^2}\,\left(y(t) + p(t)y(\tau(t))\right) \right) + f(t)y\left(\sigma(t)\right) = 0 \qquad (1N2 - B)$$

In fact , equation $(1N2 - B)$ is a general form of equation $(1N2 - A)$. Also , This form is a special case which appears in [1] which is studied by B. Baculicova and J. Dzurina when $\gamma = 1$.

This generalizes Theorem 2.2.1 for linear functions $\tau(t)$, $\sigma(t)$ where

$0 \le \tau(t) \le t$ and $0 \le \sigma(t) \le t$ for every time t to be as in the next theorem.

Theorem 2.3.1:

Assume that $(H_1) - (H_5)$ in Theorem 2.2.1 hold and

(H_6): $\lim_{t\to\infty} \ sup \ \int_{t_1}^{t} \left[2\, q(s) \left(1 - p\big(\sigma(s)\big)\right) K\, M\, \big(\sigma(s)\big)^2 - \frac{r(s)}{s} \right] ds = \infty$, for some

K , $M \in (0, 1)$ for sufficiently large $t_1 \ge t_0$.

(H_7): $\lim_{t\to\infty} \ sup \ \int_{t_2}^{t} \left[2\, q(s) \left(1 - p\big(\sigma(s)\big)\right) K\, M\, \frac{\big(\sigma(s)\big)^2}{S} - \frac{1}{R(s) r(s)} \right] ds = \infty$,

for some K , $M \in (0, 1)$ and sufficiently large $t_2 \ge t_0$ where

$$R(t) = \int_t^{\infty} \frac{1}{r(s)} \, ds ,$$

holds , then equation $(1N2 - B)$ is oscillatory.

Proof :

Let $z(t) = y(t) + p(t)\, y(\tau(t))$. (2.3.1)

Suppose that equation $(1N2 - B)$ has a nonoscillatory solution $y(t)$ and suppose $y(t)$ is positive solution . So , the rest of the proof will be similar as in Theorem 2.2.1 if we replace $t - \tau$ by $\tau(t)$ and $t - \sigma$ by $\sigma(t)$. The proof is complete . ■

2.4 Illustrating Examples :

Example 2.4.1:

Consider the linear neutral delay differential equation

$$\frac{d}{dt}\left(e^t \ \frac{d^2}{dt^2}\left(y(t)+\frac{1}{2}y(t-2\pi)\right)\right)+\frac{3}{\sqrt{2}}\,e^t\,\mathrm{y}\left(t-\frac{7\pi}{4}\right)=0\ , \tag{2.4.1}$$

where $t \geq 0$. We have

$$r(t) = e^t \in C'\big([t_0\,,\infty)\,,\ (0\,,\infty)\big)\ , r'(\mathrm{t}) = e^t > 0\,,\ p(t) = \frac{1}{2} \in C\ ([t_0\,,\infty)\,,\mathbb{R})$$

$$f(t) = \frac{3}{\sqrt{2}}e^t \in C\big([t_0\,,\infty)\,,\ [0\,,\infty)\big)\ ,\ \tau = 2\pi\ ,\ \sigma = \frac{7\pi}{4}$$

$q(t) = \dfrac{1}{t^7} > 0$, $q'(t) = -7t^{-8} < 0$ and $q(t) \leq f(t)$ for $t \in [t_0\,,\infty)$, where

$t_0 \geq 0.9$, so $q(t)$ positive decreasing function .

Applying Theorem 2.2.1 , all conditions $(H1 - H7)$ are satisfied , so every solution of equation (2.4.1) is oscillatory , and one of these solutions is

$y(t) = \sin t$.

Example 2.4.2

Consider the third order neutral delay differential equation

$$\frac{d}{dt}\left(t \ \frac{d^2}{dt^2}\left(y(t)+k_1 y\left(\frac{t}{2}\right)\right)\right)+\frac{k_2}{t^2}\ y(t)=0 \tag{2.4.2}$$

where $k_1 \in [\,0\,,1\,)$, $k_2 > 0$, $t \geq 1$.

Here $r(t) = t$, $p(t) = k_1$, $f(t) = \frac{k_2}{t^2}$, $\tau(t) = \frac{t}{2} \leq t$, $\sigma(t) = t \leq t$.

Applying Theorem (2.3.1) , and take $q(t) = \frac{k_3}{t^2}$, $0 < k_3 \leq k_2$, it is clear that $q(t)$ is decreasing function and $q(t) \leq f(t)$.

$$* \int_{t_1}^{\infty}\int_{v}^{\infty}\left[\frac{1}{u}\int_{u}^{\infty}\frac{k_2}{s^2}\,ds\right]du\,dv = k_2\int_{t_1}^{\infty}\int_{v}^{\infty}\left[\frac{1}{u}\left(-s^{-1}\Big]_{u}^{\infty}\right)\right]du\,dv = \infty .$$

$$= k_2\int_{t_1}^{\infty}\int_{v}^{\infty}\frac{1}{u^2}\,du\,dv = k_2\int_{t_1}^{\infty}\frac{-1}{u}\Big]_{v}^{\infty}dv = k_2\int_{t_1}^{\infty}\frac{1}{v}\,dv = \infty$$

$$* \int_{t_1}^{t}\left[2\,\frac{k_3}{s^2}\,(1-k_1)K\,M\,s^2 - \frac{s}{s}\right]ds = \int_{t_1}^{t}[2\,k_3\,(1-k_1)K\,M - 1]ds$$

$$= (2\,k_3\,(1-k_1)K\,M - 1)(t-t_1) \text{ , where } 2\,k_3\,(1-k_1)K\,M > 1 ,$$

so $\limsup_{t\to\infty}\int_{t_1}^{t}\left[2\,\frac{k_3}{s^2}\,(1-k_1)K\,M\,s^2 - \frac{s}{s}\right]ds = \infty$.

$$* \int_{t_2}^{t}\left[2\,\frac{k_3}{s^2}\,(1-k_1)K\,M\,\frac{s^2}{s} - \frac{1}{\int_s^{\infty}\frac{1}{u}du\,.\,s}\right]ds = \int_{t_2}^{t}\left[2\,k_3\,(1-k_1)K\,M\,\frac{1}{s}\right]ds$$

$$= 2\,k_3\,(1-k_1)K\,M\,ln\,|\,s\,|\,\Big]_{t_2}^{t} = 2\,k_3\,(1-k_1)K\,M(ln\,t - ln\,t_2) .$$

So

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left[2\,\frac{k_3}{s^2}\,(1-k_1)K\,M\,\frac{s^2}{s} - \frac{1}{\int_s^{\infty}\frac{1}{u}du\,.\,s}\right]ds = \infty .$$

Hence $(H1 - H7)$ are satisfied , so every solution $y(t)$ of equation $(2.4.2)$ is oscillatory .

Chapter Three

Oscillation of Solution of the Equation of the Form

$(2N\,1-A)$ and $(2N1-B)$

3.1 Introduction

In this chapter, we will display the main results which are concerned with the oscillatory behavior of solutions of equations

$$(2N\,1-A): \quad \frac{d^2}{dt^2}\left(r(t)\,\frac{d}{dt}\big(y(t)+p(t)y(t-\tau)\big)\right)+f(t)y(t-\sigma)=0$$

$$(2N1-B): \quad \frac{d^2}{dt^2}\left(r(t)\,\frac{d}{dt}\Big(y(t)+p(t)y\big(\tau(t)\big)\Big)\right)+f(t)y\big(\sigma(t)\big)=0$$

where $p(t)$, $f(t)\in([t_0\,,\infty),\mathbb{R}\,)$, and $f(t)\geq 0$, $r(t)\in C^2\left([t_0\,,\infty)\right)$,

$$r(t)>0\,,\ \forall\, t\geq t_0 \quad \text{and} \quad R(t)=\int_{t_0}^{\infty}\frac{1}{r(t)}\,dt\ .$$

Also, we need the following in our discussion :

(H_1): $r(t)\in C^2\big([\,t_0\ ,\infty),(\,0\,,\infty\,)\big)$, $r(t)>0$ and $r''(t)\geq 0$ for $t\geq t_0$.

(H_2): $p(t)\in C([\,t_0\ ,\infty),\ \mathbb{R})$, $0<p(t)<1$

(H_3): $f(t)\in C^1\big([\,t_0\ ,\infty),[\,0\,,\infty)\big)$

(H_4): There exists a positive decreasing function $q(t)$ such that $f(t)\geq q(t)$ for $t\in[t_0\,,\infty)$.

$$(H_5): \quad \int_{t_0}^{\infty}\left[\frac{1}{r(v)}\int_{v}^{\infty}\left(\int_{u}^{\infty} f(s)\,ds\right)du\right]dv = \infty\,,$$

3.2 Main Results :

Oscillation of Solution of the Equation of the Form $(\mathbf{2N1}-\boldsymbol{A})$

In this section , we will study the oscillation of solution of the equations of the form $(2N1-A)$ and $(2N1-B)$.

Theorem 3. 2. 1:

Assume that $(H_1)-(H5)$ hold , and

$$\int_{t_3}^{t}\left[2\,\rho(s)q(s)\big(1-p(s-\sigma)\big)\,K\,M\,\frac{(s-\sigma)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds \;=\infty$$

where $K\,,M\, \in (0\,,1)$, $\rho(t) \in C^1([t_0\,,\infty),\ (\,0\,,\infty)\,)$ for sufficiently large $t_1 > t_0$

and for $t_3 \;> t_2 \;> t_1$. Then equation $(2N1-A)$ is almost oscillatory . Where

τ , σ constants , $0 \le \tau \le t$, $0 \le \sigma \le t$ and $\lim\limits_{t\to\infty}(t-\tau) = \lim\limits_{t\to\infty}(t-\sigma) \;=\infty.$

Proof :

Let $y(t)$ be a nonoscillatory solution of equation $(2N1-A)$.

Suppose $y(t)$ is a positive solution , and suppose

$$z(t) = y(t) + p(t)y(t-\tau) \quad . \tag{3.2.1}$$

So , there exist$s$ two possible cases:

(I) $z(t) > 0$, $z'(t) \;< 0$, $(r(t)z'(t))' > 0$, $(\,r(t)z'(t))'' \le 0$,

(II) $z(t) > 0$, $z'(t) > 0$, $(r(t)z'(t))' > 0$, $(r(t)z'(t))'' \le 0$,

Case $(\boldsymbol{I})$: $z(t) > 0$, $z'(t) < 0$, $\left(r(t)z'(t)\right)' > 0$, $\left(r(t)z'(t)\right)'' \le 0$,

which means that $z(t)$ is a positive decreasing function , $r(t)\, z'(t)$ is a negative increasing function and $\left(r(t)z'(t)\right)'$ is a decreasig function .

By $(2N1 - A)$, we have

$$\left(r(t)\left(y(t) + p(t)y(t-\tau)\right)'\right)'' + f(t)y(t-\sigma) = 0 \ ,$$

so

$$\left(r(t)\left(y(t) + p(t)y(t-\tau)\right)'\right)'' = -f(t)y(t-\sigma) \ ,$$

using (3.2.1) , we have

$$\left(r(t)z'(t)\right)'' = -f(t)y(t-\sigma) ,$$

From inequality $(2.2.4)$, we have

$$y(t) \ge m\, z(t) \ ,$$

so

$$y(t-\sigma) \ge m\, z(t-\sigma) \ ,$$

multiplying this inequality by $f(t) > 0$, we have

$$-f(t)y(t-\sigma) \le -m\, f(t)\, z(t-\sigma) \ ,$$

hence

$$\left(r(t)z'(t)\right)'' \le -m\, f(t)\, z(t-\sigma) \ ,$$

so

$$-\left(r(t)z'(t)\right)'' \geq m\, f(t)\, z(t-\sigma) \ ,$$

integrating the previous inequality from t to ∞, we obtain

$$-\int_t^\infty \left(r(s)z'(s)\right)'' ds \geq \int_t^\infty m\, f(s)\, z(s-\sigma) ds$$

$$-\left(\left(\lim_{\lambda\to\infty} r(\lambda)z'(\lambda)\right)' - \left(r(t)z'(t)\right)'\right) \geq \int_t^\infty m\, f(s)\, z(s-\sigma) ds \ ,$$

hence

$$\left(r(t)z'(t)\right)' \geq \left(\lim_{\lambda\to\infty} r(\lambda)z'(\lambda)\right)' + \int_t^\infty m\, f(s)\, z(s-\sigma) ds \ ,$$

so

$$\left(r(t)z'(t)\right)' \geq \int_t^\infty m\, f(s)\, z(s-\sigma) ds \ .$$

From inequality (2.2.2) , we have $z(t-\sigma) \geq k$

hence

$$\left(r(t)z'(t)\right)' \geq \int_t^\infty m\, k\, f(s)\, ds \ ,$$

integrating this inequality from t to ∞ , we get

$$\int_t^\infty \left(r(u)z'(u)\right)' du \geq \int_t^\infty \int_u^\infty m\, k\, f(s)\, ds\, du$$

$$\lim_{\lambda \to \infty} r(\lambda)z'(\lambda) - r(t)z'(t) \geq m\,k \int_t^\infty \int_u^\infty f(s)\,ds\,du \;\;,$$

hence

$$-\,r(t)z'(t) \geq -\lim_{\lambda \to \infty} r(\lambda)z'(\lambda) + m\,k \int_t^\infty \int_u^\infty f(s)\,ds\,du\,,$$

$$-\,r(t)z'(t) \geq m\,k \int_t^\infty \int_u^\infty f(s)\,ds\,du\,,$$

dividing by $r(t) > 0$ gives

$$z'(t) \leq \frac{-m\,k}{r(t)} \int_t^\infty \int_u^\infty f(s)\,ds\,du \;\;,$$

integrating from t_0 to ∞ obtaining

$$\int_{t_0}^\infty z'\,(v)dv \;\leq\; \int_{t_0}^\infty \frac{-m\,k}{r(v)} \int_v^\infty \int_u^\infty f(s)\,ds\,du\,dv\,,$$

hence

$$z(v)\Big]_{t_0}^{\infty} \;\leq\; -m\,k \int_{t_0}^\infty \frac{1}{r(v)} \int_v^\infty \int_u^\infty f(s)\,ds\,du\,dv\,,$$

so

$$\lim_{\lambda \to \infty} z(\lambda) \;-z(t_1) \leq\; -m\,k \int_{t_0}^\infty \frac{1}{b(v)} \int_v^\infty \int_u^\infty f(s)\,ds\,du\,dv\,,$$

hence

$$m\,k \int_{t_0}^{\infty} \frac{1}{r(v)} \int_{v}^{\infty} \int_{u}^{\infty} f(s)\,ds\,du\,dv \;\leq z(t_1) - \lim_{\lambda\to\infty} z(\lambda)\,,$$

but $\lim\limits_{\lambda\to\infty} z(\lambda) > 0$, so

$$m\,k \int_{t_0}^{\infty} \frac{1}{r(v)} \int_{v}^{\infty} \int_{u}^{\infty} f(s)\,ds\,du\,dv \leq z(t_1)\,,$$

hence

$$\int_{t_0}^{\infty} \left[\frac{1}{r(v)} \int_{v}^{\infty} \left(\int_{u}^{\infty} f(s)\,ds \right) du \right] dv \leq \frac{z(t_1)}{m\,k} < \infty$$

This contradicts $(H5)$ so we must have $k = 0$.

And again from $0 \leq y(t) \leq z(t)$, we obtain $\lim\limits_{t\to\infty} y(t) = 0$

***Case*(*II*)** : $z(t) > 0$, $z'(t) > 0$, $\left(r(t)z'(t)\right)' > 0$, $\left(r(t)z'(t)\right)'' \leq 0$,

which means that $z(t)$ is a positive increasing function , $r(t)z'(t)$ is a positive increasing function and $\left(r(t)z'(t)\right)'$ is a positive decreasing function .

Let

$$\Phi(t) = \rho(t) \frac{\left(r(t)z'(t)\right)'}{r(t)z'(t)}\,, t \geq t_1\,,\ \rho(t) > 0\,,\ \rho'(t) > 0 \qquad (3.2.2)$$

it is clear that $\Phi(t) > 0$.

Integrating the function $\left(r(t)z'(t)\right)'$ from t_1 to t where $t > t_1 \geq t_0$

$$\int_{t_1}^{t} \left(r(u)z'(u)\right)' \, du = r(t)z'(t) - r(t_1)z'(t_1) \ , \ \text{where } t > t_1 \geq t_0 \, ,$$

hence

$$r(t_1)z'(t_1) + \int_{t_1}^{t} \left(r(u)z'(u)\right)' \, du = r(t)z'(t)$$

and

$$\int_{t_1}^{t} \left(r(u)z'(u)\right)' \, du \leq r(t)z'(t) \, , \quad \text{since } r(t_1)z'(t_1) > 0 \, .$$

For $t > s > t_1 \geq t_0$, we have

$$\left(r(s)z'(s)\right)' \geq \left(r(t)z'(t)\right)' \tag{3.2.3}$$

integrating the last inequality from t_1 to t , we get

$$\int_{t_1}^{t} \left(r(s)z'(s)\right)' ds \geq \left(r(t)z'(t)\right)' \int_{t_1}^{t} ds$$

$$\Longrightarrow \int_{t_1}^{t} \left(r(s)z'(s)\right)' ds \geq \left(r(t)z'(t)\right)' (t - t_1)$$

$$\Longrightarrow r(t)z'(t) - r(t_1)z'(t_1) \geq \left(r(t)z'(t)\right)' (t - t_1)$$

hence

$$r(t)z'(t) \geq r(t_1)z'(t_1) + \left(r(t)z'(t)\right)' (t - t_1)$$

so

$$r(t)z'(t) \geq \left(r(t)z'(t)\right)'(t-t_1)$$

hence

$$0 \geq \left(r(t)z'(t)\right)'(t-t_1) - r(t)z'(t)\,. \tag{3.2.4}$$

Dividing inequality (3.2.4) by the positive quantity $(t-t_1)^2$, we have

$$\frac{\left(r(t)z'(t)\right)'(t-t_1) - r(t)z'(t)}{(t-t_1)^2} \leq 0\ ,$$

hence

$$\left(\frac{\left(r(t)z'(t)\right)}{(t-t_1)}\right)' \leq 0\ . \tag{3.2.5}$$

So $\dfrac{r(t)z'(t)}{(t-t_1)}$ is a decreasing function. Hence for $t \geq s \geq 0$, we have

$$\frac{r(t)z'(t)}{(t-t_1)} \leq \frac{r(s)z'(s)}{(s-t_1)}\quad ,$$

which gives

$$z'(s) \geq \frac{r(t)z'(t)}{(t-t_1)}\,\frac{(s-t_1)}{r(s)}\quad . \tag{3.2.6}$$

Integrating (3.2.6) from t_2 to t where $t \geq t_2 \geq t_1 \geq t_0$, we get

$$z(t) - z(t_2) \geq \frac{r(t)\,z'(t)}{(t-t_1)} \int_{t_2}^{t} \frac{(s-t_1)}{r(s)}\,ds\ ,$$

hence

$$z(t) \geq z(t_2) + \frac{r(t)\, z'(t)}{(t-t_1)} \int_{t_2}^{t} \frac{(s-t_1)}{r(s)}\, ds \ ,$$

but $z(t_2) > 0$, so

$$z(t) \geq \frac{r(t)\, z'(t)}{(t-t_1)} \int_{t_2}^{t} \frac{(s-t_1)}{r(s)}\, ds$$

$$\Rightarrow \frac{z(t)}{r(t)\, z'(t)} \geq \frac{1}{(t-t_1)} \int_{t_2}^{t} \frac{(s-t_1)}{r(s)}\, ds \,, \text{ for } t > t_2 > t_1 \geq t_0 \,.$$

Differentiating $\Phi(t) = \rho(t) \dfrac{\big(r(t)z'(t)\big)'}{r(t)z'(t)}$ gives

$$\Phi'(t) = \rho(t) \left(\frac{\big(r(t)z'(t)\big)'}{r(t)z'(t)} \right)' + \rho'(t) \frac{\big(r(t)z'(t)\big)'}{r(t)z'(t)}$$

$$\Phi'(t) = \rho(t) \left(\frac{\big(r(t)z'(t)\big)\big(r(t)z'(t)\big)'' - \big(r(t)z'(t)\big)'\big(r(t)z'(t)\big)'}{\big(r(t)z'(t)\big)^2} \right) + \rho'(t) \frac{\big(r(t)z'(t)\big)'}{r(t)z'(t)} \ ,$$

so

$$\Phi'(t) = \rho'(t) \frac{\big(r(t)z'(t)\big)'}{r(t)z'(t)} + \rho(t) \frac{\big(r(t)z'(t)\big)''}{r(t)z'(t)} - \rho(t) \frac{\Big(\big(r(t)z'(t)\big)'\Big)^2}{\big(r(t)z'(t)\big)^2}$$

$$\Phi'(t) = \frac{\rho'(t)}{\rho(t)} \left(\rho(t) \frac{\big(r(t)z'(t)\big)'}{r(t)z'(t)} \right) + \rho(t) \frac{\big(r(t)z'(t)\big)''}{r(t)z'(t)} - \frac{1}{\rho(t)} \left(\big(\rho(t)\big)^2 \frac{\Big(\big(r(t)z'(t)\big)'\Big)^2}{\big(r(t)z'(t)\big)^2} \right) \tag{3.2.7}$$

From (3.2.2) and (3.2.7) we have

$$\Phi'(t) = \frac{\rho'(t)}{\rho(t)}\Phi(t) + \rho(t)\,\frac{\left(r(t)z'(t)\right)''}{r(t)z'(t)} - \frac{\left(\Phi(t)\right)^2}{\rho(t)} \quad . \tag{3.2.8}$$

$z(t)$ is increasing function, and using (2.2.6), we have

$$y(t-\sigma) \geq \left(1 - p(t-\sigma)\right) z(t-\sigma).$$

But $f(t) > 0$ and $q(t) \leq f(t) \Longrightarrow -q(t) \geq -f(t)$,

so we have

$$-f(t)y(t-\sigma) \leq -f(t)\left(1 - p(t-\sigma)\right)z(t-\sigma)$$

and since

$$\left(r(t)z'(t)\right)'' = -f(t)y(t-\sigma),$$

we have

$$\left(r(t)z'(t)\right)'' \leq -q(t)\left(1 - p(t-\sigma)\right)z(t-\sigma)\,. \tag{3.2.9}$$

Using (3.2.9) in (3.2.8), we get

$$\Phi'(t) \leq \frac{\rho'(t)}{\rho(t)}\Phi(t) + \rho(t)\,\frac{-q(t)\left(1 - p(t-\sigma)\right)z(t-\sigma)}{r(t)z'(t)} - \frac{\left(\Phi(t)\right)^2}{\rho(t)},$$

so

$$\Phi'(t) \leq \frac{\rho'(t)}{\rho(t)}\Phi(t) - \rho(t)q(t)\left(1 - p(t-\sigma)\right)\frac{z(t-\sigma)}{r(t)z'(t)} - \frac{\left(\Phi(t)\right)^2}{\rho(t)}$$

so

$$\Phi'(t) \leq \frac{\rho'(t)}{\rho(t)}\Phi(t) - \rho(t)q(t)\left(1 - p(t-\sigma)\right)\frac{z(t-\sigma)}{z'(t-\sigma)}\,\frac{z'(t-\sigma)}{z'(t)}\,\frac{1}{r(t)} - \frac{\left(\Phi(t)\right)^2}{\rho(t)}\,, \tag{3.2.10}$$

Using $r(t) > 0$, $r'(t) \geq 0$, $r''(t) \geq 0$, $\big(r(t)z'(t)\big)' > 0$, and $\big(r(t)z'(t)\big)'' \leq 0$

for $t \geq t_0$, we have $z''(t) > 0$, $z'''(t) \leq 0$. Applying Lemma 1.1.2 and

Lemma 1.1.4 , we have

$$\frac{z(t-\sigma)}{z'(t-\sigma)} \geq \frac{1}{2} M\,(t-\sigma) \text{ and } \frac{z'(t-\sigma)}{z'(t)} \geq K\,\frac{(t-\sigma)}{t}\;, \text{ where } K, M \in (0,1)\,.$$

Hence

$$\frac{z(t-\sigma)}{z'(t-\sigma)}\,\frac{z'(t-\sigma)}{z'(t)} \geq \frac{K\,M}{2}\,\frac{(t-\sigma)^2}{t}\;,$$

and using it in (3.2.10), we get

$$\Phi'(t) \leq \frac{\rho'(t)}{\rho(t)}\Phi(t) - \rho(t)q(t)\big(1-p(t-\sigma)\big)\,\frac{K\,M}{2}\,\frac{(t-\sigma)^2}{t\,r(t)} - \frac{\big(\Phi(t)\big)^2}{\rho(t)}$$

so

$$\Phi'(t) \leq -\rho(t)q(t)\big(1-p(t-\sigma)\big)\,\frac{K\,M}{2}\,\frac{(t-\sigma)^2}{t\,r(t)} + \left[\frac{\rho'(t)}{\rho(t)}\Phi(t) - \frac{1}{\rho(t)}\,\big(\Phi(t)\big)^2\right]. \tag{3.2.11}$$

Let $v = \dfrac{\rho'(t)}{\rho(t)}$, $u = \dfrac{1}{\rho(t)}$, $x = \Phi(t)$ and applying Lemma 1.1.1 ,

we have

$$\left[\frac{\rho'(t)}{\rho(t)}\Phi(t) - \frac{1}{\rho(t)}\,\big(\Phi(t)\big)^2\right] \leq \frac{1}{4}\,\frac{\left(\frac{\rho'(t)}{\rho(t)}\right)^2}{\left(\frac{1}{\rho(t)}\right)}\;,$$

so

$$\left[\frac{\rho'(t)}{\rho(t)}\Phi(t)-\frac{1}{\rho(t)}\left(\Phi(t)\right)^2\right]\leq\frac{(\rho'(t))^2}{4\rho(t)} \quad . \tag{3.2.12}$$

Using (3.2.12) in (3.2.11) , we have

$$\Phi'(t)\leq-\rho(t)q(t)\big(1-p(t-\sigma)\big)\frac{K\,M}{2}\,\frac{(t-\sigma)^2}{t\,r(t)}+\frac{(\rho'(t))^2}{4\rho(t)} \ ,$$

hence

$$2\,\rho(t)q(t)\big(1-p(t-\sigma)\big)K\,M\,\frac{(t-\sigma)^2}{t\,r(t)}-\frac{(\rho'(t))^2}{\rho(t)}\leq-4\,\Phi'(t)\,.$$

Integrating the previous inequality from t_3 to t , where $t>t_3>t_2>t_1>t_0$

$$\int_{t_3}^{t}\left[2\,\rho(s)q(s)\big(1-p(s-\sigma)\big)\,K\,M\,\frac{(s-\sigma)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds\,\leq-\int_{t_3}^{t}4\,\Phi'(s)\,ds\,,$$

so

$$\int_{t_3}^{t}\left[2\rho(s)q(s)\big(1-p(s-\sigma)\big)K\,M\,\frac{(s-\sigma)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds\,\leq 4\,\Phi(t_3)-\,4\,\Phi(t)\,.$$

But $\Phi(t)>0$, so

$$\int_{t_3}^{t}\left[2\,\rho(s)q(s)\big(1-p(s-\sigma)\big)MK\,\frac{(s-\sigma)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds\leq 4\,\Phi(t_3)\,,$$

hence

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left[2\,\rho(s)\,q(s)\big(1-p(s-\sigma)\big)MK\,\frac{(s-\sigma)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds\leq 4\,\Phi(t_3)<\infty$$

which is a contradiction to $(H6)$. Hence every solution of equation $(2N1-A)$ is oscillatory . The proof is complete . ■

Now , we can generalize this theorem for general delay terms as the next theorem .

Theorem 3.2.2:

Assume that $(H_1)-(H5)$ hold and if

$$\lim_{t\to\infty} sup \int_{t_3}^{t} \left[2\,\rho(s)q(s)\left(1-p\big(\sigma(s)\big)\right) K\,M \frac{\big(\sigma(s)\big)^2}{s\,r(s)} - \frac{(\rho'(s))^2}{\rho(s)} \right] ds = \infty \,,$$

for sufficiently large $t_1 > t_0$ and for $t_3 > t_2 > t_1$ holds , where τ , $\sigma \in C[t_0\,,\infty)$

$0 \le \tau(t) \le t$, $0 \le \sigma(t) \le t$, $\lim_{t\to\infty}(\tau(t)) = \lim_{t\to\infty}\big(\sigma(t)\big) = \infty$ and

$\rho(t) \in C'([t_0\,,\infty),(\,0\,,\infty)\,)$, then equation $(2N1-B)$ is almost oscillatory .

Proof:

Let $y(t)$ be a nonoscillatory solution of equation $(2N1-B)$. And suppose that $y(t)$ is a positive solution , where

$$z(t) = y(t) + p(t)\,y\big(\tau(t)\big) \,, \qquad (3.2.13)$$

so , there exist two possible cases :

(I) $z(t) > 0$, $z\,'(t) < 0$, $(r(t)z'(t))' > 0$, $(\,r(t)z'(t))'' \le 0$,

(II) $z(t) > 0$, $z\,'(t) > 0$, $\big(r(t)z'(t)\big)' > 0$, $\big(\,r(t)z'(t)\big)'' \le 0$.

Consider

Case $(\boldsymbol{I})$: $z(t) > 0$, $z'(t) < 0$, $\left(r(t)z'(t)\right)' > 0$, $\left(r(t)z'(t)\right)'' \leq 0$,

$$\left(r(t)\left(y(t) + p(t)y\big(\tau(t)\big)\right)'\right)'' + f(t)y\big(\sigma(t)\big) = 0 \qquad (2N1 - B)$$

so

$$\left(r(t)\left(y(t) + p(t)y\big(\tau(t)\big)\right)'\right)'' = -f(t)y\big(\sigma(t)\big) \ ,$$

using (3.2.13) we have

$$\left(r(t)z'(t)\right)'' = -f(t)y\big(\sigma(t)\big) ,$$

but $q(t) \leq f(t)$, so

$$-f(t)y\big(\sigma(t)\big) \leq \ -q(t)y\big(\sigma(t)\big)$$

But from inequality (2.2.4) , we have $y(t) \geq m\, z(t)$, so

$$y\big(\sigma(t)\big) \geq m\, z\big(\sigma(t)\big) ,$$

multiplying this inequality by $f(t) > 0$, we have

$$-f(t)y\big(\sigma(t)\big) \leq -m\, f(t)\, z\big(\sigma(t)\big) ,$$

hence

$$\left(r(t)z'(t)\right)'' \leq -m\, f(t)\, z\big(\sigma(t)\big) \ ,$$

so

$$-\left(r(t)z'(t)\right)'' \geq m\, f(t)\, z\big(\sigma(t)\big) \ ,$$

integrating the previous inequality from t to ∞

$$-\int_{t}^{\infty}\left(r(s)z'(s)\right)'' ds \geq \int_{t}^{\infty} m\, f(s)\, z\left(\sigma(s)\right)ds$$

so

$$-\left(\lim_{\lambda\to\infty}\left(r(\lambda)z'(\lambda)\right)' - \left(r(t)z'(t)\right)'\right) \geq \int_{t}^{\infty} m\, f(s)\, z\left(\sigma(s)\right)ds \ ,$$

so

$$\left(r(t)z'(t)\right)' \geq \lim_{\lambda\to\infty}\left(r(\lambda)z'(\lambda)\right)' + \int_{t}^{\infty} m\, f(s)\, z\left(\sigma(s)\right)ds \ ,$$

but $\lim_{\lambda\to\infty}\left(r(\lambda)z'(\lambda)\right)' < 0$, so

$$\left(r(t)z'(t)\right)' \geq \int_{t}^{\infty} m\, f(s)\, z\left(\sigma(s)\right)ds \, .$$

From Theorem 2.2.1, we have $z\left(\sigma(t)\right) \geq k$,

hence

$$\left(r(t)z'(t)\right)' \geq \int_{t}^{\infty} m\, k\, f(s)\, ds \, .$$

And the rest of the proof of this case is the same as the proof of case (I) in Theorem 3.2.1.

Case $(\boldsymbol{II})$: $z(t) > 0$, $z'(t) > 0$, $\left(r(t)z'(t)\right)' > 0$, $\left(r(t)z'(t)\right)'' \leq 0$,

$z(t)$ is an increasing function , so for $t \geq \sigma(t) > 0 \implies z(t) > z\left(\sigma(t)\right)$.

Also $t \geq \tau(t) > 0 \implies \sigma(t) \geq \tau(\sigma(t)) > 0$.

But

$$z(t) = y(t) + p(t)y(\tau(t)),$$

so

$$z(\sigma(t)) = y(\sigma(t)) + p(\sigma(t))\, y\left(\tau(\sigma(t))\right) , \tag{3.2.14}$$

hence

$$y(\sigma(t)) = z(\sigma(t)) - p(\sigma(t))y\left(\tau(\sigma(t))\right),$$

but $t \geq \tau(t)$ and $t \geq \sigma(t)$, so $t \geq \sigma(t) \geq \tau(\sigma(t))$, and

$$z(t) \geq z(\sigma(t)) \geq z\left(\tau(\sigma(t))\right) \geq y\left(\tau(\sigma(t))\right),$$

so

$$y(\sigma(t)) \geq z(\sigma(t)) - p(\sigma(t))z(\sigma(t)),$$

hence

$$y(\sigma(t)) \geq \left(1 - p(\sigma(t))\right) z(\sigma(t)). \tag{3.2.15}$$

Integrating $\left(r(u)z'(u)\right)'$ from t_1 to t gives

$$\int_{t_1}^{t} \left(r(u)z'(u)\right)' \, du = r(t)z'(t) - r(t_1)z'(t_1) \ , \ \text{where } t > t_1 \geq t_0$$

And following the same steps used in proving case (II) in Theorem 3.2.1 , we get again

$$\left(\frac{\left(r(t)z'(t)\right)}{(t-t_1)}\right)' \leq 0 \ . \tag{3.2.16}$$

$$\frac{z(t)}{r(t)\, z'(t)} \ \geq \ \frac{1}{(t-t_1)} \int_{t_2}^{t} \frac{(s-t_1)}{r(s)}\, ds \ , \tag{3.2.17}$$

for $t \ \geq \ t_2 \geq t_1 \geq t_0$.

Define a function

$$\varphi(t) = \rho(t) \frac{\left(r(t)z'(t)\right)'}{r(t)z'(t)} \ .$$

Differentiating $\varphi(t)$ and following the same steps used in proving case (II)

in Theorem 3.2.1 , we have again

$$\varphi'(t) = \ \frac{\rho'(t)}{\rho(t)}\varphi(t) \ + \rho(t) \ \frac{\left(r(t)z'(t)\right)''}{r(t)z'(t)} - \frac{\left(\varphi(t)\right)^2}{\rho(t)} \quad .$$

We consider $f(t) > 0$, so from (3.2.14) and $q(t) \leq f(t) \Longrightarrow -q(t) \geq -f(t)$,

we have

$$-f(t)y\left(\sigma(t)\right) \leq -f(t)\left(1-p\left(\sigma(t)\right)\right)z\left(\sigma(t)\right)$$

and since $\left(r(t)z'(t)\right)'' = -f(t)y\left(\sigma(t)\right)$

$$\Longrightarrow \left(r(t)z'(t)\right)'' \leq -q(t)\left(1-p\left(\sigma(t)\right)\right)z\left(\sigma(t)\right) \ \ , \tag{3.2.18}$$

hence

$$\varphi'(t) \leq \ \frac{\rho'(t)}{\rho(t)}\varphi(t) \ + \rho(t) \ \frac{-q(t)\left(1-p\left(\sigma(t)\right)\right)z\left(\sigma(t)\right)}{r(t)z'(t)} - \frac{\left(\varphi(t)\right)^2}{\rho(t)} \ ,$$

so

$$\varphi'(t) \le \frac{\rho'(t)}{\rho(t)}\varphi(t) - \rho(t)q(t)\Big(1 - p\big(\sigma(t)\big)\Big)\frac{z\big(\sigma(t)\big)}{r(t)z'(t)} - \frac{\big(\varphi(t)\big)^2}{\rho(t)}$$

So

$$\varphi'(t) \le \frac{\rho'(t)}{\rho(t)}\varphi(t) - \rho(t)q(t)\Big(1 - p\big(\sigma(t)\big)\Big)\frac{z\big(\sigma(t)\big)}{z'\big(\sigma(t)\big)}\frac{z'\big(\sigma(t)\big)}{z'(t)}\frac{1}{r(t)} - \frac{\big(\varphi(t)\big)^2}{\rho(t)} \; . \qquad (3.2.19)$$

Using $r(t) > 0$, $r'(t) \ge 0$, $r''(t) \ge 0$, $\big(r(t)z'(t)\big)' > 0$, and $\big(r(t)z'(t)\big)'' \le 0$ for $t \ge t_0$, we have $z''(t) > 0$, $z'''(t) \le 0$. Applying Lemma 1.1.2 and Lemma 1.1.4 , we have

$$\frac{z\big(\sigma(t)\big)}{z'\big(\sigma(t)\big)} \ge \frac{1}{2} M\big(\sigma(t)\big) \text{ and } \frac{z'\big(\sigma(t)\big)}{z'(t)} \ge K\frac{\big(\sigma(t)\big)}{t} \;, \text{ where } K, M \in (0,1) \; .$$

Hence

$$\frac{z\big(\sigma(t)\big)}{z'\big(\sigma(t)\big)}\frac{z'(t-\sigma)}{z'(t)} \ge \frac{K\,M}{2}\frac{\big(\sigma(t)\big)^2}{t} \;,$$

and using it in (3.2.19) , we get

$$\varphi'(t) \le -\rho(t)q(t)\Big(1 - p\big(\sigma(t)\big)\Big)\frac{K\,M}{2}\frac{\big(\sigma(t)\big)^2}{t\,r(t)} + \left[\frac{\rho'(t)}{\rho(t)}\varphi(t) - \frac{1}{\rho(t)}\big(\varphi(t)\big)^2\right]. \qquad (3.2.20)$$

Now let $v = \dfrac{\rho'(t)}{\rho(t)}$, $u = \dfrac{1}{\rho(t)}$, $x = \varphi(t)$ and applying Lemma 1.1.1 ,

we have

$$\left[\frac{\rho'(t)}{\rho(t)}\varphi(t)-\frac{1}{\rho(t)}\left(\varphi(t)\right)^2\right] \leq \frac{1}{4}\frac{\left(\frac{\rho'(t)}{\rho(t)}\right)^2}{\left(\frac{1}{\rho(t)}\right)} \quad ,$$

or

$$\left[\frac{\rho'(t)}{\rho(t)}\varphi(t)-\frac{1}{\rho(t)}\left(\varphi(t)\right)^2\right] \leq \frac{(\rho'(t))^2}{4\rho(t)} \quad , \tag{3.2.21}$$

and using (3.2.21) in (3.2.20) , we have

$$\varphi'(t) \leq -\rho(t)q(t)\left(1-p\big(\sigma(t)\big)\right)\frac{K\,M}{2}\frac{\left(\sigma(t)\right)^2}{t\,r(t)}+\frac{(\rho'(t))^2}{4\rho(t)} \quad ,$$

hence

$$\rho(t)q(t)\left(1-p\big(\sigma(t)\big)\right)\frac{K\,M}{2}\frac{\left(\sigma(t)\right)^2}{t\,r(t)}-\frac{\left(\rho^{'}(t)\right)^2}{4\rho(t)} \leq -\ \varphi^{'}(t)\,.$$

Integrating the last inequality from t_3 to t where $t > t_3 > t_2 > t_1 \geq t_0$

$$\int_{t_3}^{t}\left[2\,\rho(s)q(s)\left(1-p\big(\sigma(s)\big)\right)K\,M\frac{\left(\sigma(s)\right)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds \leq -\int_{t_3}^{t}4\,\varphi'(s)\,ds\,,$$

so

$$\int_{t_3}^{t}\left[2\,\rho(s)q(s)\left(1-p\big(\sigma(s)\big)\right)K\,M\frac{\left(\sigma(s)\right)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds \leq 4\varphi(t_3)-4\varphi(t)$$

but $\varphi(t) > 0$, hence

$$\int_{t_3}^{t}\left[2\,\rho(s)q(s)\left(1-p\big(\sigma(s)\big)\right)K\,M\frac{\left(\sigma(s)\right)^2}{s\,r(s)}-\frac{(\rho'(s))^2}{\rho(s)}\right]ds \leq 4\,\varphi(t_3)\,.$$

Take the $\lim_{t\to\infty} \sup$, we get

$$\lim_{t\to\infty} \sup \int_{t_3}^{t} \left[2\,\rho(s) q(s) \left(1 - p(\sigma(s))\right) K\, M \frac{\left(\sigma(s)\right)^2}{s\, r(s)} - \frac{\left(\rho'(s)\right)^2}{\rho(s)} \right] ds \le 4\, \varphi(t_3) < \infty$$

which is a contradiction to $(H6)$. Hence every solution of equation $(2N1 - B)$ is oscillatory. The proof is complete. ■

3.3 Illustrating Examples :

Example 3.3.1:

Consider the neutral delay differential equation ,

$$\frac{d^2}{dt^2}\left(e^{-t}\frac{d}{dt}\left(y(t) + \frac{1}{2}\, y(t - 2\pi)\right)\right) + 3e^{-t}\, y(t - \pi) = 0 \qquad (3.3.1)$$

Here $\sigma = \pi$, $\tau = 2\pi$, $r(t) = e^{-t}$, $p(t) = \frac{1}{2}$, $f(t) = e^{-t}$, $q(t) = \frac{s}{(s - \pi)^2}$,

and take $\rho(t) = 1$. $R(t) = \int_{t}^{\infty} \frac{1}{e^{-s}}\, ds = \int_{t}^{\infty} e^{s}\, ds = \infty$, and applying

Theorem 3.2.1 , then all conditions are satisfied. Hence all solutions of (3.3.1) are oscillatory or converge to zero. One of these solutions is $y(t) = \sin t$.

Example 3.3.2 :

Consider the third order neutral delay differential equation

$$\frac{d^2}{dt^2}\left(t\ \frac{d}{dt}\left(y(t) + k_1\, y\left(\frac{t}{2}\right)\right)\right) + k_2 t\, y(t) = 0 \qquad (3.3.2)$$

$k_1 \in [\,0\,,1\,)$, $k_2 \geq 0$, $t \geq 1$.

Here , $\rho(t) = 1$, $r(t) = t$, $p(t) = k_1$, $f(t) = k_2 t$, $q(t) = \dfrac{1}{t^2}$

and $R(t) = \int_t^{\infty} \frac{1}{t}\, ds = \infty$,

applying Theorem 3.2.1 , then all conditions are satisfied . Hence all solutions of equation (3.3.2) are oscillatory or converge to zero .

Chapter Four

Oscillation of Solution of the Equation of the Form

$(\mathbf{1N1-A})$ and $(\mathbf{1N1-B})$

4.1 Introduction :

This chapter discusses the oscillation of solutions for the third order neutral delay differential equation of the form

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\big(y(t)+p(t)y(t-\tau)\big)\right)\right)+f(t)y(t-\sigma)=0 \qquad (1N1-A)$$

which is appeared at the end of the paper of K. V. V. Seshagiri Rao [13] .

So , we will prove theorems concerning this form . And we will also consider the more general delay form

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\big(y(t)+p(t)y(\tau(t))\big)\right)\right)+f(t)y(\sigma(t))=0\ . \qquad (1N1-B)$$

In this chapter we always assume that

$(H1)$: $r_1(t)$, $r_2(t)$, $p(t)$, $f(t)\in C\left([t_0,\infty)\right)$, $r_1(t)>0$, $r_2(t)>0$, $p(t)>0$,

$f(t)>0$, $r_1(t)\in C^2\big([t_0,\infty)\big)$, $r_2(t)\in C^1\big([t_0,\infty)\big)$, ${r_2}'(t)\geq 0$, ${r_1}''(t)\geq 0$

$(H2)$: $0\leq p(t)\leq p_1<1$, $p(t)\in C\left(\,[\,t_0,\infty)\right)$.

$(H3)$: There exists a positive decreasing function $q(t)$ such that $f(t)\geq q(t)$

for $t \in [t_0, \infty)$. (i e : $q(t) > 0$, $q'(t) < 0$, $q(t) \le f(t)\ \forall\, t \ge t_0$)

And set

$$R_1(t) = \int_t^{\infty} \frac{1}{r_1(s)}\, ds \quad , \quad R_2(t) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds\,.$$

There are three possibilities for $R_1(t)$ and $R_2(t)$

$$R_1(t) = \int_t^{\infty} \frac{1}{r_1(s)}\, ds = \infty \quad , \quad R_2(s) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds = \infty\,, \tag{4.1.1}$$

$$R_1(t) = \int_t^{\infty} \frac{1}{r_1(s)}\, ds = \infty \quad , \quad R_2(t) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds < \infty\,, \tag{4.1.2}$$

$$R_1(t) = \int_t^{\infty} \frac{1}{r_1(s)}\, ds < \infty \quad , \quad R_2(t) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds < \infty\ . \tag{4.1.3}$$

4.2 Oscillation of Solution Conditions of the Equation $(\mathbf{1}\boldsymbol{N}\mathbf{1} - \boldsymbol{A})$

Theorem 4.2.1 :

Assume that (4.1.1) and $(H1) - (H3)$ hold and if for some function

$\rho(t) \in C^1\big([t_0, \infty)\ , (0, \infty)\big)$ for all sufficiently large $t_1 \ge t_0$ and for $t_3 > t_2 > t_1$

where $\rho(t) > 0$, $\rho'(t) \ge 0$, one has

$$\lim_{t\to\infty} sup \int_{t_3}^{t} \left[\rho(s) q(s) (1 - p(s-\sigma)\ \frac{\int_{t_2}^{s-\sigma} \left[\int_{t_1}^{v} \frac{1}{r_2(u)} du\ \frac{1}{r_1(v)} \right] dv}{\int_{t_1}^{s} \frac{1}{r_2(u)} du} - \frac{r_2(s) \big(\rho'(s)\big)^2}{4\rho(s)} \right] ds = \infty$$

(4.2.1)

or

$$\int_{t_0}^{\infty} \frac{1}{r_1(v)} \int_{v}^{\infty} \frac{1}{r_2(u)} \int_{u}^{\infty} f(s)ds\ du\, dv = \infty . \tag{4.2.2}$$

Then $(1N1-A)$ is almost oscillatory , where $0 \le \tau \le \mathrm{t}$, $0 \le \sigma \le t$,

σ , τ constants and $\lim_{t\to\infty}(t-\tau) = \lim_{t\to\infty}(t-\sigma) = \infty$

Proof :

Suppose that $y(t)$ is a positive solution and

$$z(t) = y(t) + p(t)y(t-\tau) . \tag{4.2.3}$$

There exists two possible cases:

(I) $z(t) > 0$, $z'(t) > 0$, $\left(r_1(t)z'(t)\right)' > 0$, $\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' \le 0,$

(II) $z(t) > 0$, $z'(t) < 0$, $\left(r_1(t)z'(t)\right)' > 0$, $\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' \le 0,$

for $t > t_1$, t_1 is large enough .

Case $(\boldsymbol{I})$: $z(t) > 0$, $z'(t) > 0$, $\left(r_1(t)z'(t)\right)' > 0$, $\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' \le 0$.

Let

$$\omega(t) = \rho(t)\frac{r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)} , \quad t \ge t_1 , \quad \rho(t) > 0 , \quad \rho'(t) > 0 \tag{4.2.4}$$

it is clear that $\varphi(t) > 0$,

$z(t)$ is an increasing function , so for $t \ge t-\sigma \ge 0 \Longrightarrow z(t) \ge z(t-\sigma)$

$z(t) = y(t) + p(t)y(t-\tau) \Rightarrow z(t) \ge y(t)$ and $z(t-\sigma) \ge y(t-\sigma)$, hence $z(t) \geq y(t-\sigma)$

So , $z(t) \leq y(t) + p(t)z(t-\sigma)$ and so $z(t) \leq y(t) + p(t)z(t)$,

hence $z(t) - p(t)z(t) \leq y(t)$ which implies

$$y(t) \geq \big(1 - p(t)\big)\, z(t) \tag{4.2.5}$$

$$\int_{t_1}^{t} \big(r_1(s)z'(s)\big)'\, ds = r_1(s)z'(s)\Big]_{t_1}^{t} = r_1(t)z'(t) - r_1(t_1)z'(t_1) \quad ,$$

the function $r_1(t)z'(t)$ is positive increasing and $t > t_1$, so we get

$$\int_{t_1}^{t} \big(r_1(s)z'(s)\big)'\, ds \leq r_1(t)z'(t) \tag{4.2.6}$$

Since $r_2(t)\big(r_1(t)z'(t)\big)'$ is positive decreasing function and

$\big(r_1(t)z'(t)\big)' \geq 0$ we have

$$t \geq s \implies r_2(t)\big(r_1(t)z'(t)\big)' \leq r_2(s)\big(r_1(s)z'(s)\big)'$$

$$\implies \big(r_1(s)z'(s)\big)' \geq r_2(t)\big(r_1(t)z'(t)\big)' \frac{1}{r_2(s)}$$

$$\int_{t_1}^{t} \big(r_1(s)z'(s)\big)'\, ds \geq \int_{t_1}^{t} r_2(t)\big(r_1(t)z'(t)\big)' \frac{1}{r_2(s)}\, ds \,,$$

hence

$$\int_{t_1}^{t} \big(r_1(s)z'(s)\big)'\, ds \geq r_2(t)\big(r_1(t)z'(t)\big)' \int_{t_1}^{t} \frac{1}{r_2(s)}\, ds. \tag{4.2.7}$$

From (4.2.6) and (4.2.7) , we get

$$r_1(t)z'(t) \geq \int_{t_1}^{t} \frac{r_2(s)\left(r_1(s)z'(s)\right)'}{r_2(s)}\,ds \geq r_2(t)\left(r_1(t)z'(t)\right)' \int_{t_1}^{t} \frac{1}{r_2(s)}\,ds \;. \tag{4.2.8}$$

From (4.2.8) , we have

$$r_2(t)\left(r_1(t)z'(t)\right)' \int_{t_1}^{t} \frac{1}{r_2(s)}\,ds - r_1(t)z'(t) \leq 0 \;,$$

dividing by $r_2(t)\left(\int_{t_1}^{t} \frac{1}{r_2(s)}\,ds\right)^2 > 0$, we get

$$\frac{r_2(t)\left(r_1(t)z'(t)\right)' \int_{t_1}^{t} \frac{1}{r_2(s)}\,ds - r_1(t)z'(t)}{r_2(t)\left(\int_{t_1}^{t} \frac{1}{r_2(s)}\,ds\right)^2} \leq 0 \;.$$

So we have

$$\left(\frac{r_1(t)z'(t)}{\int_{t_1}^{t} \frac{1}{r_2(s)}\,ds}\right)' \leq 0 \quad , \tag{4.2.9}$$

hence $\dfrac{r_1(t)z'(t)}{\int_{t_1}^{t} \frac{1}{r_2(s)}\,ds}$ is positive nonincreasing function for $t \geq t_1$.

For $t \geq s \geq t_1$ we have

$$\frac{r_1(t)z'(t)}{\int_{t_1}^{t} \frac{1}{r_2(u)}\,du} \leq \frac{r_1(s)z'(s)}{\int_{t_1}^{s} \frac{1}{r_2(u)}\,du} \;. \tag{4.2.10}$$

But $\int_{t_2}^{t} z'(s)ds = z(t) - z(t_2)$, so $z(t) = z(t_2) + \int_{t_2}^{t} z'(s)ds$

since $z(t)$ is increasing and $t \geq t_2$, we have

$$z(t) \geq \int_{t_2}^{t} z'(s)ds \quad ,$$

multiplying and dividing by $r_1(s) \int_{t_1}^{t} \frac{1}{r_2(u)}\, du$ the last inequality becomes

$$z(t) \geq \int_{t_2}^{t} z'(s) \frac{r_1(s) \int_{t_1}^{t} \frac{1}{r_2(u)}\, du}{r_1(s) \int_{t_1}^{t} \frac{1}{r_2(u)}\, du}\, ds \; .$$

Arranging the last inequality gives

$$z(t) \geq \int_{t_2}^{t} \frac{r_1(s) z'(s)}{\int_{t_1}^{s} \frac{1}{r_2(u)}\, du} \; \frac{\int_{t_1}^{s} \frac{1}{r_2(u)}\, du}{r_1(s)}\, ds \; ,$$

using the previous inequality and (4.2.10) , we get

$$z(t) \geq \int_{t_2}^{t} \frac{r_1(t) z'(t)}{\int_{t_1}^{t} \frac{1}{r_2(u)}\, du} \; \frac{\int_{t_1}^{s} \frac{1}{r_2(u)}\, du}{r_1(s)}\, ds \; ,$$

so , we have

$$z(t) \geq \frac{r_1(t) z'(t)}{\int_{t_1}^{t} \frac{1}{r_2(u)}\, du} \int_{t_2}^{t} \frac{\int_{t_1}^{s} \frac{1}{r_2(u)}\, du}{r_1(s)}\, ds,$$

hence

$$\frac{z(t)}{r_1(t)z'(t)} \geq \frac{\int_{t_2}^{t}\left[\int_{t_1}^{s}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(s)}\,ds}{\int_{t_1}^{t}\frac{1}{r_2(u)}\,du} \tag{4.2.11}$$

where $t \geq t_2 > t_1$.

Differentiating $\omega(t) = \rho(t)\frac{\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)}$ gives

$$\omega'(t) = \rho(t)\left(\frac{r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)}\right)' + \rho'(t)\,\frac{r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)}$$

$$\omega'(t) = \rho(t)\frac{r_1(t)z'(t)\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' - r_2(t)\left(r_1(t)z'(t)\right)'\left(r_1(t)z'(t)\right)'}{\left(r_1(t)z'(t)\right)^2}$$

$$+\rho'(t)\,\frac{r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)}$$

$$\omega'(t) = \rho(t)\frac{r_1(t)z'(t)\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{\left(r_1(t)z'(t)\right)^2} - \rho(t)\frac{r_2(t)\left(r_1(t)z'(t)\right)'\left(r_1(t)z'(t)\right)'}{\left(r_1(t)z'(t)\right)^2}$$

$$+\rho'(t)\,\frac{r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)}$$

$$\omega'(t) = \rho(t)\frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \rho(t)\frac{r_2(t)\left[\left(r_1(t)z'(t)\right)'\right]^2}{\left(r_1(t)z'(t)\right)^2} + \frac{\rho'(t)}{\rho(t)}\,\frac{\rho(t)r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)}$$

$$\omega'(t) = \rho(t)\frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{\left(\rho(t)\right)^2}{\rho(t)\,r_2(t)}\,\frac{\left(r_2(t)\right)^2\left[\left(r_1(t)z'(t)\right)'\right]^2}{\left(r_1(t)z'(t)\right)^2}$$

$$+\frac{\rho'(t)}{\rho(t)}\,\rho(t)\frac{\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)} \, .$$

Using $\omega(t)$ in $\omega'(t)$ gives

$$\omega'(t) = \rho(t)\frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{\left(\omega(t)\right)^2}{\rho(t)\, r_2(t)} + \frac{\rho'(t)}{\rho(t)}\,\omega(t)$$

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\,\omega(t) + \rho(t)\frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{\left(\omega(t)\right)^2}{\rho(t)\, r_2(t)} \;. \tag{4.2.12}$$

But

$$\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' = -f(t)y(t-\sigma)$$

and using (4.2.5) , we have

$$-\rho(t)\, f(t)y(t-\sigma) \le -\rho(t)\, f(t)\big(1-p(t-\sigma)\big)z(t-\sigma). \tag{4.2.13}$$

Putting (4.2.13) in (4.2.12) , we get

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}.\,\omega(t) - \rho(t)\, f(t)\big(1-p(t-\sigma)\big)\frac{z(t-\sigma)}{r_1(t)z'(t)} - \frac{\left(\omega(t)\right)^2}{\rho(t)\, r_2(t)} \;,$$

so

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\,\omega(t) - \rho(t)\, f(t)\big(1-p(t-\sigma)\big)\frac{z(t-\sigma)}{r_1(t)z'(t)}\,\frac{r_1(t-\sigma)z'(t-\sigma)}{r_1(t-\sigma)z'(t-\sigma)} - \frac{\left(\omega(t)\right)^2}{\rho(t)\, r_2(t)} \;,$$

hence

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\,\omega(t) - \rho(t)\, f(t)\big(1-p(t-\sigma)\big)\frac{z(t-\sigma)}{r_1(t-\sigma)z'(t-\sigma)}\,\frac{r_1(t-\sigma)z'(t-\sigma)}{r_1(t)z'(t)}$$

$$- \frac{\left(\omega(t)\right)^2}{\rho(t)\, r_2(t)} \;. \tag{4.2.14}$$

Using (4.2.9) and definition of nonincreasing functions and positivity, we have for

$$t \ge t-\sigma \Longrightarrow \; 0 < \frac{r_1(t)z'(t)}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds} \le \frac{r_1(t-\sigma)z'(t-\sigma)}{\int_{t_1}^{t-\sigma}\frac{1}{r_2(s)}\,ds}$$

$$\Longrightarrow \quad 0 < \frac{r_1(t-\sigma)z'(t-\sigma)}{r_1(t)z'(t)} \leq \frac{\int_{t_1}^{t-\sigma} \frac{1}{r_2(s)}\, ds}{\int_{t_1}^{t} \frac{1}{r_2(s)}\, ds} \ . \tag{4.2.15}$$

Using (4.2.11) and (4.2.15) in (4.2.14) , we have

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\big(1-p(t-\sigma)\big)\frac{\int_{t_2}^{t-\sigma}\left[\int_{t_1}^{s}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(s)}\,ds}{\int_{t_1}^{t-\sigma}\frac{1}{r_2(u)}\,du}\,\frac{\int_{t_1}^{t-\sigma}\frac{1}{r_2(s)}\,ds}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds} - \frac{\big(\omega(t)\big)^2}{\rho(t)\,r_2(t)}$$

hence

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}.\omega(t) - \rho(t)\,f(t)\big(1-p(t-\sigma)\big)\frac{\int_{t_2}^{t-\sigma}\left[\int_{t_1}^{s}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(s)}\,ds}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds} - \frac{\big(\omega(t)\big)^2}{\rho(t)\,r_2(t)} \quad ,$$

so

$$\omega'(t) \leq -\rho(t)\,f(t)\big(1-p(t-\sigma)\big)\frac{\int_{t_2}^{t-\sigma}\left[\int_{t_1}^{s}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(s)}\,ds}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}$$

$$+\left[\frac{\rho'(t)}{\rho(t)}.\omega(t) - \frac{1}{\rho(t)\,r_2(t)}\big(\omega(t)\big)^2\right] \tag{4.2.16}$$

Using $u = \dfrac{1}{\rho(t)\,r_2(t)}$, $v = \dfrac{\rho'(t)}{\rho(t)}$, $x = \omega(t)$ and applying Lemma 1.1.1 ,

we have

$$\frac{\rho'(t)}{\rho(t)}\ \omega(t) - \frac{1}{\rho(t)\,r_2(t)}\big(\omega(t)\big)^2 \leq \frac{1}{4}\ \frac{\left(\frac{\rho'(t)}{\rho(t)}\right)^2}{\left(\frac{1}{\rho(t)\,r_2(t)}\right)}$$

which gives

$$\frac{\rho'(t)}{\rho(t)}.\,\omega(t)-\frac{1}{\rho(t)\,r_2(t)}\big(\,\omega(t)\big)^2 \;\leq\; \frac{1}{4}\;\frac{r_2(t)\big(\rho'(t)\big)^2}{\rho(t)}\;. \tag{4.2.17}$$

Substituting (4.2.17) in (4.2.16) and using $f(t)\geq q(t)$ gives

$$\omega'(t)\leq -\rho(t)\,q(t)\big(1-p(t-\sigma)\big)\frac{\int_{t_2}^{t-\sigma}\left[\int_{t_1}^{s}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(s)}\,ds}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}+\frac{1}{4}\;\frac{r_2(t)\big(\rho'(t)\big)^2}{\rho(t)}\;,$$

hence

$$\rho(t)\,q(t)\big(1-p(t-\sigma)\big)\frac{\int_{t_2}^{t-\sigma}\left[\int_{t_1}^{s}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(s)}\,ds}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}-\frac{r_2(t)\big(\rho'(t)\big)^2}{4\,\rho(t)}\leq\;-\omega'(t).$$

Integrating the last inequality from t_3 to t where $t>t_3>t_2\geq 0$, we get

$$\int_{t_3}^{t}\left[\rho(s)\,q(s)\big(1-p(s-\sigma)\big)\frac{\int_{t_2}^{s-\sigma}\left[\int_{t_1}^{v}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(v)}\,dv}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}-\frac{r_2(s)\big(\rho'(s)\big)^2}{4\,\rho(s)}\right]ds\leq-\int_{t_3}^{t}\omega'(s)\,ds$$

$$\int_{t_3}^{t}\left[\rho(s)\,q(s)\big(1-p(s-\sigma)\big)\frac{\int_{t_2}^{s-\sigma}\left[\int_{t_1}^{v}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(v)}\,dv}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}-\frac{r_2(s)\big(\rho'(s)\big)^2}{4\,\rho(s)}\right]ds\leq\int_{t}^{t_3}\omega'(s)\,ds$$

$$\int_{t_3}^{t}\left[\rho(s)\,q(s)\big(1-p(s-\sigma)\big)\frac{\int_{t_2}^{s-\sigma}\left[\int_{t_1}^{v}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(v)}\,dv}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}-\frac{r_2(s)\big(\rho'(s)\big)^2}{4\,\rho(s)}\right]ds\;\leq\omega(t_3)-\omega(t)$$

But $\omega(t)>0$, so

$$\int_{t_3}^{t}\left[\rho(s)\,q(s)\big(1-p(s-\sigma)\big)\frac{\int_{t_2}^{s-\sigma}\left[\int_{t_1}^{v}\frac{1}{r_2(u)}\,du\right]\frac{1}{r_1(v)}\,dv}{\int_{t_1}^{t}\frac{1}{r_2(s)}\,ds}-\frac{r_2(s)\big(\rho'(s)\big)^2}{4\,\rho(s)}\right]ds\leq\omega(t_3)<\infty$$

Which contradicts (4.2.1) .

Case $(\boldsymbol{II})$: $z(t) > 0$, $z'(t) < 0$, $\left(r_1(t)z'(t)\right)' > 0$, $\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' \leq 0$.

From equation $(1N1 - A)$, we have

$$\left[r_2(t)\left(r_1(t)\ \left(y(t) + p(t)y(t-\tau)\right)'\right)'\right]' = -f(t)y(t-\sigma)$$

and using equation (4.2.3) , we have

$$\left[r_2(t)\left(r_1(t)\, z'(t)\right)'\right]' = -f(t)y(t-\sigma)\ .$$

In proving $case\,(I)$ in Theorem 2.2.1 , we had equation (2.2.4) which is

$$y(t) > m\, z(t),$$

so

$$y(t-\sigma) \geq m\, z(t-\sigma)\ ,$$

multiplying this inequality by $f(t) > 0$, we have

$$-f(t)y(t-\sigma) \leq -m\, f(t)\, z(t-\sigma)\ ,$$

hence

$$\left[r_2(t)\left(r_1(t)\, z'(t)\right)'\right]' \leq -m\, f(t)\, z(t-\sigma)\ ,$$

so

$$-\left[r_2(t)\left(r_1(t)\, z'(t)\right)'\right]' \geq m\, f(t)\, z(t-\sigma)\ .$$

Integrating the previous inequality from t to ∞

$$-\int_t^{\infty} \left[r_2(s)\big(r_1(s)\, z'(s)\big)'\right]' ds \geq \int_t^{\infty} m\, f(s)\, z(s-\sigma) ds \ ,$$

so

$$-\Big(\lim_{\lambda\to\infty} r_2(\lambda)\big(r_1(\lambda)\, z'(\lambda)\big)' - r_2(t)\big(r_1(t)\, z'(t)\big)' \Big) \geq \int_t^{\infty} m\, f(s)\, z(s-\sigma) ds \ ,$$

hence

$$r_2(t)\big(r_1(t)\, z'(t)\big)' \ \geq \ \lim_{\lambda\to\infty} r_2(\lambda)\big(r_1(\lambda)\, z'(\lambda)\big)' + \int_t^{\infty} m\, f(s)\, z(s-\sigma) ds \, .$$

Using inequality (2.2.2) , we have

$$z(t-\sigma) \geq k \ \ ,$$

hence

$$r_2(t)\big(r_1(t)\, z'(t)\big)' \ \geq \int_t^{\infty} m \ k\, f(s)\, ds \ ,$$

dividing by $r_2(t) > 0$, we get

$$\big(r_1(t)\, z'(t)\big)' \ \geq \ \frac{m\, k}{r_2(t)} \int_t^{\infty} f(s)\, ds \, .$$

Integrating this inequality from t to ∞ gives

$$\int_t^{\infty} \big(r_1(u)\, z'(u)\big)' \ du \ \geq \int_t^{\infty} \frac{m\, k}{r_2(u)} \left[\int_u^{\infty} f(s) \ ds \right] du$$

$$\lim_{\lambda\to\infty} r_1(\lambda)z'(\lambda) - r_1(t)z'(t) \geq m\,k \int_t^\infty \frac{1}{r_2(u)} \left[\int_u^\infty f(s)\,ds \right] du \; .$$

So

$$- r_1(t)z'(t) \geq - \lim_{\lambda\to\infty} r_1(\lambda)z'(\lambda) + m\,k \int_t^\infty \frac{1}{r_2(u)} \left[\int_u^\infty f(s)\,ds \right] du \; ,$$

$r_1(u)z'(u)$ is a negative continuous increasing function , hence

$$- r_1(t)z'(t) \geq m\,k \int_t^\infty \frac{1}{r_2(u)} \left[\int_u^\infty f(s)\,ds \right] du \; ,$$

dividing by $- r_1(t) < 0$, we get

$$\frac{- r_1(t)z'(t)}{- r_1(t)} \leq \frac{- m\,k}{r_1(t)} \int_t^\infty \frac{1}{r_2(u)} \left[\int_u^\infty f(s)\,ds \right] du$$

$$z'(t) \leq \frac{- m\,k}{r_1(t)} \int_t^\infty \frac{1}{r_2(u)} \left[\int_u^\infty f(s)\,ds \right] du \; .$$

Integrating the last inequality from t_0 to t , we get

$$\int_{t_0}^t z'(v)\,dv \leq - m\,k \int_{t_0}^t \left[\frac{1}{r_1(v)} \int_v^\infty \left(\frac{1}{r_2(u)} \int_u^\infty f(s)\,ds \right) du \right] dv$$

so

$$z(t) - z(t_1) \leq - m\,k \int_{t_0}^t \left[\frac{1}{r_1(v)} \int_v^\infty \left(\frac{1}{r_2(u)} \int_u^\infty f(s)\,ds \right) du \right] dv$$

hence

$$z(t) + m\,k \int_{t_0}^{t} \left[\frac{1}{r_2(v)} \int_{v}^{\infty} \left(\frac{1}{r_1(u)} \int_{u}^{\infty} f(s)\,ds \right) du \right] dv \leq z(t_1)$$

But $z(t) > 0$, so

$$m\,k \int_{t_1}^{t} \left[\frac{1}{r_1(v)} \int_{v}^{\infty} \left(\frac{1}{r_2(u)} \int_{u}^{\infty} f(s)\,ds \right) du \right] dv \leq z(t_1) \ .$$

Thus

$$\int_{t_0}^{t} \left[\frac{1}{r_1(v)} \int_{v}^{\infty} \left(\frac{1}{r_2(u)} \int_{u}^{\infty} f(s)\,ds \right) du \right] dv \leq \frac{z(t_1)}{m\,k} < \infty$$

This is a contradiction to (4.2.2) , hence k must be zero .

But $0 \leq y(t) \leq z(t)$, so

$$\lim_{t\to\infty} 0 \leq \lim_{t\to\infty} y(t) \leq \lim_{t\to\infty} z(t) = 0 \implies \lim_{t\to\infty} y(t) = 0 \ .$$

And the proof is complete . ■

Theorem 4.2.2 :

Assume that (4.1.2) and $(H1)-(H3)$ hold , if for some function

$\rho(t) \in C^1([t_0,\infty),(0,\infty))$ for all sufficiently large $t_1 > t_0$ and for $t_3 > t_2 > t_1$

where $\rho(t) > 0$, $\rho'(t) \geq 0$, one has equations (4.2.1) and (4.2.2) . If

$$\lim_{t\to\infty} \sup \int_{t_2}^{t} \left(R_2(s)\, q(s)(1-p(s-\sigma) \int_{t_1}^{s-\sigma} \frac{dv}{r_1(v)} - \frac{1}{4R_2(s) r_1(s)} \right) ds = \infty \qquad (4.2.18)$$

where $R_2(t) := \int_{t}^{\infty} \frac{1}{r_2(s)}\,ds$, $0 \leq \tau \leq t$, $0 \leq \sigma \leq t$, σ , τ constants and

$\lim_{t \to \infty} (t-\tau) = \lim_{t \to \infty} (t-\sigma) = \infty$. Then equation $(1N1-A)$ is almost oscillatory.

Proof:

Suppose that $y(t)$ is a positive solution and

$$z(t) = y(t) + p(t)y(t-\tau). \qquad (4.2.19)$$

There are three possible cases $(I), (II)$ as in Theorem 4.2.1, and

(III) $z(t) > 0$, $z'(t) > 0$, $\left(r_1(t)z'(t)\right)' < 0, \left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' \leq 0.$

Assume that case (I) and case (II) hold, respectively. We can obtain the conclusion of Theorem 4.2.2 by applying the proof of Theorem 4.2.1.

Assume case (III) holds. $r_2(t)\left(r_1(t)z'(t)\right)'$ is a negative continuous decreasing function from $\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' \leq 0$. So for $s \geq t \geq t_1 \geq 0$, we have

$$r_2(s)\left(r_1(s)z'(s)\right)' \leq r_2(t)\left(r_1(t)z'(t)\right)',$$

dividing by $r_2(s) > 0$ we have

$$\left(r_1(s)z'(s)\right)' \leq r_2(t)\left(r_1(t)z'(t)\right)' \frac{1}{r_2(s)},$$

integrating from t to ∞ we have

$$\int_t^\infty \left(r_1(s)z'(s)\right)' ds \leq r_2(t)\left(r_1(t)z'(t)\right)' \int_t^\infty \frac{1}{r_2(s)} ds$$

$$\lim_{\lambda\to\infty} r_1(\lambda)z'(\lambda) - r_1(t)z'(t) \leq r_2(t)\big(r_1(t)z'(t)\big)' \int_t^{\infty} \frac{1}{r_2(s)}\, ds,$$

hence

$$-r_1(t)z'(t) \leq -\lim_{\lambda\to\infty} r_1(\lambda)z'(\lambda) + r_2(t)\big(r_1(t)z'(t)\big)' \int_t^{\infty} \frac{1}{r_2(s)}\, ds\,,$$

so

$$-r_1(t)z'(t) \leq r_2(t)\big(r_1(t)z'(t)\big)' \int_t^{\infty} \frac{1}{r_2(s)}\, ds,$$

hence

$$- r_2(t)\big(r_1(t)z'(t)\big)' \int_t^{\infty} \frac{1}{r_2(s)}\, ds \leq r_1(t)z'(t)$$

dividing by $r_1(t)z'(t) > 0$ gives

$$-\frac{r_2(t)\big(r_1(t)z'(t)\big)'}{r_1(t)z'(t)} \int_t^{\infty} \frac{1}{r_2(s)}\, ds \leq \frac{r_1(t)z'(t)}{r_1(t)z'(t)}$$

so

$$-\frac{r_2(t)\big(r_1(t)z'(t)\big)'}{r_1(t)z'(t)} \int_t^{\infty} \frac{1}{r_2(s)}\, ds \leq 1\ . \tag{4.2.20}$$

Using $R_2(t) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds$ in (4.2.20) we have

$$-\frac{r_2(t)\big(r_1(t)z'(t)\big)'}{r_1(t)z'(t)} R_2(t) \leq 1\ . \tag{4.2.21}$$

We define the function ψ by

$$\psi(t) = \frac{r_2(t)\left(r_1(t)z'(t)\right)'}{r_1(t)z'(t)} \ , \tag{4.2.22}$$

where $t \geq t_1 \geq 0$.

Using (4.2.21) and (4.2.22) , we get

$$-R_2(t)\ \psi(t) \leq 1 \tag{4.2.23}$$

Differentiating (4.2.22) , we obtain

$$\psi'(t) = \frac{r_1(t)z'(t)\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]' - r_2(t)\left(r_1(t)z'(t)\right)'\left(r_1(t)z'(t)\right)'}{\left(r_1(t)z'(t)\right)^2}$$

$$\psi'(t) = \frac{r_1(t)z'(t)\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{\left(r_1(t)z'(t)\right)^2} - \frac{r_2(t)\left(r_1(t)z'(t)\right)'\left(r_1(t)z'(t)\right)'}{\left(r_1(t)z'(t)\right)^2}$$

$$\psi'(t) = \frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{r_2(t)\left[\left(r_1(t)z'(t)\right)'\right]^{\mathbf{2}}}{\left(r_1(t)z'(t)\right)^2}$$

$$\psi'(t) = \frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{r_2(t)}{r_2(t)}\ \frac{r_2(t)\left[\left(r_1(t)z'(t)\right)'\right]^{\mathbf{2}}}{\left(r_1(t)z'(t)\right)^2}$$

$$\psi'(t) = \frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{1}{r_2(t)}\ \frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]^{\mathbf{2}}}{\left(r_1(t)z'(t)\right)^2} \ . \tag{4.2.24}$$

Using (4.2.22) in (4.2.24) , we get

$$\psi'(t) = \frac{\left[r_2(t)\left(r_1(t)z'(t)\right)'\right]'}{r_1(t)z'(t)} - \frac{1}{r_2(t)}\left(\psi(t)\right)^2 \ . \tag{4.2.25}$$

Since

$$\left[r_2(t)\big(\, r_1(t)z'(t)\big)'\right]' < 0 \ , \ r_1(t)z'(t) > 0 \ \text{ and } \ r_2(t) > 0 \,,$$

we have

$$\psi'(t) < 0 \ ,$$

hence $\psi(t)$ is negative decreasing function.

But $\left[r_2(t)\big(\, r_1(t)z'(t)\big)'\right]' = -f(t)y(t-\sigma)$ and using (4.2.5), we have

$$-f(t)y(t-\sigma) \le -f(t)\big(1-p(t-\sigma)\big)z(t-\sigma) \tag{4.2.26}$$

putting (4.2.26) in (4.2.25), we obtain

$$\psi'(t) \le -f(t)\big(1-p(t-\sigma)\big)z(t-\sigma) \ \frac{1}{r_1(t)z'(t)} - \frac{1}{r_2(t)}\big(\psi(t)\big)^2$$

$$\psi'(t) \le -f(t)\big(1-p(t-\sigma)\big) \ \frac{z(t-\sigma)}{r_1(t)z'(t)} - \frac{1}{r_2(t)}\big(\, \psi(t)\big)^2 . \tag{4.2.27}$$

But $r_1(t)z'(t)$ is a positive continuous decreasing function, so

$$\forall\, s \le t \ \Longrightarrow \ r_1(s)z'(s) \ \ge \ r_1(t)z'(t)$$

$$\Longrightarrow z'(s) \ \ge \ r_1(t)z'(t)\,\frac{1}{r_1(s)}$$

$$\Longrightarrow \int_{t_1}^{t} z'(s)\, ds \ \ge \int_{t_1}^{t} r_1(t)z'(t)\,\frac{1}{r_1(s)}\, ds \ , \ \text{where } t \ge t_1 \ge 0$$

$$\Longrightarrow z(t) - z(t_1) \ \ge \ r_1(t)\, z'(t) \int_{t_1}^{t} \frac{1}{r_1(s)}\, ds \ ,$$

hence

$$z(t) \ \geq \ z(t_1) + \ r_1(t)\, z'(t) \int_{t_1}^{t} \frac{1}{r_1(s)}\, ds \ ,$$

so

$$z(t) \ \geq \ r_1(t)\, z'(t) \int_{t_1}^{t} \frac{1}{r_1(s)}\, ds \ , \qquad (4.2.28)$$

so

$$\frac{z(t)}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds} \ \geq \ r_1(t)\, z'(t)\,,$$

but $r_1(t)\, z'(t) \ > 0$, so

$$r_1(t) z'(t) \ - \ \frac{z(t)}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds} \ \leq 0 \ ,$$

dividing the last inequality by the positive quantity $r_1(t) \left[\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds \right]^2$ gives

$$\frac{r_1(t) z'(t) \ - \ \dfrac{z(t)}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds}}{r_1(t) \left[\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds \right]^2} \ \leq \ 0 \ ,$$

hence

$$\left(\frac{z(t)}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds} \right)' \ \leq \ 0\,.$$

Thus , $\dfrac{z(t)}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds}$ is a positive nonincreasing function ,

so for $t-\sigma \le t$, we have $\dfrac{z(t-\sigma)}{\int_{t_1}^{t-\sigma} \frac{1}{r_1(s)}\, ds} \ge \dfrac{z(t)}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds} > 0$,

hence

$$\frac{z(t-\sigma)}{z(t)} \ge \frac{\int_{t_1}^{t-\sigma} \frac{1}{r_1(s)}\, ds}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds} \quad . \tag{4.2.29}$$

Refaring to (4.2.27)

$$\psi'(t) \le -f(t)\big(1-p(t-\sigma)\big)\,\frac{z(t-\sigma)}{r_1(t)z'(t)}\,\frac{z(t)}{z(t)} - \frac{1}{r_2(t)}\big(\psi(t)\big)^2$$

$$\psi'(t) \le -f(t)\big(1-p(t-\sigma)\big)\,\frac{z(t)}{r_1(t)z'(t)}\,\frac{z(t-\sigma)}{z(t)} - \frac{1}{r_2(t)}\big(\psi(t)\big)^2 . \tag{4.2.30}$$

Using (4.2.28) and (4.2.29) in (4.2.30) , we get

$$\psi'(t) \le -f(t)\big(1-p(t-\sigma)\big)\,\frac{r_1(t)\,z'(t)\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds}{r_1(t)z'(t)}\,\frac{\int_{t_1}^{t-\sigma} \frac{1}{r_1(s)}\, ds}{\int_{t_1}^{t} \frac{1}{r_1(s)}\, ds} - \frac{1}{r_2(t)}\big(\psi(t)\big)^2$$

$$\psi'(t) \le -f(t)\big(1-p(t-\sigma)\big)\int_{t_1}^{t-\sigma} \frac{1}{r_1(s)}\, ds - \frac{1}{r_2(t)}\big(\psi(t)\big)^2 .$$

Multiplying the last inequality by $R_2(t)$, we have

$$R_2(t)\psi'(t) \le -R_2(t)\,f(t)\big(1-p(t-\sigma)\big)\int_{t_1}^{t-\sigma} \frac{1}{r_1(s)}\, ds - \frac{R_2(t)}{r_2(t)}\big(\psi(t)\big)^2 \tag{4.2.31}$$

$$R_2(t)\,f(t)\big(1-p(t-\sigma)\big)\int_{t_1}^{t-\sigma}\frac{1}{r_1(s)}\,ds+\frac{R_2(t)}{r_2(t)}\big(\psi(t)\big)^2\ \le\ -R_2(t)\psi'(t)\ .$$

Using $f(t)\ge q(t)$ and ntegrating (4.2.31) from t_2 to t where $t_2 > t_1 > 0$

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\big(\psi(s)\big)^2\right]ds\ \le\int_{t_2}^{t}-R_2(s)\psi'(s)\,ds$$

$$\int_{t_2}^{t}\left[R_2(s)\,f(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\big(\psi(s)\big)^2\right]ds\ \le\int_{t}^{t_2}R_2(s)\psi'(s)\,ds$$

(4.2.32)

Using integration by parts with $u=R_2(s)\ \Longrightarrow\ du=R'_2(s)\,ds$

$$dv=\psi'(s)\,ds\quad\Longrightarrow\quad v=\psi(s)$$

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\big(\psi(s)\big)^2\right]ds$$

$$\le R_2(s)\,\psi(s)\Big]_{t}^{t_2}-\int_{t}^{t_2}\psi(s)R_2{}'(s)\,ds$$

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\big(\psi(s)\big)^2\right]ds$$

$$\le\ R_2(t_2)\,\psi(t_2)-R_2(t)\,\psi(t)-\int_{t}^{t_2}\psi(s)R_2{}'(s)\,ds\ ,$$

using (4.2.23) which is $-R(t)\,\psi(t)\le 1$ in the last inequality , we have

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\,\big(\psi(s)\big)^2\right]ds$$

$$\leq\ R_2(t_2)\,\psi(t_2)+1+\int_{t_2}^{t}\psi(s)R_2{}'(s)\,ds$$

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\,\big(\psi(s)\big)^2\right]ds\ -\int_{t_2}^{t}\psi(s)R_2{}'(s)\,ds$$

$$\leq\ R_2(t_2)\,\psi(t_2)+1$$

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\frac{R_2(s)}{r_2(s)}\,\big(\psi(s)\big)^2\right]ds\ -\int_{t_2}^{t}\psi(s)R_2{}'(s)\,ds$$

$$\leq\ R_2(t_2)\,\psi(t_2)+1$$

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\left(\frac{R_2(s)}{r_2(s)}\,\big(\psi(s)\big)^2-R_2{}'(s)\psi(s)\right)\right]ds$$

$$\leq\ R_2(t_2)\,\psi(t_2)+1\quad .$$

Using $R_2(s)=\int_{s}^{\infty}\frac{1}{r_2(s)}\,ds\ \Rightarrow\ R_2{}'(s)=\ -\frac{1}{r_2(s)}$,

and substituting it in the previous inequality , we get

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv+\left(\frac{R_2(s)}{r_2(s)}\,\big(\psi(s)\big)^2-\frac{1}{r_2(s)}\psi(s)\right)\right]ds$$

$$\leq\ R_2(t_2)\,\psi(t_2)+1\,. \tag{4.2.33}$$

Applying Lemma 1.1.1 with $u=\ \frac{R_2(s)}{r_2(s)}$, $v=\ \frac{1}{r_2(s)}$ and $x=\ \psi(s)$, we have

$$\frac{R_2(s)}{r_2(s)}\left(\psi(s)\right)^2-\frac{1}{r_2(s)}\psi(s)\ \le\ -\ \frac{1}{4}\,\frac{\left[\frac{1}{r_2(s)}\right]^2}{\left[\frac{R_2(s)}{r_2(s)}\right]}\ ,$$

hence

$$\frac{R_2(s)}{r_2(s)}\left(\psi(s)\right)^2-\frac{1}{r_2(s)}\psi(s)\ \le\ -\ \frac{1}{4\,R_2(s)\,r_2(s)}\ . \tag{4.2.34}$$

Putting (4.2.34) in (4.2.33) and we have

$$\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv\ -\ \frac{1}{4\,R_2(s)\,r_2(s)}\right]ds$$

$$\le\ R_2(t_2)\,\psi(t_2)+1\ . \tag{4.2.35}$$

Take $\lim\limits_{t\to\infty}\,sup$ to (4.2.35) , we get

$$\lim_{t\to\infty}\,sup\int_{t_2}^{t}\left[R_2(s)\,q(s)\big(1-p(s-\sigma)\big)\int_{t_1}^{s-\sigma}\frac{1}{r_1(v)}\,dv\ -\frac{1}{4\,R_2(s)\,r_2(s)}\right]ds\ <\infty,$$

and this is a contradiction to (4.2.18) , hence equation $(1N1-A)$ is almost oscillatory . And this completes the proof . ■

Theorem 4.2.3:

Assume that (4.1.3) and $(H1)-(H3)$ are holds , if for some function $\rho(t)\in C^1\big([t_0\,,\infty\,)\ ,(\,0\,,\infty\,)\big)$ for all sufficiently large $t_1>t_0$ and for $t_3>t_2>t_1$ where $\rho(t)\ >0$, $\rho'(t)\ge 0$, one has equations (4.2.1) , (4.2.2) and (4.2.18) . If

$$\lim_{t \to \infty} \sup \int_{t_1}^{t} \left(\frac{1}{r_1(v)} \int_{t_1}^{v} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)} p(s-\sigma) \right) R_1(s-\sigma)\, ds \right] du \right) dv = \infty$$

(4.2.36)

or $$\int_{t_1}^{\infty} \frac{1}{r_1(v)} \int_{t_1}^{v} \frac{1}{r_2(u)} \int_{t_1}^{u} f(s) ds\ du\, dv = \infty \qquad (4.2.37)$$

where $R_1(t) = \int_t^{\infty} \frac{1}{r_1(s)}\, ds$, $R_2(t) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds$, $0 \le \tau \le \mathrm{t}$, $0 \le \sigma \le t$,

σ , τ constants and $\lim_{t \to \infty}(t-\tau) = \lim_{t \to \infty}(t-\sigma) = \infty$. Then equation $(1N1 - A)$ is almost oscillatory.

Proof:

Suppose that $y(t)$ is a positive solution and

$$z(t) = y(t) + p(t)y(t-\tau) . \qquad (4.2.38)$$

There are four possible cases (I) , (II) , (III) as in Theorem 4.2.2 , and

(IV) $z(t) > 0$, $z'(t) < 0$, $\left(r_1(t)z'(t) \right)' < 0$, $\left[r_2(t) \left(r_1(t)z'(t) \right)' \right]' \le 0.$

Assume that case (I) , case (II) and *case* (III) hold , respectively. We can obtain the conclusion of Theorem 4.2.3 by applying the proof of Theorem 4.2.2 .

Assume case (IV) holds . Since $\left[r_2(t) \left(r_1(t)z'(t) \right)' \right]' < 0$ and $r_2(t) > 0$, the function $r_2(t)\left(r_1(t)z'(t) \right)'$ is negative continuous decreasing .

So for $s \ge t \ge t_1 \ge 0$, we have

$$r_2(s)\big(r_1(s)z'(s)\big)' \le r_2(t)\big(r_1(t)z'(t)\big)' ,$$

$r_1(t)\, z'(t)$ is a negative decreasing function, so when

$$s \ge t > 0 \implies r_1(s)\, z'(s) \le r_1(t) z'(t)$$

or

$$-r_1(s)\, z'(s) \ge -r_1(t) z'(t) \ge L \,, \text{ where } L > 0 ,$$

hence

$$-z'(s) \ge -r_1(t) z'(t) \frac{1}{r_1(s)} \ge L \frac{1}{r_1(s)} \,.$$

Integrating the previous inequality from t to ∞, we get

$$-\int_t^\infty z'(s)\, ds \ge -\int_t^\infty r_1(t) z'(t) \frac{1}{r_1(s)}\, ds \ge \int_t^\infty L \frac{1}{r_1(s)}\, ds$$

$$-\left(\lim_{\lambda\to\infty} z(\lambda) - z(t)\right) \ge -r_1(t) z'(t) \int_t^\infty \frac{1}{r_1(s)} ds \ge L \int_t^\infty \frac{1}{r_1(s)} ds$$

$$\implies z(t) \ge \lim_{\lambda\to\infty} z(\lambda) - r_1(t) z'(t) \int_t^\infty \frac{1}{r_1(s)} ds \ge \lim_{\lambda\to\infty} z(\lambda) + L \int_t^\infty \frac{1}{r_1(s)} ds \,.$$

Using $z(t)$ positive decreasing function and $R_1(t) = \int_t^\infty \frac{1}{r_1(s)}\, ds$, so

$$z(t) \ge -r_1(t)\, z'(t)\, R_1(t) \ge L\, R_1(t) > 0 \,, \tag{4.2.39}$$

hence

$$r_1(t)\, z'(t)\, R_1(t) + z(t) \ge 0 \,,$$

dividing by the positive quantity $r_1(t)\, {R_1}^2(t)$ and using ${R_1}'(t) = \dfrac{-1}{r_1(t)}$,

we get

$$\frac{r_1(t)\, z'(t)\, R_1(t) - (-z(t)\,)}{r_1(t)\, {R_1}^2(t)} \geq 0\,,$$

hence

$$\left(\frac{z(t)}{R_1(t)}\right)' \geq 0 \quad , \tag{4.2.40}$$

hence $\dfrac{z(t)}{R_1(t)}$ is a positive increasing function.

Using $z(t)$ is positive decreasing function and the previous result implies for

$$t \geq t - \tau > 0\,,$$

we have

$$\frac{z(t)}{R_1(t)} \geq \frac{z(t-\tau)}{R_1(t-\tau)} > 0 \implies \frac{R_1(t-\tau)}{R_1(t)} \geq \frac{z(t-\tau)}{z(t)} \geq 1. \tag{4.2.41}$$

Using equation (4.2.38) , we have

$$y(t) = z(t) - p(t)y(t-\tau) \ .$$

And using $z(t) \geq y(t) \implies z(t-\tau) \geq y(t-\tau)$,

hence

$$y(t) \geq z(t) - p(t)\, z(t-\tau)$$

$$y(t) \geq \left(1 - p(t)\, \frac{z(t-\tau)}{z(t)}\right) z(t)\,,$$

using (4.2.40) in the previous inequality gives

$$y(t) \geq \left(1 - p(t) \frac{R_1(t-\tau)}{R_1(t)} \right) z(t) \quad , \tag{4.2.42}$$

hence

$$y(t-\sigma) \geq \left(1 - p(t-\sigma) \frac{R_1((t-\sigma)-\tau))}{R_1(t-\sigma))} \right) z(t-\sigma) \quad . \tag{4.2.43}$$

Using (4.2.42) and $f(t) > 0$ we have

$$-f(t) y(t-\sigma) \leq -f(t) \left(1 - p(t-\sigma) \frac{R_1(t-\tau)}{R_1(t)} \right) z(t) \; . \tag{4.2.44}$$

From (4.2.44) and the equation

$$\left[r_2(t) \big(r_1(t) \, z'(t) \big)' \right]' = -f(t) \, y(t-\sigma) \;\; \text{and} \;\; z(t) < z(t-\sigma) \quad ,$$

we have

$$\left[r_2(t) \big(r_1(t) \, z'(t) \big)' \right]' \leq -f(t) \left(1 - \; p(t-\sigma) \frac{R_1\big((t-\sigma)-\tau\big)}{R_1(t-\sigma)} \right) z(t-\sigma) \; . \tag{4.2.45}$$

Using (4.2.39) we have

$$z(t-\sigma) \geq \; L \;\; R(t-\sigma) \; > 0 \;\; ,$$

and putting it in (4.2.45) , we get

$$\left[r_2(t) \big(r_1(t) \, z'(t) \big)' \right]' \leq -f(t) \left(1 - \; p(t-\sigma) \frac{R_1((t-\sigma)-\tau))}{R_1(t-\sigma)} \right) L \;\; R_1(t-\sigma)$$

$$\left[r_2(t) \big(r_1(t) \, z'(t) \big)' \right]' + L \, f(t) \left(1 - \; p(t-\sigma) \frac{R_1((t-\sigma)-\tau)}{R_1(t-\sigma)} \right) R_1(s-\sigma) \; \leq 0 \tag{4.2.46}$$

Integrating (4.2.46) from t_1 to t and using $f(t) \geq \mathrm{q(t)}$, we have

$$\int_{t_1}^{t} \left[\, r_2(s)\big(\, r_1(s)\, z'(s)\big)'\right]' ds + L \int_{t_1}^{t} q(s) \left(1 - \, p(s-\sigma) \frac{R_1\big((s-\sigma)-\tau\big)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds \ \le 0$$

$$r_2(t)\big(\, r_1(t)\, z'(t)\big)' - r_2(t_1\,)\big(\, r_1(t_1\,)\, z'(t_1\,)\big)'$$

$$+L \int_{t_1}^{t} q(s) \left(1 - \, p(s-\sigma) \frac{R_1\big((s-\sigma)-\tau\big)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds \quad \le \ 0\,,$$

$$r_2(s)\big(\, r_1(s)\, z'(s)\big)' + L \int_{t_1}^{t} q(s) \left(1 - \, p(s-\sigma) \frac{R_1\big((s-\sigma)-\tau\big)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds$$

$$\le r_2(t_1\,)\big(\, r_1(t_1\,)\, z'(t_1\,)\big)' < 0 \quad ,$$

hence

$$r_2(t)\big(\, r_1(t)\, z'(t)\big)' + L \int_{t_1}^{t} q(s) \left(1 - \, p(s-\sigma) \frac{R_1\big((s-\sigma)-\tau\big)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\ ds \le 0 \ \ ,$$

dividing the previous inequality by $r_2(t) > 0$, we have

$$\big(\, r_1(t)\, z'(t)\big)' + \frac{L}{r_2(t)} \int_{t_1}^{t} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds \ \le 0 \ .$$

Integrating the previous inequality again from t_1 to t

$$\int_{t_1}^{t} \big(\, r_1(s)\, z'(s)\big)' ds + L \int_{t_1}^{t} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - \, p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds \right] du \le 0$$

$$r_1(t)\, z'(t) - r_1(\,t_1)\, z'(\,t_1) + L \int_{t_1}^{t} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds \right] du \le 0$$

$$r_1(t)\, z'(t) + L \int_{t_1}^{t} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)}\right) R_1(s-\sigma)\, ds \right] du \le r_1(\,t_1)\, z'(\,t_1) < 0\,,$$

hence

$$r_1(t)\, z'(t) + L \int_{t_1}^{t} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)} \right) R_1(s-\sigma)\ ds \right] du < 0\ .$$

Dividing the previous inequality by $r_1(t) > 0$ gives

$$z'(t) + \frac{L}{r_1(t)} \int_{t_1}^{t} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)} \right) R_1(s-\sigma)\ ds \right] du < 0\, .$$

Integrating the previous inequality from t_1 to t , we get

$$\int_{t_1}^{t} z'(v)\, dv + L \int_{t_1}^{t} \left(\frac{1}{r_1(v)} \int_{t_1}^{v} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)} \right) R_1(s-\sigma)\, ds \right] du \right) dv < 0$$

$$z(t) - z(t_1) + L \int_{t_1}^{t} \left(\frac{1}{r_1(v)} \int_{t_1}^{v} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1((s-\sigma)-\tau)}{R_1(s-\sigma)} \right) R(s-\sigma)\, ds \right] du \right) dv < 0$$

$$\int_{t_1}^{t} \left(\frac{1}{r_1(v)} \int_{t_1}^{v} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1\big((s-\sigma)-\tau\big)}{R_1(s-\sigma)} \right) R_1(s-\sigma)\, ds \right] du \right) dv$$

$$< \frac{z(t_1) - z(t)}{L} \le \frac{z(t_1)}{L}\ ,\ \ L > 0\ .$$

Taking the $\lim\limits_{t \to \infty} sup$

$$\lim_{t \to \infty} sup \int_{t_1}^{t} \left(\frac{1}{r_1(v)} \int_{t_1}^{v} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - p(s-\sigma) \frac{R_1\big((s-\sigma)-\tau\big)}{R_1(s-\sigma)} \right) R_1(s-\sigma)\, ds \right] du \right) dv$$

$$< \lim_{t \to \infty} sup \frac{z(t_1)}{L} < \infty\, ,$$

which is a contradiction to (4.2.36) , so L must be zero .

Return to inequality (2.2.4) , we have

$$y(t) > m\, z(t)$$

so

$$y(s-\sigma) \geq m\, z(s-\sigma) \ ,$$

multiply this inequality by $f(t) > 0$, we have

$$-f(t) y(s-\sigma) \leq -m\, f(t)\, z(s-\sigma) \ ,$$

hence

$$\left[r_2(t) \big(r_1(t)\, z'(t) \big)' \right]' \leq -m\, f(t)\, z(s-\sigma) \ ,$$

multiplying by -1 and integrating the previous inequality from t_1 to t

$$-\int_{t_1}^{t} \left[r_2(s) \big(r_1(s)\, z'(s) \big)' \right]' ds \geq \int_{t_1}^{t} m\, f(s)\, z(s-\sigma) ds \quad , \text{ where} \quad t > t_1 > 0 \ ,$$

so

$$- r_2(s) \big(r_1(s)\, z'(s) \big)' \Big]_{t_1}^{t} \geq \int_{t}^{t} m\, f(s)\, z(s-\sigma) ds$$

$$-\Big(r_2(t) \big(r_1(t)\, z'(t) \big)' - r_2(t_1) \big(r_1(t_1)\, z'(t_1) \big)' \Big) \geq \int_{t_1}^{t} m\, f(s)\, z(s-\sigma) ds \ ,$$

but from a negative continuous decreasing function , we have

$$r_2(t) \big(r_1(t)\, z'(t) \big)' \leq r_2(t_1) \big(r_1(t_1)\, z'(t_1) \big)'$$

$$\Rightarrow -\Big(r_2(t) \big(r_1(t)\, z'(t) \big)' - r_2(t_1) \big(r_1(t_1)\, z'(t_1) \big)' \Big) \leq -r_2(t) \big(r_1(t)\, z'(t) \big)'$$

$$- r_2(t)\big(r_1(t)\, z'(t)\big)' \;\geq\; \int_{t_1}^{t} m\, f(s)\, z(s-\sigma) ds \,,$$

but from inequality (2.2.2) , we have

$$z(s-\sigma) \geq k \;\;,$$

hence

$$-r_2(t)\big(r_1(t)\, z'(t)\big)' \;\geq\; \int_{t_1}^{t} m\;\, k\, f(s)\, ds \;\;,$$

dividing by $r_1(t) > 0$, we get

$$- \big(r_1(t)\, z'(t)\big)' \;\geq\; \frac{m\, k}{r_2(t)} \int_{t_1}^{t} f(s)\, ds \;\;,$$

integrating this inequality from t_1 to t , we get

$$- \int_{t_1}^{t} \big(r_1(u)\, z'(u)\big)' \; du \;\geq\; \int_{t_1}^{t} \frac{m\, k}{r_2(u)} \left[\int_{u}^{u} f(s)\;\, ds \right] du$$

$$- \big(r_1(t) z'(t) - \; r_1(\, t_1) z'(\, t_1)\big) \geq \;\; m\, k \int_{t_1}^{t} \frac{1}{r_2(u)} \left[\int_{t_1}^{u} f\,(s)\, ds \right] du \;\; .$$

Since $r_1(u) z'(u)$ is negative continuous decreasing function , we have

$$r_1(t) z'(t) \leq \; r_1(\, t_1) z'(\, t_1) < 0 \,,$$

hence

$$0 > r_1(\, t_1) z'(\, t_1) \;\geq\; r_1(t) z'(t) + \;\; m\, k \int_{t_1}^{t} \frac{1}{r_2(u)} \left[\int_{t_1}^{u} f\,(s)\, ds \right] du \;\;,$$

$$- r_1(t)z'(t) \geq m\,k \int_{t_1}^{t} \frac{1}{r_2(u)} \left[\int_{t_1}^{u} f\,(s)\,ds \right] du \ ,$$

dividing by $r_1(t) > 0$, we get

$$\frac{-\; r_1(t)z'(t)}{r_1(t)} \geq \frac{m\,k}{r_1(t)} \int_{t_1}^{t} \frac{1}{r_2(u)} \left[\int_{t_1}^{u} f\,(s)\,ds \right] du$$

$$- z'(t) \geq \frac{m\,k}{r_1(t)} \int_{t_1}^{t} \frac{1}{r_2(u)} \left[\int_{t_1}^{u} f\,(s)\,ds \right] du$$

Integrating the previous inequality from t_1 to t , we get

$$- \int_{t_1}^{t} z'(v)\,dv \ \geq \ m\,k \int_{t_1}^{t} \left[\frac{1}{r_1(v)} \int_{t_1}^{v} \left(\frac{1}{r_2(u)} \int_{t_1}^{u} f\,(s)\,ds \right) du \right] dv \ ,$$

so

$$-\big(z(t) - z(t_1)\big) \geq m\,k \int_{t_1}^{t} \left[\frac{1}{r_1(v)} \int_{t_1}^{v} \left(\frac{1}{r_2(u)} \int_{t_1}^{u} f\,(s)\,ds \right) du \right] dv \ ,$$

hence

$$m\,k \int_{t_1}^{t} \left[\frac{1}{r_1(v)} \int_{t_1}^{v} \left(\frac{1}{r_2(u)} \int_{t_1}^{u} f\,(s)\,ds \right) du \right] dv \leq -\big(z(t) - z(t_1)\big) \ .$$

But $z(t) > 0$, so

$$m\,k \int_{t_1}^{t} \left[\frac{1}{r_1(v)} \int_{t_1}^{v} \left(\frac{1}{r_2(u)} \int_{t_1}^{u} f\,(s)\,ds \right) du \right] dv \leq z(t_1)$$

Thus ,

$$\int_{t_1}^{t}\left[\frac{1}{r_1(v)}\int_{t_1}^{v}\left(\frac{1}{r_2(u)}\int_{t_1}^{u} f(s)\,ds\right)du\right]dv \leq \frac{z(t_1)}{m\,k} < \infty\ .$$

This contradiction (4.2.37). Thus, we must have $k = 0$.

So every solution of equation $(1N1 - A)$ is oscillatory. The proof is complete. ■

4.3 Oscillation of Solution Conditions of the Equation $(\mathbf{1N1 - B})$

In this section we will introduce four theorems that generalize Theorem 4. 2. 1, Theorem 4. 2. 2 and Theorem 4. 2. 3.

Theorem 4. 3. 1 :

Assume that (4.1.1) and $(H1) - (H3)$ hold, if for some function $\rho(t) \in C^1([t_0, \infty), (0, \infty))$ for all sufficiently large $t_1 > t_0$ and for $t_3 > t_2 > t_1$ where $\rho(t) > 0$, $\rho'(t) \geq 0$, one has

$$\lim_{t\to\infty} \sup \int_{t_3}^{t}\left[\rho(s)q(s)(1-p(\sigma(s)))\ \frac{\int_{t_2}^{\sigma(s)}\left[\int_{t_1}^{v}\frac{1}{r_2(u)}du\ \frac{1}{r_1(v)}\right]dv}{\int_{t_1}^{s}\frac{1}{r_2(u)}du} - \frac{r_2(s)(\rho'(s))^2}{4\rho(s)}\right]ds = \infty \tag{4.3.1}$$

$$\int_{t_0}^{\infty}\frac{1}{r_1(v)}\int_{v}^{\infty}\frac{1}{r_2(u)}\int_{u}^{\infty} f(s)ds\ du\,dv = \infty\ , \tag{4.3.2}$$

where $\tau(t)$, $\sigma(t) \in C([t_0, \infty))$, $0 \leq \tau(t) \leq \mathrm{t}$, $0 \leq \sigma(t) \leq t$ and $\lim_{t\to\infty}(\tau(t)) = \lim_{t\to\infty}(\sigma(t)) = \infty$. Then $(1N1 - B)$ is almost oscillatory.

Theorem 4.3.2 :

Assume that (4.1.2) and $(H1)-(H4)$ hold, if for some function

$\rho(t) \in C^1([t_0, \infty), (0, \infty))$, for all sufficiently large $t_1 > t_0$ and for $t_3 > t_2 > t_1$

where $\rho(t) > 0$, $\rho'(t) \geq 0$ one has equations (4.3.1) and (4.3.2). If

$$\lim_{t\to\infty} \sup \int_{t_2}^{t} \left(R_2(s)q(s)(1-p(\sigma(s)) \int_{t_1}^{\sigma(s)} \frac{dv}{r_1(v)} - \frac{1}{4R_2(s)r_1(s)} \right) ds = \infty, \qquad (4.3.3)$$

where $R_2(t) := \int_t^\infty \frac{1}{r_2(s)} ds$, $\tau(t)$, $\sigma(t) \in C([t_0, \infty))$, $0 \leq \tau(t) \leq \mathrm{t}$,

$0 \leq \sigma(t) \leq t$, and $\lim_{t\to\infty}(\tau(t)) = \lim_{t\to\infty}(\sigma(t)) = \infty$. Then equation $(1N1-B)$ is almost oscillatory.

Theorem 4.3.3 :

Assume that (4.1.3) and $(H1)-(H3)$ hold, if for some function

$\rho(t) \in C^1([t_0, \infty), (0, \infty))$ for all sufficiently large $t_1 > t_0$ and for $t_3 > t_2 > t_1$

where $\rho(t) > 0$, $\rho'(t) \geq 0$, one has equations (4.3.1), (4.3.2) and (4.3.18). If

$$\lim_{t\to\infty} \sup \int_{t_1}^{t} \left(\frac{1}{r_1(v)} \int_{t_1}^{v} \left[\frac{1}{r_2(u)} \int_{t_1}^{u} q(s) \left(1 - \frac{R_1(\tau(\sigma))}{R_1(\sigma(s))} p(\sigma(s)) \right) R(\sigma(s)\, ds \right] du \right) dv = \infty$$

(4.3.4)

where $R_1(t) := \int_t^\infty \frac{1}{r_1(s)} ds$, $R_2(t) := \int_t^\infty \frac{1}{r_2(s)} ds$, $\tau(t)$, $\sigma(t) \in C([t_0, \infty))$,

$0 \leq \tau(t) \leq \mathrm{t}$, $0 \leq \sigma(t) \leq t$, and $\lim_{t\to\infty}(\tau(t)) = \lim_{t\to\infty}(\sigma(t)) = \infty$. Then equation

$(1N1-B)$ is almost oscillatory.

Theorem 4.3.4 :

Assume that (4.1.3) and $(H1)-(H3)$ hold, if for some function $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ for all sufficiently large $t_1 \geq t_0$ and for $t_3 > t_2 > t_1$ where $\rho(t) > 0$, $\rho'(t) \geq 0$, one has equations (4.3.1), (4.3.2) and (4.3.3). If

$$\int_{t_1}^{\infty} \frac{1}{r_1(v)} \int_{t_1}^{v} \frac{1}{r_2(u)} \int_{t_1}^{u} f(s)ds\ du\ dv = \infty \tag{4.3.5}$$

where $R_1(t) := \int_t^{\infty} \frac{1}{r_1(s)}\, ds$, $R_2(t) := \int_t^{\infty} \frac{1}{r_2(s)}\, ds$, $\tau(t)$, $\sigma(t) \in C([t_0,\infty))$,

$0 \leq \tau(t) \leq t$, $0 \leq \sigma(t) \leq t$, and $\lim\limits_{t\to\infty} \tau(t) = \lim\limits_{t\to\infty} \sigma(t) = \infty$. Then equation $(1N1-B)$ is almost oscillatory.

Theorem 4.3.1 and Theorem 4.3.2 have the same as the proofs of Theorem 4.2.1 and Theorem 4.2.2, respectively. Theorem 4.3.3 and Theorem 4.3.4 have the same as the proof of Theorem 4.2.3, just replace $\tau(s)$, $\sigma(s)$ and $\tau(\sigma(s))$ instead of $s-\tau$, $s-\sigma$ and $(s-\sigma)-\tau$, respectively, in the proofs. For more details to the proofs see [30].

4.4 Illustrating Examples :

Here are some illustrative examples.

Example 4.3.1:

Consider the third order neutral delay differential

$$\frac{d}{dt}\left(t\frac{d}{dt}\left(\frac{1}{\sqrt{t}}\frac{d}{dt}\left(y(t)+\frac{1}{2}\,y(t-3\pi)\right)\right)\right)+\frac{1}{2}\left(\sqrt{t}+\frac{1}{4\sqrt{t^3}}\right)y\left(t-\frac{3\pi}{2}\right)=0\,, \qquad (4.3.6)$$

where $t \geq 1$.

Let $\rho(t) = 1$. It is clear that $r_1(t) = \dfrac{1}{\sqrt{t}}$, $r_2(t) = t$, $p(t) = \dfrac{1}{2}$,

$f(t) = \dfrac{1}{2}\left(\sqrt{t}+\dfrac{1}{4\sqrt{t^3}}\right)$, $\tau = 3\pi$, and $\sigma = \dfrac{3\pi}{2}$. Applying Theorem 4.2.1 ,

then every solution y(t) of (4.3.6) is oscillatory or converge to zero . One such

solution is $y(t) = cos\, t$.

Example 4.3.2 :

Consider the third order neutral delay differential equation

$$\frac{d}{dt}\left(t^2\,\frac{d}{dt}\left(t^2\,\frac{d}{dt}\left(y(t)+\frac{1}{3}\,y\left(\frac{t}{2}\right)\right)\right)\right)+\lambda\, t^2\, y\,(t)=0 \qquad (4.3.7)$$

where $\lambda > 0$, $t \geq 1$.

$r_2(t) = t^2$, $r_1(t) = t^2$, $\tau(\,t) = \dfrac{t}{2} \leq \mathrm{t}$, $\sigma\,(t) = t \leq t$, $p(t) = \dfrac{1}{3} \in [\,0\,,1\,)$

and $f(t) = \lambda\, t^2$.

Let $\rho(t) = 1$ and applying Theorem 4.3.3 , then every solution $y(t)$ of

(4.3.7) is oscillatory or $\lim\limits_{t\,\to\,\infty} y(t) = 0$.

4.5 Remarks :

Remark 4.5.1 :

It is interesting to study equations $(2N1 - A)$ and $(2N1 - B)$ for the case when

$$R_1(t) = \int_t^{\infty} \frac{1}{r_1(s)}\, ds < \infty \quad , \quad R_2(s) = \int_t^{\infty} \frac{1}{r_2(s)}\, ds = \infty \, .$$

Remark 4.5.2 :

It is interesting to find other conditions which guarantee that every solution of equations $(1N1 - A)$, $(1N1 - B)$, $(2N1 - A)$ and $(2N1 - B)$, $(1N2 - A)$ and $(1N2 - B)$ is oscillatory .

References :

[1] B. Baculikova, J. Dzurina, " Oscillation of third-order neutral differential equations", J Math. and Comp. Modeling 52(2010), pp. 215-226..

[2] Baker, C.T.H., Paul, C.A.H., and Willé , D.R., Issues in the numerical solution of evolutionary delay differential equations, Adv. Comput. Math., 3:171-196, 1995.

[3] Cemil Tunc, On the nonoscillation of solutions of some nonlinear differential equations of third order , Dynamics and systems theory , 7(4), (2007), 419-430

[4] Ch. G. Philo; Oscillation theorems for linear differential equation of second order, Arch Math., 53(1989), 482–492.

[5] D. G. Saari, Dynamical systems and mathematical economics, in “Models of Economic Dynamics” (H. F. Sonnenschein, Ed.), pp. l-24, Lecture Notes in Economics and Mathematical Systems, Vol. 264, Springer-Verlag, Berlin/New York, 1986.

[6] G. Ladas, Y. G. Sficas; Oscillation of higher-order neutral delay differential equations; J Austral. Math. Soc. Ser. B 27, PP. 502-511, 1986.

[7] G. S. Ldde, V. Lakshmikantham, B. G. Zhang; Oscillation Theory of Differential Equation with Deviating Arguments, Marcel Dekker, Inc. New York, 1987.

[8] I. Geory. Ladas; Oscillation Theory of Delay Differential Equations with Applications, Clarendan Press, Oxford, 1991.

[9] I. Gyori, Oscillation and comparison results in neutral differential equations with Application to delay logistic equation, in " Proceedings of the Conference on Mathematical Problems in Population Dynamics", Oxford, Mississippi, 1986.

[10] I. T. Kiguradze , T. A. Chaturia; Asymptotic properties of solution of nonautonomous ordinary differential equations, Kluwer Acad. Publ. ,Dordrecht, 1993.

[11] J. Dzurina and I. P. Stavroulakis, Oscillation criteria for second order delay differential equations, Appl. Math. Comput., 140(2003), 445–453.

[12] J.K. Hale , S.M. Verduynl Unel, Introduction to Functional Differential Equations, Springer, New York, 1993.

[13] K.V.V Seshagiri Rao and P.V.H. S. Sai Kumar, "On the solution of Third Order Linear neutral delay differential equations ", Int. Journal of Mathematics Vol. 4 , Iss. 11, Dec. 2013, pp. 357-368

[14] K.V.V Seshagiri Rao and P.V.H.S.Sai Kumar, "Existence of positive solutions of a class of neutral delay differential equations of second order", Int. Journal of Math. Analysis, Vol.3, 2009,no. 28, 1399-1404.

[15] L. Erbe, Oscillation & Nonoscillation and Asymptotic behaviour for third order nonlinear differential equations, Annal di Mathematica Pure ed Applicata , Vol. 110, pp. 373-391.

[16] L. H. Erbe, Q. Kong, B. G. Zhang; Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.

[17] M. C. Mackey;" Commodity Price Fluctuations: Price Dependent Delays and Nonlinearities as Explanatory Factors", J. Economic Theory 48, Canada (1989),pp. 497-509.

[18] M. Gregus; Third Order Linear Differential Equations, Reidel, Dordrecht, 1985.

[19] M. K. Gramatikopoulos, E. A. Grove and G. Ladas; "Oscillations of first order neutral delay differential equations". J. (to appear).

[20] M. K. Gramatikopoulos, G. Ladas , and A. Meimaridou;"Oscillations of second order neutral delay differential equations", Rad. Mat. 1(1985), 267-274.

[21] M. R. S. Kulenovic and G. Ladas, Linearized Oscillations in Population Dynamics Bull Math. Biol. 49 (1987) , 615-617.

[22] M. Slemrod and E. F. Infante, "Asymptotic stability criteria for linear systems of difference-differential equations on neutral type and their discrete analogues", J. Math. Anal. Appl. 38(1972), 399-415.

[23] P. M. Nisbet, and W. S. C. Gurney Modelling Fluctuating Populations, " Wiley, New York , 1982.

[24] Paul Waltman, Oscillation criteria for third order nonlinear differential equations,Pacific Journal of Mathematics, Vol. 18, No: 2, (1966), 385- 389.

[25] R. D. Driver, "A mixed neutral system", Nonlinear Anal.-TMA 8 (1984), 155-158.

[26] R. D. Driver, "Existence and continuous dependence of solutions of a neutral functional Differential equation", Arch. Rational Mech. Anal. 19 (1965), 149-166.

[27] R. Franke ," A Prototype Model of Speculative Dynamics With Position-Based Trading"; Kiel, Germany, December 2007.

[28] R. K. Brayton and R. A. Willoughby, " On the numerical integration of a symmetric system of difference -differential equations of neutral type", J. Math. Anal. Appl. 18 (1967), 182-189.

[29] R. P. Agarwal, S. L. Shieh and C. C. Yeh, Oscillation criteria for second order retarded differential equations, Math. Comput. Model., 26(1997), 1–11.

[30] Tongxing Li, Chenghui Zhang, and Guojing Xing," Oscillation of Third-Order Neutral delay Differential equations",Hindawi Publishing Corporation Abstract and Applied Analysis Volume 2012, Article ID 569201, 11 pages doi:10.1155/2012/569201.

[31] W.E.Boyce ,R.C.Diprima,Elementary differential Equations and Boundary Value Problems, 2nd edition ,John Willy and Sons, New York ,New York 10158,ISBN:0-471-83824-1, 1986.

[32] Y. Zhang , Y. Wang; A new oscillation criterion for first order neutral delay differential equation, Ann. of Diff. Eqs. 22 (3) (2006), 473–476.

الملخص

من المعروف ، إن المعادلات التفاضلية لها صور وأشكال مختلفة ، وكل شكل من أشكالها له تطبيقاته العملية الخاصة به وطرق حله الخاصة به ، ومن هذه الأنواع المعادلة التفاضلية الخطية المتعادلة . وهذا النوع من المعادلات حله يكون على إحدى الصور التالية :

إما متذبذب (أي أن له عدد لا نهائي من الأصفار) ، أو شبه متذبذب (أي أن له عدد كبير من الأصفار وفي آخر المطاف يؤول إلى الصفر) ، وإما غير تذبذبي . وعندما يكون على الأقل أحد حلول هذه المعادلة غير متذبذب ، تكون المعادلة غير تذبذبية . وللحكم على المعادلة بأنها تذبذبية أو لا، هناك شروط إذا توافرت فيها يكون بالإمكان الحكم عليها بأنها معادلة تذبذبية أو غير تذبذبية .

وفي هذه الرسالة تم دراسة التذبذب في الحلول الحقيقية $y(t)$ ذات الأهمية للمعادلة التفاضلية التي على الصورة

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\big(y(t)+p(t)y(t-\tau)\big)\right)\right)+f(t)y(t-\sigma)=0 \qquad (1N1-A)$$

$$\frac{d}{dt}\left(r_2(t)\,\frac{d}{dt}\left(r_1(t)\,\frac{d}{dt}\big(y(t)+p(t)y(\tau(t))\big)\right)\right)+f(t)y(\sigma(t))=0 \qquad (1N1-B)$$

$$\frac{d}{dt}\left(r(t)\,\frac{d^2}{dt^2}\Big(\big(y(t)+p(t)y(\tau(t))\big)\Big)\right)+f(t)y(\sigma(t))=0 \qquad (1N2-B)$$

$$\frac{d^2}{dt^2}\left(r(t)\,\frac{d}{dt}\big(y(t)+p(t)y(t-\tau)\big)\right)+f(t)y(t-\sigma)=0 \qquad (2N\,1-A)$$

$$\frac{d^2}{dt^2}\left(r(t)\,\frac{d}{dt}\Big(y(t)+p(t)y\big(\tau(t)\big)\Big)\right)+f(t)y\big(\sigma(t)\big)=0 \qquad (2N1-B)$$

حيث

$$p(t)\,,f(t)\in C\,([t_0\,,\infty),\mathbb{R}\,)\,,f(t)\geq 0\,,r_1(t)\,,r_2(t),r(t)\in C^1\,([t_0\,,\infty)\,,\,\mathbb{R}^+\,)$$

$$\tau\ ,\ \sigma\in[0\,,t\,)\,.$$

وكان هدف هذه الرسالة هو اختبار شروط التذبذب لحلول المعادلات التفاضلية $(1N1-A)$

$(1N2-B)$ ، $(1N2-A)$ ، $(2N1-B)$ ، $(2N1-A)$ ، $(1N1-B)$

أو التذبذب والتقارب من الصفر من خلال اتباع نهج مماثل لخطوات العالم

"K. V. V. Seshagiri Rao" في بحثه للمعادلة التفاضلية الخطية المتعادلة ذات الصورة

$$\frac{d}{dt}\left(r(t)\,\frac{d^2}{dt^2}\left(\left(y(t)+p(t)y(t-\tau)\right)\right)\right)+f(t)y(t-\sigma)=0 \qquad (1N2-A)$$

وشروط تذبذب حلولها ، وأيضا من خلال اتباع نفس نهج خطوات العالم " Tongxing Li " في بحثه المستخدم في دراسة المعادلة $(1N1-A)$ وشروط تذبذب حلولها تم وضع شروط لتذبذب حلول المعادلة التفاضلية $(1N1-A)$ عندما يكون الاقتران $r_1(t)=1$. وفي سياق أدبيات هذه الرسالة شرح كافٍ لبيان ذلك . كما تضمنت هذه الرسالة برهان فرضية المقارنة المساعدة التي استخدمها العالم " K. V. V. Seshagiri Rao " في بحثه ، بالإضافة إلى بعض الأمثلة التوضيحية لمغزى الرسالة .