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On Applying Partial Differential Equations in Image Processing

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On Applying Partial Differential Equations in Image Processing

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On Applying Partial Differential Equations in Image Processing

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Dedication

To my parents, my husband, my son, and to all those who supported me in completing this research.

Sireen Nasr

Declaration

I certify that this submitted for the degree of master is the result of my own research, except where otherwise acknowledge. And that this (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: Sireen ....

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Date: 13/7/2019

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Abstract

In this thesis we present a brief review of the assumed prerequisites in signals, linear time-invariant (LTI) systems and Fourier discrete-time systems with some of its properties. We derive the convolution formula, which allows us to determine the output of an LTI system to any given arbitrary input signal. Furthermore, we discuss the use of operators that can be used to remove noise and enhance signal, and we evolve the main principles for designing noise reduction and signal enhancement filters in the frequency and time domains. We introduce the design of notch and comb filters for removing periodic interference, enhancing periodic signals, and separating the luminance and chrominance components in digital color TV systems.

A major aim to the thesis is to use geometrical and variational partial differential equations (PSEs) in the process of image processing. We first derive and relate the p-Laplacians in terms of the concepts of gauge coordinates. We also present several PDE-based image denoising and enhancement models to evolve images, motivated by combining Gaussian blurring, the Mean Curvature Motion, for denoising and edge enhancement. We introducing the qualitative behavior by deriving a solution of the equations and mention its properties briefly. We also discuss the notions of curvature motion and edge affected diffusion filtering. Further in this thesis we implement and obtain quantitative and numerical results on a real-life image, showing the predictable behaviour on both noise-constrained and unconstrained evolution, one can vary the desired amount of blurring and denoising. At the end of the thesis the optimal homotopy asymptotic method (OHAM) is applied to derive a solution of the second order Gauge coordinate equation.

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Chapter 1

Introduction

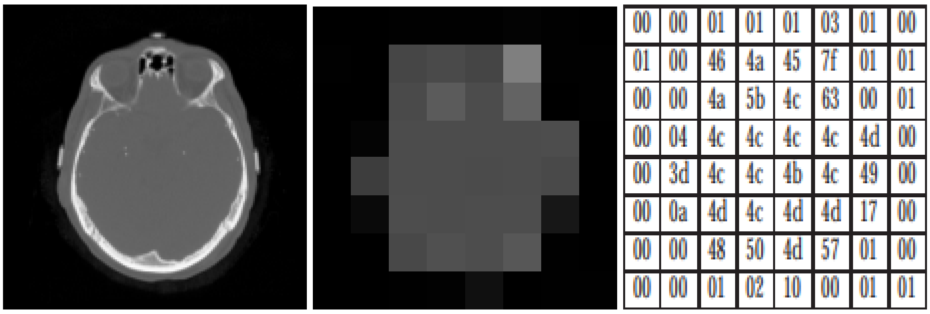
1.1 Digital Image

Images are everywhere, since we rely on the images we see with our eyes more than any other tangible motivation. Almost all of the information we realize comes to us in the form of an image. In order to admire the theory and application of the image processing, we first need to understand the digital image. A digital image comes from a continuous one. We sampling and quantize analogue images to obtain such digital images. It is converted to a discretized form by a digitizer, which performs two tasks, known as sampling and quantization. Then the result is a digital image, and it can be used in digital image processing applications.

Example 1.1.1 *Fig. (1.1) shows sampling and quantization, in the sampling process, the values of the continuous signal are sampled at specific locations in the image. In the quantization process, the real values are discretized into digital numbers.*

A digital image is a matrix of real numbers. Each matrix element, i.e., a quantized sample, is called a picture element or a pixel, and its value is the gray-level or brightness, and it characterised by matrix size, pixel depth and resolution. The matrix size is determined from the number of the columns n and the number of rows m of the image matrix $m \times n$ as shown in Fig. (1.2). Generally, as the matrix dimension increases the resolution is getting better. Pixel or bit depth refers to the number of bits per pixel that represent the colour levels of each pixel in an image.

The term resolution of the image refers to the number of pixels per unit length of the image. In digital images the spatial resolution depends on pixel size. The pixel size is calculated by the Field of View (FoV) which is a measure of the ground area viewed by a single detector element in a given instant in time- divided by the



00	00	01	01	01	03	01	00
01	00	46	4a	45	7f	01	01
00	00	4a	5b	4c	63	00	01
00	04	4c	4c	4c	4c	4d	00
00	3d	4c	4c	4b	4c	49	00
00	0a	4d	4c	4d	4d	17	00
00	00	48	50	4d	57	01	00
00	00	01	02	10	00	01	01

Figure 1.1: Example of sampling and quantization. On the left an original image with continuous intensity values . On the middle a sampling using 8×8 locations, the image has real intensity values at specific locations only. And a quantization on the right (these real values are converted to discrete numbers)[17].

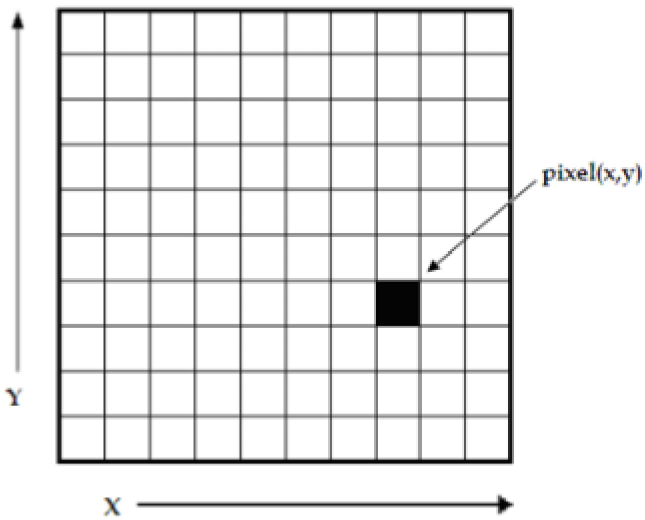


Figure 1.2: A digital image is a 2D array of pixels. Each pixel is characterised by its (x, y) coordinates and its value

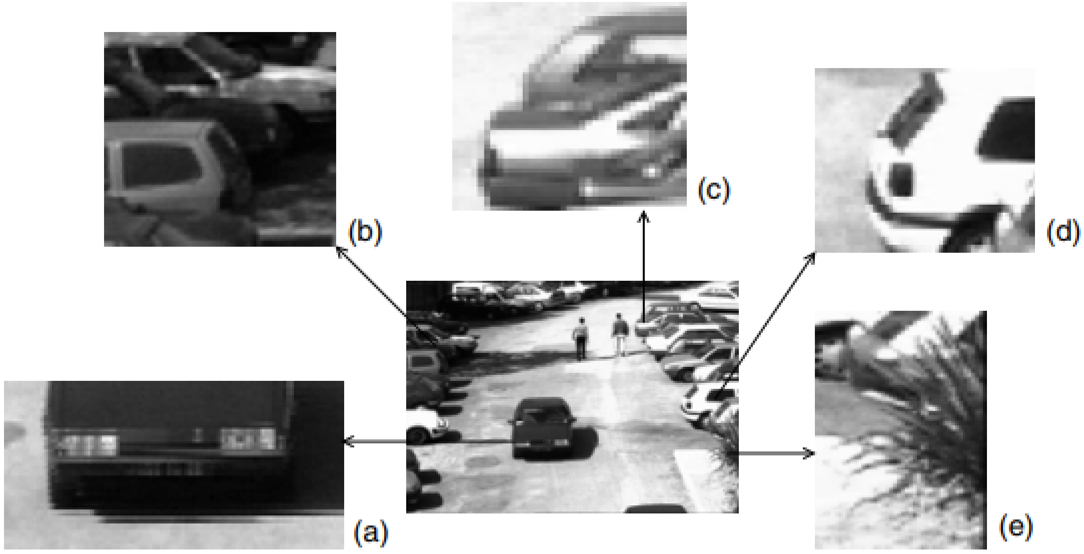


Figure 1.3: Different parts with different structures of the image [4]

number of pixels across the matrix. For a standard FoV, an increase of the matrix size decreases the pixel size and the ability to see details is improved. In a way, the higher the resolution, the closer the digital image is to the physical world.

To describe a pixel, one need several bands, as an example, a vector field has two components, a color image is described with three bands, red, green and blue. Another important characteristic concerning an image is the number of rows and the number of columns in that image, it is called the size of the image.

Example 1.1.2 *As in the real life, an image is composed of a wide variety of structures, and this is even more complex because of the digitalization and the limited number of gray levels to represent it, see [4].To have a better look, an image and some close-ups on different parts are shown in Fig. (1.3).*

The standard slot shape in todays photographic equipments is square, as it is easy to fabricate on a detector chip. The effect appears clearly when we zoom in to pixel level.

Example 1.1.3 *The face in Fig.* (1.4) *certainly has no square corners all over and sharp edge discontinuities. So what is the shape of the optimal aperture?*

The derivation of the optimal shape can be based on the measurements taken

Figure 1.4: Typically pixels are measured with the wrong aperture function, such as squares. Sharp edges and corners that are not in the original scene [8].

by a finite aperture, all locations are treated similarly, (this leads to translation invariance), and the using of the superposition principle (the measurement should be linear). see [8].

By these principles the observation must be a convolution, for simplicity consider the 1D convolution example :

$$h(x) = \int_{-\infty}^{\infty} L(y)g(x-y)dy,$$

where $L(x)$ is the luminance, $g(x)$ is the unknown aperture (positive function) and $h(x)$ the result of the measurement.

Definition 1.1.1 *The entropy H of the filter is defined as*

$$H = \int_{-\infty}^{\infty} -g(x)log(g(x))dx$$

It measures the spurious extras (the maximal disorder) when filter is used.

We need to determine the $g(x)$, for which the entropy is minimal given the constraints:

$$\int_{-\infty}^{\infty} g(x)dx = 1, \qquad \text{(normalization constraint)},$$

$$\int_{-\infty}^{\infty} xg(x)dx = 0, \qquad \text{(first moment of } g(x)\text{)},$$

and

$$\int_{-\infty}^{\infty} x^2 g(x)dx = \sigma^2 \qquad \text{(second moment of } g(x)\text{)}.$$

Applying Lagrange multipliers method, The entropy under the given constraints reads

$$\tilde{H} = \int_{-\infty}^{\infty} -g(x)log(g(x))dx + \lambda_1 \int_{-\infty}^{\infty} g(x)dx + \lambda_2 \int_{-\infty}^{\infty} xg(x)dx + \lambda_3 \int_{-\infty}^{\infty} x^2 g(x)dx$$

where λ_1, λ_2, and λ_3 are scalars (Lagrange multipliers). According to this method $\tilde{H}$ (minimal) when $\dfrac{\partial \tilde{H}}{\partial g} = 0$. This gives

$$\frac{\partial H}{\partial g} = \int_{-\infty}^{\infty} -1 - log[g(x)]dx + \int_{-\infty}^{\infty} \lambda_1 dx + \int_{-\infty}^{\infty} \lambda_2 x dx + \int_{-\infty}^{\infty} \lambda_3 x^2 dx = 0,$$

$$-1 - log[g(x)] + \lambda_1 + x\lambda_2 + x^2\lambda_3 = 0$$

or the last equations leads to

$$g(x) = e^{-1+\lambda_1+x\lambda_2+x^2\lambda_3} \tag{1.1}$$

The scalar λ_3 must be less than zero, otherwise the function $g(x)$ explodes as x approaches infinity, which is physically irrelevant. By substituting from (1.1) into the three constaints and noting that λ_3 is negative we get respectively

$$e\sqrt{-\lambda_3} = e^{\lambda_1 - \frac{\lambda_2^2}{4\lambda_3}} \sqrt{\pi},$$

$$e^{\lambda_1 - \frac{\lambda_2^2}{4\lambda_3}} \lambda_2 = 0,$$

$$\text{and} \quad \frac{\mathrm{e}^{-1+\lambda_1 - \frac{\lambda_2^2}{4\lambda_3}} \sqrt{\pi}(\lambda_2^2 - 2\lambda_3)}{4(-\lambda_3)^{5/2}} = \sigma^2.$$

Solving the three equations for λ_1, λ_2 and λ_3 gives

$$\lambda_1 = \frac{1}{4} log\left[\frac{e^4}{4\pi^2\sigma^4}\right],$$

$$\lambda_2 = 0,$$

$$\lambda_3 = -\frac{1}{2\sigma^2 e},$$

clearly λ_3 is negative. Using Eq.(1.1) the aperture function $g(x)$ reads

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

which is the Gaussian kernel.

It is important to note that the Gaussian kernel is the only solution of the previous constaints. Further, it is circular and generates no suprious resolution, being very smooth.

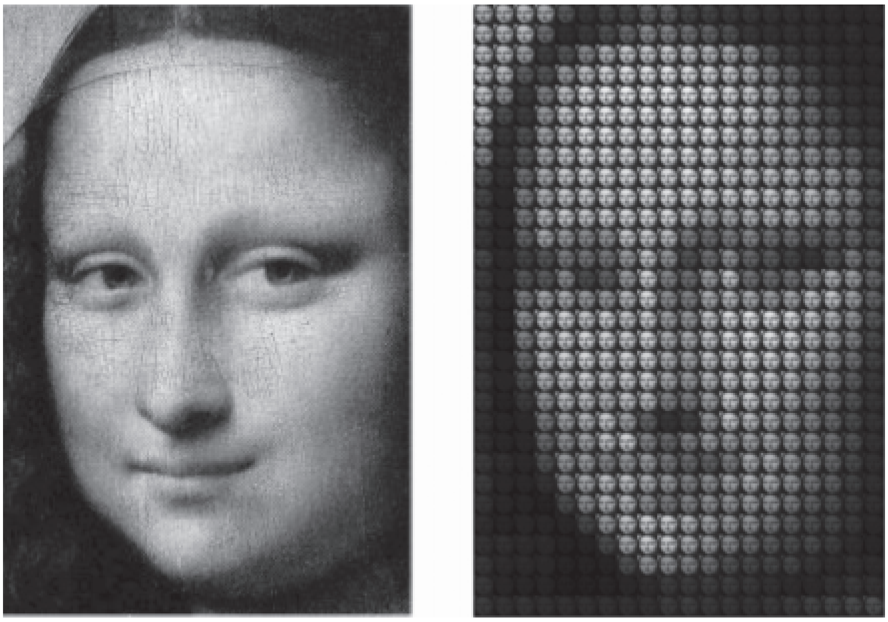


Figure 1.5: Mona Lisa at different scales [8].

Example 1.1.4 *Fig.* (1.5) *shows Mona Lisa at different scales, the Gaussian kernel blurs the image, but that is the natural consequence of an observation with a finite aperture. We cannot see molecules with our naked eye, [8].*

1.2 The Differential Structure of Images

Details of images exist only over a restricted range of scale. Hence it is important to study the image structure, which can be described as the local multi-scale derivatives of the image, using heightlines, local coordinate systems and independence of the choice of coordinates. We will also use the tools of differential geometry, which is defined as the framework of finite-dimensional real manifolds, employing methods from analysis to investigate geometric problems, involving the shape of smooth curves and surfaces, lines, spaces, surfaces etc.

An important property of real-world images is that they are exist as significative structure over definite ranges of scale. For example, the notion of a section of a tree, which makes sense only at a scale from, say, a few centimeters to at most a few meters. It is meaningless to discuss the tree notion at the nanometer or kilometer level. At those scales, it is more relevant to talk about the molecules that form the leaves of the tree, and the forest in which the tree grows, respectively. This fact, that images are shown in different ways according on the scale of observation, has important implications if one aims at describing them. It shows that the notion of scale is of utmost importance.

Scale-space theory provides a mathematically convenient method to generate representations of images at multiple scales, which also handle the above-mentioned multi-scale nature of image data. The reasoning behind its construction is that if no prior information is available about what are the appropriate scales for a given data set, then the only reasonable approach for an uncommitted vision system is to represent the input data at multiple scales.

Example 1.2.1 *Take a look at Fig.* (1.6). *This painting contains a wealth of objects "living" at a wide range of scales, from small-scale objects such as the cracks in the pavement to large-scale structures such as the building in the background.*

An essential requirement is that structures at coarse scales in the multi−scale representation should frame simplifications of corresponding structures at finer scales−they should not be occasional phenomena formed by the method for repress fine-scale structures. This idea has been formalized in a different ways by different Researchers. A noteworthy coincidence is that similar results can be observed from several different points. A main result is that if rather general conditions are imposed on the types of computations that are to be performed, then convolution by the Gaussian kernel and its derivatives is singled out as a canonical class of smoothing transformations. The requirements (scale−space axioms) that specify the uniqueness are essentially linearity and spatial shift invariance, combined with different ways of formalizing the notion that new structures hould not be created in the transformation from fine to coarse scales, [15].

In summary, for any N-dimensional signal $f : R^N \longrightarrow R$, its scale-space representation $L : R^N \times R_+ \longrightarrow R$ is defined by

$$L(x;t) = \int_{\xi \in R^N} f(x-\xi) g(\xi, t) d\xi$$

where $g : R^N \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ denotes the Gaussian kernel.

$$g(x;t) = \frac{1}{(2\pi t^2)^{N/2}} e^{-(x_1^2+\ldots+x_N^2)/2t}$$

and the variance t of this kernel is referred to as the scale parameter. And the scale space can be viewed as a stack of images, where the original image is at the bottom of the stack $f_0(x,y) = f(x,y)$, and the image resolution gets lower as we rise in the stack, see Fig. (1.7).