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An Iterative Technique for Solving Nonlinear Quadratic Optimal Control Problem Using Orthogonal Functions

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Dedication

To the late memory of my Father

Amjad Ahed Mustafa Majdalawi

Declaration:

I certify that this thesis submitted for the degree of Master, is the result of my own research, except where otherwise acknowledged, and that this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed:……………………..

Amjad Ahed Mustafa Majdalawi

Date: /   /

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My deep thanks go to my wife for her support and encouragement. I also would like to thank my great children: Mohammad, Nada, and Jana for their continuous smiles which gave me a great support.

Thank you all.

Abstract

Over the recent years, many techniques and methods have been proposed to solve the difficult nonlinear optimal control problem. These methods can be classified as: direct and indirect methods. The proposed method in this work is classified as a direct method in which the optimal control problem is directly converted into a mathematical programming problem. As its name implies, direct methods are employed by directly substituting the control and state variables into the performance index.

Direct methods can be implemented using either discretization or parameterization. Parameterization can be implemented using one of three ways: (1) Control parameterization, (2) Control-state parameterization and (3) State parameterization. The proposed method in this work is based on state parameterization which is employed by parameterizing the system state variables by a finite length series Chebyshev or Legendre polynomials with unknown parameters.

The proposed method in this work is also based on the iteration technique which replaces the nonlinear state equations by an equivalent sequence of linear time-varying state equations. Then, state parameterization is applied on this sequence. By this, the original nonlinear quadratic optimal control problem is directly converted into quadratic linear programming problems, which are easier to solve.

To show the effectiveness of the proposed method, several optimal control problems free and subject to different types of constraints were solved, and the simulation results show that the proposed method gives good and comparable results with some other methods. Among the optimal control problems which were solved is the complex containers crane problem.

Contents

Chapter One

Introduction

1.1 Motivations

Optimal control problem is to find an optimal controller u^* that minimizes a certain cost function (specification) keeping at the same time the system state equations, initial condition, and any other constraints of the system satisfied. Examples of optimal control applications include environment, engineering, economics etc.

Unlike linear optimal control problem which has an analytical solution that is given as a closed loop feedback control law, nonlinear optimal control problem does not. This motivates many researchers to try to find a solution to this problem. In most cases, if not all, these solutions are numerical i.e. approximate or suboptimal solutions.

Generally, solution methods to optimal control problem are classified as direct and indirect methods. Indirect methods are usually employed by converting the optimal control problem into a two-point boundary value problem TPBVP and solving this new problem which is easier than the original problem or finding a solution that satisfies the Hamilton-Jacobi-Bellman equation. The main advantage of indirect methods is that the resulted solutions produce a closed loop or feedback control law. However, indirect methods suffer some drawbacks which are [8]: (1) There is no solution to the Hamilton-Jacobi-Bellman equation of general nonlinear optimal control problem. (2) The introduction of artificial costates. (3) The lack of robustness. (4) A deep knowledge of the system model (mathematical and physical) is required.

For those reasons and others direct methods were proposed to solve optimal control problems. Direct methods are employed by either discretization or parameterization of the state and/or the control variables. In discretization, many discrete points (samples) of the state and/or control variable are required in order to produce accurate results. This would ends up with a system of large dimension (curse of dimensionality). On the other hand, parameterization can be implemented by one of the three ways: (1) Control parameterization is employed by approximating the control variables by a finite series of known functions with unknown parameters, then the state variables are obtained as a function of the unknown parameters by integrating the system state equation. This process is computationally expensive [15]. (2) Control-state parameterization is employed by approximating both state and control variables by a finite series of known functions with unknown parameters. The resulted system would ends up with large unknown parameters. (3) State parameterization is the least used method compared with control parameterization and control-state parameterization. In state parameterization, only some state variables are directly approximated by a finite series of known functions with unknown parameters. The remaining state and control variables are obtained as a function of the unknown parameters directly from the state equation(s).

Though, state parameterization is not used extensively in optimal control. Our choice in this work is to use state parameterization because it has some advantages over both control parameterization and control-state parameterization. These advantages are: (1) There is no need as in control parameterization to integrate the system state equations. (2) The number of unknown parameters is smaller compared with control-state parameterization. (3) The state constraints can be handled directly.

State parameterization requires known functions for the approximation of the state variables. To simplify computation of the optimal control problem, the known functions are usually chosen to be orthogonal. In this work, we choose two orthogonal functions; Chebyshev and Legendre polynomials. These polynomials offer some advantages over other orthogonal functions. Fast convergence and good min-max properties [17] are only few advantages that both functions offer.

1.2 Thesis Goals

The main goal of this work is to apply the iteration technique developed by Banks [3-7] on the optimal control problem under consideration to directly obtain a numerical solution to this problem. As a result, state parameterization via Legendre or Chebyshev polynomials will be applied on the resulted optimal control problems of the iteration technique. In the application of state parameterization, we will follow Jaddu method [8].

1.3 Thesis Contribution

The main contribution of this work is the introduction of a new technique for solving the nonlinear quadratic optimal control problem, both free and subject to different types of constrains. As a result, other contribution can be stated as follows:

- Introducing a new Legendre polynomial property called the differentiation operational matrix. This matrix is used to approximate the derivatives of the state polynomials using Legendre polynomials.

- Introducing a new formula for the approximated performance index using Legendre polynomials.

- Introducing a new method for solving the linear quadratic optimal control problem using state parameterization via Legendre polynomials.

1.4 Thesis Organization

The remaining chapters of this thesis are organized as follows:

Chapter two reviews the optimal control problem in general and discusses some of the important previous works that are proposed to handle the optimal control problem. In this chapter, the computational techniques and methods used to handle optimal control problems are classified into direct and indirect methods.

Chapter three describes a method for solving the linear quadratic optimal control problems. In spite of the fact that this work is intended for nonlinear optimal control problems, it is necessary to solve linear optimal control problems, because as will be demonstrated in chapters four and five, the solution method for nonlinear optimal control problems is based on converting the nonlinear optimal control problem into a sequence of linear time-varying optimal control problems. All aspects of state parameterization via Legendre polynomials are discussed in this chapter. In addition, some of the important properties of Legendre polynomials are reviewed. One of them is a newly introduced property for the Legendre polynomials called the differentiation operational matrix. This property is used to approximate the derivative of the state variables. An explicit formula to approximate the quadratic performance index using Legendre polynomials is introduced. Finally, computational results of a standard example are introduced and the results are compared with some other methods.

Chapter four presents the main idea of this work, where a computational method for solving the nonlinear quadratic optimal control problem is introduced. In this chapter, the concept of the iteration technique is presented. Also introduced in this chapter is state parameterization via Chebyshev polynomials developed by Jaddu [8]. In this chapter, the steps of converting the nonlinear quadratic optimal control problem into a sequence of quadratic programming problems are introduced. To verify the proposed method, a standard example is solved for the purpose of comparison with other methods.

Chapter five is an extension of chapter four, where the optimal control problem under consideration is subject to different types of constraints. These constraints include: Terminal state constraints, State saturation constraints and Control saturation constraints. In this chapter, the constrained nonlinear quadratic optimal control problem is converted into a constrained sequence of standard quadratic programming problems solved using the active set method. To show the effectiveness of the proposed method, a typical Van der Pol problem subject to different types of constrained is solved and the results are compared with the results of other methods. Also introduced in this chapter is the complex problem of transferring containers from ships to trucks at the port of Kobe.

Finally, Chapter six presents some of the important conclusions of this work and a suggestion of the future work that can be built over this thesis.

Chapter two

Optimal Control Problem: Literature Review

2.1 Introduction

In this chapter, we present a review of the optimal control problem in general. We discuss some of the important previous works presented to handle the optimal control problem. Many textbooks [1-2] and survey papers [31-32] that handled optimal control problem were published.

Basically, the main objective of optimal control is to find a controller that can be an open loop (off- line) controller denoted as $u^*(t)$ or a closed loop (on-line) controller denoted as $u^*(x(t),t)$. This controller is optimal because when applied to the dynamic system, it minimizes (maximizes) a certain function called the cost function or performance index keeping at the same time the system physical constraints unviolated. The performance index or cost function can be considered as the desired specifications of the system.

The basic optimal control problem consists of three elements:

1. Plant model: This is the system to be controlled. Mathematically, it is represented as a set of state equations which are a set of first order differential equations

   $$\dot{x} = f(x(t),u(t),t) \quad , t \in [t_0, t_f] \tag{2.1}$$

   where $x \in R^n$ is the state vector, $u \in R^m$ is the control vector. f is assumed continuous differentiable function with respect to all its arguments.

2. Initial plant state: A set of initial conditions which indicate the system state values at initial time

   $$x(t_0) = x_0 \tag{2.2}$$

   where $x_0 \in R^n$ represents a known initial condition vector.

3. Plant performance index (specifications): The desired specifications of the system that needs to be minimized (or maximized). Mathematically, the performance index is represented by a scalar function given by

$$J = h\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{2.3}$$

where t_0 and t_f are the initial and final time; h and g are scalar functions. t_f may be specified or "free", depending on the problem statement. Figure (2.1) shows the elements of a basic optimal control problem.

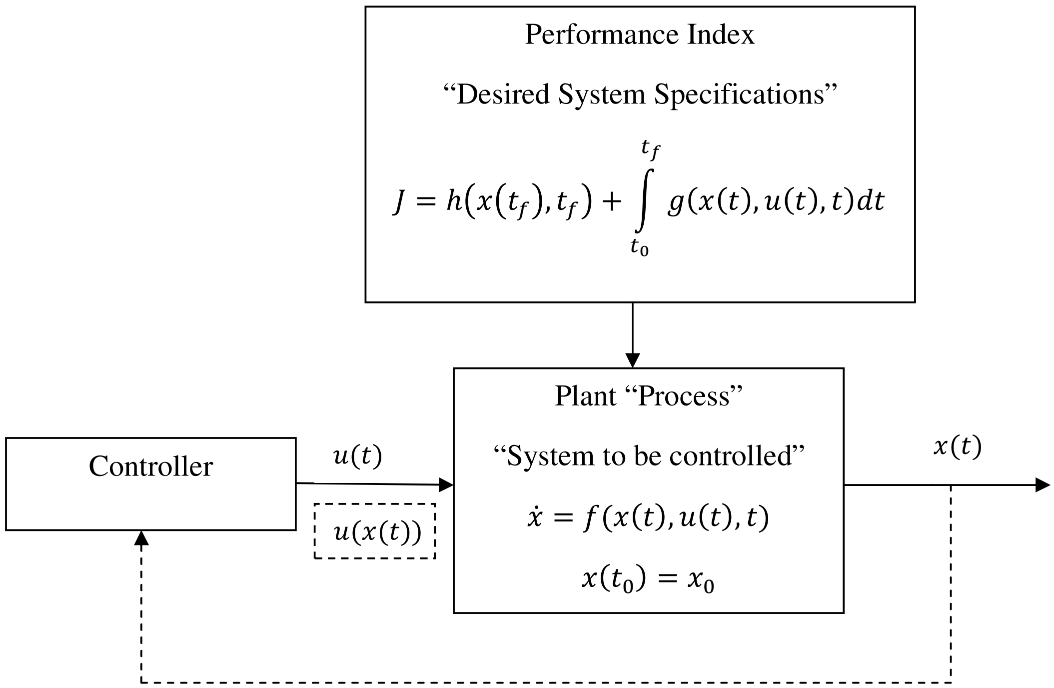


Figure (2.1) Elements of an optimal control problem

2.2 Problem Statement

The general unconstrained optimal control problem can be stated as follows:
Find an optimal controller, feedback $u(x(t), t)$ if possible, or if not an open loop $u(t)$ that minimizes the following performance index

$$J = h\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{2.4}$$

subject to

$$\dot{x} = f(x(t), u(t), t) \qquad x(t_0) = x_0 \tag{2.5}$$

Many methods have been proposed to solve the problem (2.4)-(2.5). More or less, these methods can be categorized into one of the following tracks:

- Dynamic programming (Hamilton-Jacobi-Bellman HJB Equation).
- Calculus of Variation (Euler-Lagrange Equations).
- Parameterization or discretization (nonlinear mathematical programming).

Dynamic programming is based on methods that satisfy HJB equation. The optimal controller resulted from these methods is a closed loop or feedback controller $u(x(t))$. Methods that are based on the calculus of variation (Euler-Lagrange equations) convert the optimal control problem into a Two-Point Bounded Value Problem (TPBVP). The optimal controller resulted from using these methods would also produce a feedback or closed loop controller $u(x(t))$. Methods that are based on HJB equation or Euler-Lagrange equations are usually classified as indirect methods.

Methods that are based on parameterization or discretization are classified as direct methods. Direct methods usually produce an open loop optimal controller $u(t)$. Direct methods are based on solving the optimal control problem by converting it into a nonlinear programming problem. The proposed method in this work is classified as a direct method.

In the following sections, we discuss these methods and review some of the important papers that were published. Figure (2.2) shows a block diagram that illustrates the computational methods of optimal control problem.

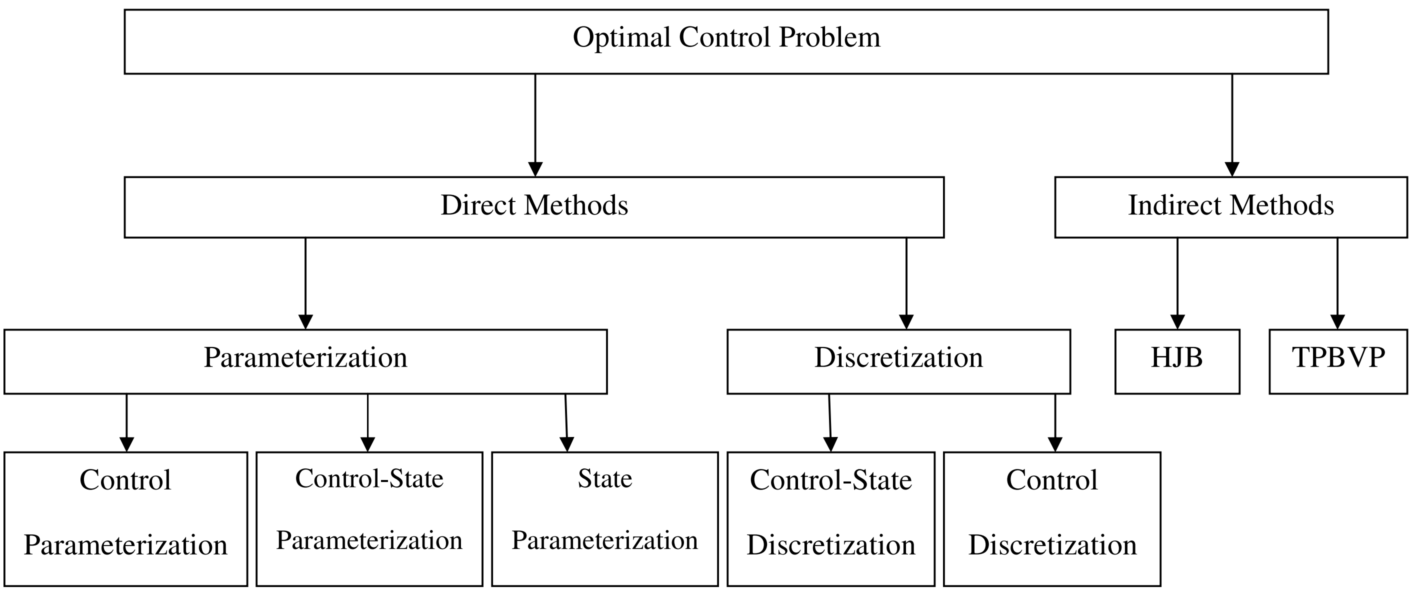


Figure (2.2) Computation methods of optimal control problem

2.3 Indirect Methods

In this section, we review some of the important methods that are classified as indirect method. As indicated earlier, these methods are based on solutions that satisfy the HJB equation or on solutions that convert the optimal control problem into a TPBVP. In what follows is a review of these methods:

1. Power series approach: This approach is based on finding an approximate solution to the Hamilton-Jacobi-Bellman equation or the nonlinear two-point boundary value

problem by using power series expansion. The approximated feedback control law obtained by this technique is solved successively. The pioneers of this method are:

- Lukes [33]: Applied this method to obtain an approximated feedback control law of the HJB equation. Lukes assumed a general nonlinear infinite horizon (regulator) optimal control problem. The problem treated by Lukes can be stated as

  Find a feedback controller $u(x)$ that minimizes the following performance index

  $$J = \int_0^\infty (x^T Q x + 2x^T M u + u^T R u + g(x,u))dt \tag{2.8}$$

  subject to

  $$\dot{x} = Ax(t) + Bu(t) + f(x(t), u(t)) \tag{2.9}$$

  where Q, R are real symmetric positive definite matrices and A, B, M are real matrices. $f(x,u)$ is a higher order term in (x,u) of order two and above and $g(x,u)$ is also a higher order term in (x,u) of order three and above. Lukes assumed that the expected feedback control law will be of the form

  $$u(x) = Dx + h(x) \tag{2.10}$$

  where $h(x)$ is a higher order term in x of order two and above. The optimal controller $u^*(x)$ and the optimal performance index $J^*(x)$ can be found successively as

  $$u^*(x) = u_*^{[1]}(x) + u_*^{[2]}(x) + \cdots + u_*^{[m]}(x) \tag{2.11}$$

  $$J^*(x) = J_*^{[2]}(x) + J_*^{[3]}(x) + \cdots + J_*^{[k]}(x) \tag{2.12}$$

  By this, the original difficult optimal control problem reduced to solving successively systems of linear algebraic equations.

- Willemstein [34]: Extended the work of Lukes to handle finite time optimal control problems both fixed end and free end. The optimal control problem reduced to solving successively systems of ordinary differential equations.
- Garrard and Jordan [35]: Applied the work done by Lukes to control a complex dynamic system of an F8 fighter jet.
- Yoshida and Loparo [36]: Apply a similar idea of Lukes to solve a nonlinear optimal control problem with quadratic performance index for both finite and infinite time problems. The problem treated by Yoshida can be stated as

  Find an approximate feedback optimal controller $u(x)$ that minimizes

  $$J = \int_0^{t_f} (x^T Q x + u^T R u)dt \tag{2.13}$$

  subject to the following dynamic system

$$\dot{x} = f(x) + Bu \tag{2.14}$$

where $f(x)$ includes the linear and nonlinear terms of the dynamic system. The function $f(x)$ was expanded by a power series around the origin. Also, the costates were expanded by a power series with unknown parameters. By this, the finite time optimal control problem was reduced to solving a Riccati equation and a sequence of ordinary linear differential equations. Whereas, the solution to the infinite time optimal control problem was reduced to solving a sequence of algebraic equations.

2. Extended linearization method [9, 37]: In this method, the nonlinear dynamic system expressed as a nonlinear state equations of the form

$$\dot{x} = f(x, u, t) \tag{2.15}$$

where $f(x, u, t)$ is a nonlinear function in x is to be rewritten in a “pseudo” linear form

$$\dot{x} = A(x)x + B(x)u \tag{2.16}$$

where the matrices $A(x), B(x)$ are called State Dependant Coefficient (SDC) matrices.

Using this method, the quadratic nonlinear optimal control problem has an approximate feedback controller of the form

$$u(x) = -R^{-1}B^{T}P(x)x(t) \tag{2.17}$$

where $P(x)$ is the solution of the following State Dependant Riccati Equation (SDRE)

$$\dot{P}(x) = P(x)A(x) + A^{T}(x)P(x) - P(x)B(x)R^{-1}(x)B^{T}(x)P(x) + Q(x) \tag{2.18}$$

and for the infinite horizon problem $\dot{P}(x) = 0$, and (2.18) becomes Algebraic State Dependent Riccati Equation (ASDRE) written as

$$P(x)A(x) + A^{T}(x)P(x) - P(x)B(x)R^{-1}(x)B^{T}(x)P(x) + Q(x) = 0 \tag{2.19}$$

3. Inverse optimal control problem: In this method, an optimal feedback control is obtained by finding a solution to the inverse optimal control problem. For more details of this method, the reader can refer to [38] and the references therein.

2.4 Direct Methods

As their name implies, direct methods are employed by direct substitution of the state and control variables into the performance index without constructing the Hamiltonian of the system as in indirect methods. This would produce many advantages of direct methods over indirect methods. Some of these advantages are: (1) There is no need to find a costates variables. (2) Using direct methods, the dynamic optimal control problem is converted into a static optimization problem. (3) As a result, many software packages are available to solve this static problem. (4) Different constraints can be handled directly. Due to these advantages and to the drawbacks of the indirect methods mentioned earlier, many techniques and methods were proposed that are based on direct methods.

By using direct methods, the difficult nonlinear dynamic optimal control problem is converted into a nonlinear mathematical programming problem. Direct methods can be implemented by either using discretization or parameterization methods. In this work, we will use parameterization technique to convert the difficult nonlinear quadratic optimal control problem into linear time-varying quadratic control problems which are much easier than the original problem. In the following sections, we will briefly discuss both discretization and parameterization technique with some focus on parameterization.

2.4.1 Discretization:

Discretization is a process in which the time interval $t \in [t_0, t_f]$ is to be divided into an equal n time segments. Mathematically, this can be given as

$$t_0 < t_1 < t_2 < t_3 < \cdots < t_n = t_f \tag{2.20}$$

As a result, and depending on the discretization technique, the variable(s) is (are) sampled at each time point in (2.20). Basically, there are two discretization technique used in optimal control problem: Control-state discretization and control discretization.

1. Control-State Discretization:

In this method, both state and control variables are to be discretized. As a result, the following vector which contains a sequence of unknown state and control variables will be produced

$$y = (x_0, x_1, \dots, x_n, u_0, u_1, \dots, u_{n-1}) \tag{2.21}$$

By this, the system state equations are replaced by algebraic equations which are treated as equality constraints. This would convert the original optimal control problem into a static optimization problem that can be solved using any available software packages like MATLAB. Note that in order to have accurate results, large amount of samples should be taken, this would result in a system that is highly dimensional. In literature, this is referred to as the “curse of dimensionality” [45]. More details about this method can be found in [10]

2. Control Discretization:

In this method, only the control variables are to be discretized. As a result, the following vector is obtained

$$y = (u_1, u_2, \dots, u_{n-1}) \tag{2.22}$$

In order to get the state variables, it is necessary to integrate the system state equations. This would produce state variables that are a function of the control variables. An advantage of this method over control-state discretization is that the resulted system is lower in dimension. For more details of these techniques and method, the reader can refer to [10].

2.4.2 Parameterization:

Parameterization is a process in which a function or a variable is approximated using known functions with known or unknown parameters. Parameterization can be employed by one of the three forms: Control parameterization, control-state parameterization and state parameterization. In this work, we use state parameterization.

1. Control Parameterization:

In this method, only the control variables are approximated by a finite length series of known functions with unknown parameters. Mathematically, this can be formulated as follows

$$u_k = \sum_{i=0}^{N} b_i^{(k)} f_i\,(t) \qquad k = 1,2,\dots,m \tag{2.23}$$

where N is the order of approximation, $b_i's$ are the unknown parameters and $f_i(t)$ is a suitably selected set of functions forming a basis of the control space.

By integrating the state equation, the state variables can be obtained as a function of the unknown parameters of the control variables. Both control and state variables are then directly substituted into the performance index. By this, the original difficult optimal control problem is converted into a static optimization problem of the unknown parameters which can be solved using any available software packages.

This method is the most widely used method compared to the other parameterization techniques. But, integration of the state equations to get the state variables is an expensive computation process [15]. More details about this method can be found in [12, 18, 23].

2. Control-State Parameterization:

Using this method, both control and state variables are approximated by a finite length series of known functions with unknown parameters of its own. Mathematically, this can formulated as follows

$$x_k = \sum_{i=0}^{N} a_i^{(k)} f_i\,(t) \qquad k = 1,2,\dots,n \tag{2.24}$$

$$u_l = \sum_{i=0}^{N} b_i^{(l)} f_i\,(t) \qquad l = 1,2,\dots,m \tag{2.25}$$

where a_i's, b_i's are the unknown parameters, N is the order of approximation and f_i is a suitably selected set of functions forming a basis. By this, the optimal control problem is converted into a nonlinear mathematical programming problem. Since both state and control variables are parameterized, the resulted system would ends up with a large number of unknown parameters. More details about this method can be found in [25, 26].

3. State Parameterization:

In this method, only the state variables are to be approximated by a finite length series of known functions with unknown parameters. Mathematically, this can be formulated as follows

$$x_k = \sum_{i=0}^{N} a_i^{(k)} f_i(t) \qquad k = 1,2,\dots,n \tag{2.26}$$

The control variables can be obtained from the state equations. The work of thesis is based on using state parameterization. The idea is to choose a set of state variables that are to be approximated directly by a finite length series of known functions with unknown parameters. The remaining state and control variables can be obtained as a function of the directly approximated state variables parameters from the system state equations. This would decrease the resulted system dimension dramatically. If any state equation remains unsatisfied, it will be considered as an equality constraint.

By using state parameterization, drawbacks of control parameterization and control-state parameterization can be overcome. In state parameterization, there is no need to integrate the state equations to get the state variables as in control parameterization. The dimension of the resulted system is much smaller compared to control-state parameterization. In control-state parameterization, all system state equations will be replaced by equality constraints, while in state parameterization only unsatisfied state equation(s) will be replaced by equality constraints. It is difficult to handle state constraints in control parameterization and control-state parameterization, while in state parameterization all state constraints can be treated directly.

All these advantage of state parameterization makes it almost perfect for solving linear optimal control problems. However, in nonlinear systems it is difficult to apply state parameterization [14]. One of the reasons for this is the difficulty of getting the control variables out of the state equations. In this work, we overcome this problem by using the iteration technique [3-7] which will replace the original nonlinear state equations by an equivalent sequence of linear time-varying state equations.

Application of state parameterization requires basis functions. State parameterization can be implemented using different basis function [23]. In this work, we will use Chebyshev and Legendre polynomials as the basis functions to parameterize the state variables.

The use of Chebyshev polynomial in optimal control is not new. Many papers have been published to handle linear and nonlinear optimal control problems. Examples of papers that use Chebyshev polynomials to handle optimal control problems are: Jaddu [8, 16, 24] used state parameterization and quasilinearization to solve a general nonlinear optimal control problems. Vlassenbroeck and Van Dooren [13] handled the nonlinear optimal control problems subject to different type of constraints by parameterizing both state and control variables using Chebyshev polynomials.

On the other hand, the use of Legendre polynomials in optimal control problems is rare. Up to our knowledge, most of the applications are of other versions of Legendre polynomials and in particular shifted Legendre polynomials. Examples can be found in [39, 41] and the references therein.

In this thesis, we will propose a method that can handle both linear and nonlinear optimal control problems using state parameterizations and Legendre or Chebyshev polynomials.

Chapter Three

Linear Quadratic Optimal Control Problem

3.1 Introduction

In spite of the fact that this work is intended for nonlinear optimal control problems, it is necessary to study linear optimal control problems, because as will be seen in the next chapter, the main idea proposed in this work to handle nonlinear optimal control problems is to replace the original nonlinear state equations by an equivalent sequence of linear time-varying state equations [3-7].

It is well known that the linear optimal control problem is one of the few optimal control problems that can be solved analytically. The solution of this problem gives a feedback control law. This solution can be found in many text books like [1-2]. However, this solution is not that easy. In order to solve this problem, it is necessary to solve either the nonlinear matrix Riccati equation or to convert the problem into Two-Point Boundary Value Problem (TPBVP).

In order to avoid difficulties associated with solving linear optimal control problems using indirect methods, some researchers proposed direct methods by using either discretization or parameterization. Razzaghi and Elnagar [41] parameterize the derivative of the state variables using shifted Legendre polynomials. Jaddu [8, 40] proposed a method that is based on state parameterization using Chebyshev polynomials. Other researchers proposed methods that are based on converting the linear two-point boundary value problem into a set of linear algebraic equations by parameterizing the state and costate variables [42].

In this chapter, we will propose a new method for handling linear quadratic optimal control problems using state parameterization via Legendre polynomials. This method will be used in the next chapter to handle nonlinear optimal control problems.

3.2 Statement of Linear Quadratic Optimal Control Problem

The linear quadratic optimal control problem can be stated as follows:
Find an optimal controller $u^*(t)$ that minimizes the following quadratic performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{3.1}$$

subject to the following linear dynamic system and initial conditions

$$\dot{x} = Ax(t) + Bu(t), \;\; x(0) = x_0 \tag{3.2}$$

where $x \in R^n, u \in R^m, x_0 \in R^n$, A, B are $n \times n$ and $n \times m$ real-valued matrices respectively. Q is an $n \times n$ positive semidefinite matrix and R is an $m \times m$ positive definite matrix. We will assume that $m \le n$ and $t \in [0, t_f]$.

The method proposed to handle problem (3.1)-(3.2) is based on directly parameterizing the state variables by a finite length series of Legendre polynomials with unknown parameters.

3.3 State Parameterization via Legendre Polynomials

In this section, we will discuss all aspects of state parameterization using Legendre polynomials. Before doing this, it is worth to review some of the important properties of Legendre polynomials.

3.3.1 Legendre Polynomials:

Due to many advantages that Legendre polynomials offer in comparison with other orthogonal polynomials, we will use them in this work to perform state parameterization. Fast convergence and good min-max properties are examples of these advantages [17]. In this section, some background review of Legendre polynomials is given to help presenting the upcoming materials in the next sections and chapters.

The Legendre polynomials of the first kind are defined on the interval $\tau \in [-1,1]$. Many formulas can be given to generate these polynomials. In this work, we will use the Rodrigues formula [19] to generate the Legendre polynomials as follows

$$P_n(\tau) = \frac{1}{2^n n!}\frac{d^n}{d\tau^n}(\tau^2 - 1)^n \tag{3.3}$$

From which the first three Legendre polynomials can be given

$$\begin{aligned} P_0(\tau) &= 1 \\ P_1(\tau) &= \tau \\ P_2(\tau) &= \tfrac{1}{2}(3\tau^2 - 1) \end{aligned} \tag{3.4}$$

The Legendre polynomials are solution to the Legendre differential equation

$$(1 - \tau^2)\frac{d^2y}{d\tau^2} - 2\tau\frac{dy}{d\tau} + n(n+1)y = 0 \tag{3.5}$$

Two Legendre polynomials $P_n(\tau)$ and $P_m(\tau)$ are orthogonal on the interval $\tau \in [-1,1]$ with respect to a unity weighing function

$$\int_{-1}^{1} P_n(\tau)P_m(\tau) = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases} \quad (3.6)$$

In this work, we will use some of the important properties of Legendre polynomials which are

- Initial and final value

$$P_n(1) = 1 \quad (3.7)$$

$$P_n(-1) = (-1)^n \quad (3.8)$$

- Product relation [20]

$$P_n(\tau)P_m(\tau) = \sum_{r=0}^{m} \frac{A_r A_{n-r} A_{m-r}}{A_{n+m-r}} \frac{2n+2m-4r+1}{2n+2m-2r+1} P_{n+m-2r}(\tau) \quad (3.9)$$

where $n \geq m$, $A_r = \frac{\left(\frac{1}{2}\right)r}{r!}$; where $(a)r = a(a+1)(a+2) \ldots (a+r-1)$, and $(a)_0 = 1$.

A function $x(\tau)$ can be approximated using Legendre series of length N as follows

$$x(\tau) = \sum_{i=0}^{N} a_i P_i(\tau) \quad (3.10)$$

where

$$a_i = \frac{2i+1}{2} \int_{-1}^{1} x(\tau)P_i(\tau)d\tau \quad i = 0,1, \ldots, N \quad (3.11)$$

In state parameterization, it is necessary to approximate the derivative of the state variables and since we were unable to find a formula that govern this relation between the state variable and its derivative; we introduce a new formula for this relation by introducing a new Legendre polynomial property called the differentiation operational matrix D. This property is given in lemma 1.

Lemma 1 *The matrix D is called the differentiation operational matrix of Legendre polynomials and is given by*

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 7 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 5 & 0 & 9 & 0 & 0 & 0 & \ldots \\ 0 & 3 & 0 & 7 & 0 & 11 & 0 & 0 & \ldots \\ 1 & 0 & 5 & 0 & 9 & 0 & 13 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \quad (3.12)$$

Proof:

It can be shown that the relationship between $\dot{P}$ and P can be given by

$$\dot{P} = DP \tag{3.13}$$

or in matrix form

$$\begin{bmatrix} \dot{P}_0 \\ \dot{P}_1 \\ \dot{P}_2 \\ \vdots \\ \dot{P}_N \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1,N+1} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2,N+1} \\ d_{31} & d_{32} & d_{33} & \cdots & d_{3,N+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ d_{N+1,1} & d_{N+1,2} & d_{N+1,3} & \cdots & d_{N+1,N+1} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} \tag{3.14}$$

where $\dot{P}_0 = 0, \dot{P}_1 = P_0 = 1$. Using the recurrence relation between $\dot{P}$ and P of Legendre polynomials, the elements of the matrix D can be found easily as follows

$$\dot{P}_{n+1} = (n+1)P_n + P_1\dot{P}_n \tag{3.15}$$

Now we are ready to use this property to write a formula for the derivative of the state variable as follows

$$\dot{x} = [a_0 \;\; a_1 \ldots \; a_N]DP(\tau) \tag{3.16}$$

where $P(\tau) = [P_0 \;\; P_1 \;\; \ldots P_N]^T$.

3.3.2 State Parameterization:

The basic idea behind state parameterization is to approximate the state variables by a finite length series of Legendre polynomials of the first kind as follows

$$x_j = \sum_{i=0}^{N} a_i^{(j)} P_i(\tau) \quad j = 1,2, \ldots, n \tag{3.17}$$

where $P_i(\tau)$ is the Legendre polynomial of the ith order of the first kind and a_i's are the unknown parameters. The control parameters are determined as a function of the state variables unknown parameters from the state equations. The initial conditions of the system are replaced by equality constraints. These approximated state and control variables are then directly substituted into the system performance index.

If all the state variables of the system are to be approximated directly, then all the state equation of the system should be replaced by equality constraints. This would increase the system dimension. In order to avoid this, and to decrease the system dimension we will follow Jaddu [8] procedure. A set of state variables are directly approximated which enable us to find the remaining state and control variables as a function of this set variables. If any state equation remains unsatisfied, it will be replaced by $N+1$ equality constraints.

Assume in any given problem that the number of state equations that can approximated directly is z, where $z < n$, then these z equations can be approximated using a finite series of Legendre polynomials with unknown parameters as follows

$$x_j = \sum_{i=0}^{N} a_i^{(j)} P_i(\tau) \quad j = 1,2, \dots, z \tag{3.18}$$

The remaining state and control variables can be found from the system state equations as a function of the state parameters in (3.18) and can be expressed in terms of finite series of Legendre polynomials with unknown parameters as follows

$$x_j = \sum_{i=0}^{N} a_i^{(j)} P_i(\tau) \quad j = z+1, z+2, \dots, n \tag{3.19}$$

$$u_l = \sum_{i=0}^{N} b_i^{(l)} P_i(\tau) \quad l = 1,2, \dots, m \tag{3.20}$$

where the unknown parameters in (3.19) and (3.20) are functions of the unknown parameters in (3.18). By this, the dimension of the optimal control problem under consideration is reduced.

3.4 Optimal Control Problem Reformulation

As indicated earlier, Legendre polynomials of the first kind are defined on the time interval $\tau \in [-1,1]$ and since the optimal control problem under consideration is defined on the time interval $t \in [0, t_f]$, it is necessary before using Legendre polynomials to transform the time interval of the optimal control problem into the interval $\tau \in [-1,1]$. This can be achieved using the following transformation

$$\tau = \frac{2t}{t_f} - 1 \tag{3.21}$$

From which we can write

$$dt = \frac{t_f}{2} d\tau \tag{3.22}$$

Using (3.21) and (3.22), the optimal control problem (3.1)-(3.2) can be reformulated as follows:
Find an optimal controller $u^*(\tau)$ that minimizes the following performance index

$$J = \frac{t_f}{2} \int_{-1}^{1} (x^T Q x + u^T R u) d\tau \tag{3.23}$$

subject to the following state equations

$$\frac{dx}{d\tau} = \frac{t_f}{2} (Ax(\tau) + Bu(\tau)) \tag{3.24}$$

and initial conditions

$$x(-1) = x_0 \tag{3.25}$$

After transforming the time interval of the optimal control problem into the time interval of the Legendre polynomials, we can know parameterize the state variables using a finite length series of Legendre polynomial with unknown parameters. As indicated earlier, this can be done using the following

$$x_j = \sum_{i=0}^{N} a_i^{(j)} P_i(\tau) \quad j = 1,2, \dots, n \tag{3.26}$$

and for the control variables

$$u_l = \sum_{i=0}^{N} b_i^{(l)} P_i(\tau) \quad l = 1,2, \dots, m \tag{3.27}$$

(3.26) and (3.27) can be rewritten in matrix form as follows

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_0^{(1)} & a_1^{(1)} & \dots & a_N^{(1)} \\ a_0^{(2)} & a_1^{(2)} & \dots & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_0^{(n)} & a_1^{(n)} & \dots & a_N^{(n)} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_N \end{bmatrix} \tag{3.28}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} b_0^{(1)} & b_1^{(1)} & \dots & b_N^{(1)} \\ b_0^{(2)} & b_1^{(2)} & \dots & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ b_0^{(m)} & b_1^{(m)} & \dots & b_N^{(m)} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_N \end{bmatrix} \tag{3.29}$$

(2.28) and (2.29) can be also written in compact form

$$x = \alpha P \qquad u = \beta P \tag{3.30}$$

The next step in converting the optimal control problem into a quadratic programming problem is to approximate the performance index. Substituting (3.30) into (3.23) yields

$$\hat{J} = \frac{t_f}{2} \int_{-1}^{1} (P^T \alpha^T Q \alpha P + P^T \beta^T R \beta P) d\tau \tag{3.31}$$

where $\hat{J}$ is the approximated performance index of J. To simplify computation of (3.31), let $M = \alpha^T Q \alpha$ and $Z = \beta^T R \beta$. Then, the value of the approximated performance index $\hat{J}$ used in (3.31) can be given by the following theorem.

Theorem 1 *The value of the approximated performance index $\hat{J}$ given in (3.31) is given by*

$$\hat{J} = \frac{t_f}{2} \sum_{i=1}^{N+1} \frac{2}{2i-1} (m_{ii} + z_{ii}) \tag{3.32}$$

or

$$\hat{J} = \frac{t_f}{2} (trace(M) + trace(Z)) \sum_{i=1}^{N+1} \frac{2}{2i-1} \tag{3.33}$$

where m_{ii} and z_{ii} are the diagonal elements of the symmetrical matrices $M = \alpha^T Q \alpha$ and $Z = \beta^T R \beta$ respectively.

Proof: (3.31) can be rewritten as

$$\hat{J} = \frac{t_f}{2} \int_{-1}^{1} (P^T M P + P^T Z P) d\tau \tag{3.34}$$

The term $P^T MP$ can be written as

$$\begin{aligned} P^T MP = m_{11}P_0P_0 + 2m_{12}P_0P_1 + 2m_{13}P_0P_2 + \cdots + 2m_{1,N+1}P_0P_N \\ +m_{22}P_1P_1 + 2m_{23}P_1P_2 + \cdots + 2m_{2,N+1}P_1P_N \\ +m_{33}P_2P_2 + \cdots + 2m_{3,N+1}P_2P_N \\ \ddots \qquad \vdots \\ + m_{N+1,N+1}P_NP_N \end{aligned} \tag{3.35}$$

Using the integration property of Legendre polynomials

$$\int_{-1}^{1} P_n(\tau)P_m(\tau)d\tau = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} & n = m \end{cases} \tag{3.36}$$

Integration of all terms in (3.35) where P_nP_m such that $m \neq n$ is zero. The remaining parts of (3.35) are given by

$$P^T MP = m_{11}P_0P_0 + m_{22}P_1P_1 + m_{33}P_2P_2 + \cdots + m_{N+1,N+1}P_NP_N \tag{3.37}$$

Integration of (3.37) can be given by

$$\int_{-1}^{1}(m_{11}P_0P_0 + m_{22}P_1P_1 + m_{33}P_2P_2 + \cdots + m_{N+1,N+1}P_NP_N) = \sum_{i=1}^{N+1} m_{ii} \int_{-1}^{1} P_{i-1}P_{i-1}d\tau \tag{3.38}$$

and

$$\sum_{i=1}^{N+1} m_{ii} \int_{-1}^{1} P_{i-1}P_{i-1}d\tau = \sum_{i=1}^{N+1} m_{ii} \frac{2}{2i-1} = trace(M) \sum_{i=1}^{N+1} \frac{2}{2i-1} \tag{3.39}$$

This gives the first part of the previous theorem. ■ Following the same procedure, integration of the second part $P^T ZP$ can be computed.

The last step in converting the optimal control problem into a quadratic programming problem is to approximate the initial conditions. This can be done using the initial value property of Legendre polynomials. By substituting $\tau = -1$ into (3.26), all initial conditions are transformed into algebraic equations of the form

$$a_0^{(k)} - a_1^{(k)} + a_2^{(k)} - a_3^{(k)} + \cdots + (-1)^N a_N^{(k)} - x_k(-1) = 0 \quad k = 1,2, \dots, n \tag{3.40}$$

(3.40) will be considered as an equality constraints imposed on the system under consideration.

(3.32) or (3.33) can be rewritten into a standard quadratic performance index as follows

$$\hat{J} = \frac{1}{2} a^T Ha \tag{3.41}$$

where $a^T = [a_0^{(1)} a_1^{(1)} \dots a_N^{(1)}\ a_0^{(2)} a_1^{(2)} \dots a_N^{(2)} \dots a_0^{(q)} \dots a_N^{(q)}]$ is the unknown parameter vector and H is appositive definite [8] Hessian matrix given by

$$H = \frac{\partial^2 \hat{J}}{\partial a_i^{(k)} \partial a_j^{(k)}} \tag{3.42}$$

where $i, j = 0,1, \dots, N$ and $k = 1,2, \dots, q$.

By this, the optimal control problem (3.1)–(3.2) is converted into parameters optimization problem which is quadratic in terms of the unknown parameters and the new problem can be stated as,

$$min_a \quad \frac{1}{2} a^T H a \tag{3.43}$$

subject to the linear constraints

$$Fa - b = 0 \tag{3.44}$$

where the linear constraints are due to initial conditions and in some cases are due to some system state equations which are not satisfied. The optimal values of the unknown parameters vector a^* can be obtained from the standard quadratic programming method [18]

$$a^* = H^{-1} F^T (F H^{-1} F^T)^{-1} b \tag{3.45}$$

Before giving an example that clarifies the proposed method, it is worth to summarize the steps that are required to solve a linear quadratic optimal control problem. These steps are:

1. Change the time interval of the original optimal control problem into $\tau \in [-1,1]$.

2. Choose a set of state variables that are to be approximated directly using finite series of Legendre polynomials with unknown parameters such that the remaining state and control variables can be obtained as a function of this set unknown parameters from the system state equations.

3. Find the control variables and the state variables that are not directly approximated as a function of the directly approximated state variables unknown parameters.

4. Replace state equations that are not yet satisfied by $N + 1$ equality constraints.

5. Calculate the matrix M from $\alpha^T Q \alpha$ and the matrix Z from $\beta^T R \beta$.

6. Calculate the approximated performance index $\hat{J}$ from (3.32) or (3.33).

7. Determine the Hessian matrix H from (3.42)

8. Determine the set of equality constraints due to system initial conditions, final conditions and the equality constraints of step 4.

9. Find the optimal parameters a^* from (3.45).

10. Substitute the optimal parameters from step 9 into (3.26) and (3.27) to find the different optimal trajectories (state and control).

3.5 Numerical Example

Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = \int_0^1 (x_1^2 + x_2^2 + 0.005u^2)dt \tag{3.46}$$

subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0 \tag{3.47}$$

$$\dot{x}_2 = -x_2 + u \qquad x_2(0) = -1 \tag{3.48}$$

The first step is to convert the time interval into $\tau \in [-1,1]$. The new problem can be restated as

Minimize

$$J = \frac{1}{2}\int_{-1}^1 (x_1^2 + x_2^2 + 0.005u^2)d\tau \tag{3.49}$$

subject to

$$\frac{dx_1}{d\tau} = \frac{1}{2}x_2 \qquad x_1(-1) = 0 \tag{3.50}$$

$$\frac{dx_2}{d\tau} = \frac{1}{2}(-x_2 + u) \quad x_2(-1) = -1 \tag{3.51}$$

We choose to approximate x_1 by a finite series of order $N = 9$ (to compare results with other methods which use $N = 9$) with Legendre polynomials of unknown parameters. This is done by

$$x_1(\tau) = \sum_{i=0}^9 a_i P_i(\tau) \tag{3.52}$$

$\dot{x}_1$ can be calculated using (3.16),

$$\dot{x}_1(\tau) = (a_1 + a_3 + a_5 + a_7 + a_9)P_0 + 3(a_2 + a_4 + a_6 + a_8)P_1 + 5(a_3 + a_5 + a_7 + a_9)P_2 + 7(a_4 + a_6 + a_8)P_3 + 9(a_5 + a_7 + a_9)P_4 + 11(a_6 + a_8)P_5 + 13(a_7 + a_9)P_6 + 15a_8P_7 + 17a_9P_8 \tag{3.53}$$

From (3.50) x_2 can be determined,

$$x_2(\tau) = 2(a_1 + a_3 + a_5 + a_7 + a_9)P_0 + 6(a_2 + a_4 + a_6 + a_8)P_1 + 10(a_3 + a_5 + a_7 + a_9)P_2 + 14(a_4 + a_6 + a_8)P_3 + 18(a_5 + a_7 + a_9)P_4 + 22(a_6 + a_8)P_5 + 26(a_7 + a_9)P_6 + 30a_8P_7 + 34a_9P_8 \tag{3.54}$$

From (3.54) we can determined $\dot{x}_2$,

$$\dot{x}_2(\tau) = (6a_2 + 20a_4 + 42a_6 + 72a_8)P_0 + (30a_3 + 84a_5 + 162a_7 + 264a_9)P_1 + (70a_4 + 180a_6 + 330a_8)P_2 + (126a_5 + 308a_7 + 546a_9)P_3 + (198a_6 + 468a_8)P_4 + (286a_7 + 660a_9)P_5 + 390a_8P_6 + 510a_9P_7 \quad (3.55)$$

Substituting $x_2(\tau)$ and $\dot{x}_2(\tau)$ into (3.51), the control $u(\tau)$ can be determined,

$$u(\tau) = 2\dot{x}_2 + x_2 \quad (3.56)$$

From which $u(\tau)$ can be given as

$$\begin{aligned} u(\tau) = {} & (2a_1 + 12a_2 + 2a_3 + 40a_4 + 2a_5 + 84a_6 + 2a_7 + 144a_8 + 2a_9)P_0 \\ & + (6a_2 + 60a_3 + 6a_4 + 168a_5 + 6a_6 + 324a_7 + 6a_8 + 528a_9)P_1 \\ & + (10a_3 + 140a_4 + 10a_5 + 360a_6 + 10a_7 + 660a_8 + 10a_9)P_2 \\ & + (14a_4 + 252a_5 + 14a_6 + 616a_7 + 14a_8 + 1092a_9)P_3 \\ & + (18a_5 + 396a_6 + 18a_7 + 936a_8 + 18a_9)P_4 \\ & + (22a_6 + 572a_7 + 22a_8 + 1320a_9)P_5 + (26a_7 + 780a_8 + 26a_9)P_6 \\ & + (30a_8 + 1020a_9)P_7 + 34a_9P_8 \end{aligned} \quad (3.57)$$

From the above approximations of x_1, x_2 and u, the matrices M and Z can be calculated and using the result of theorem 1; the value of the approximated performance index can be determined as

$$\hat{J} = 34.2a_2a_6 + a_0^2 + 1.68a_1a_6 + 4.3533a_1^2 + 0.24a_1a_2 + 8.04a_1a_3 + 0.8a_1a_4 + 8.04a_1a_5 + 8.04a_1a_7 + 2.88a_1a_8 + 8.04a_1a_9 + 12.98a_2^2 + 1.44a_2a_3 + 28.92a_2a_4 + 3.6a_2a_5 + 6.72a_2a_7 + 41.4a_2a_8 + 10.8a_2a_9 + 30.2629a_3^2 + 4.8a_3a_4 + 81.84a_3a_5 + 10.08a_3a_6 + 113.04a_3a_7 + 17.28a_3a_8 + 153.84a_3a_9 + 67.9111a_4^2 + 12a_4a_5 + 214.8a_4a_6 + 22.4a_4a_7 + 322.8a_4a_8 + 36a_4a_9 + 152.7909a_5^2 + 25.2a_5a_6 + 523.8a_5a_7 + 43.2a_5a_8 + 809.4a_5a_9 + 336.4969a_6^2 + 47.04a_6a_7 + 1176.84a_6a_8 + 75.6a_6a_9 + 707.3467a_7^2 + 80.64a_7a_8 + 129.6a_8a_9 + 2442.72a_7a_9 + 1404.7788a_8^2 + 2636.1526a_9^2 \quad (3.58)$$

Using (3.42), the Hessian can be calculated and eventually the optimal parameters vector a^* can determined using (3.45).

The last step is to approximate the initial conditions of the problem. This can be done using (3.40) and the resulted equality constraints equations are

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - a_9 = 0 \quad (3.59)$$

$$2a_1 - 6a_2 + 12a_3 - 20a_4 + 30a_5 - 42a_6 + 56a_7 - 72a_8 + 90a_9 = -1 \quad (3.60)$$

The difficult dynamic linear quadratic optimal control problem was converted into a simpler quadratic programming problem solved by a simple MATLAB program. The resulted optimal performance index obtained using this method is illustrated in table (3.1) together with the exact value of the problem under consideration and the values of the performance index obtained by different methods for comparison.

Table (3.1) Minimum values of J for the example

Source	J	Deviation error
Exact value	0.06936094	0
Hsieh [43]	0.0702	8.4×10^{-4}
Neuman and Sen [25]	0.06989	5.3×10^{-4}
Vlassenbroeck [44]	0.069368	7.1×10^{-6}
Jaddu [8]	0.0693689	7.96×10^{-6}
This research	0.0693689	7.96×10^{-6}

The optimal state trajectories and control law are shown in figures (3.1) and (3.2)

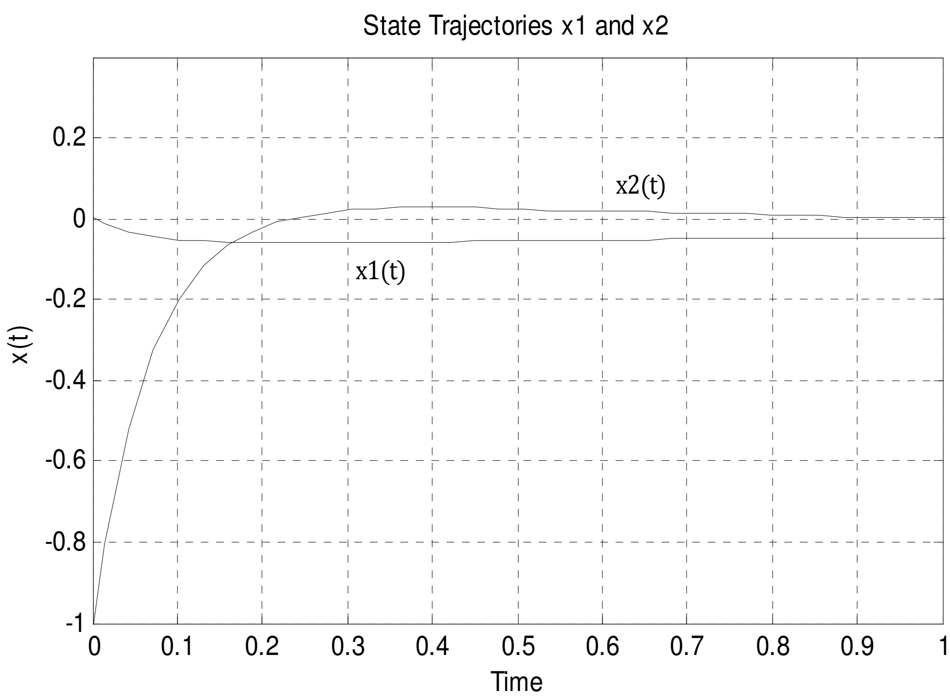


Figure (3.1) optimal state trajectories $x_1(t)$ and $x_2(t)$

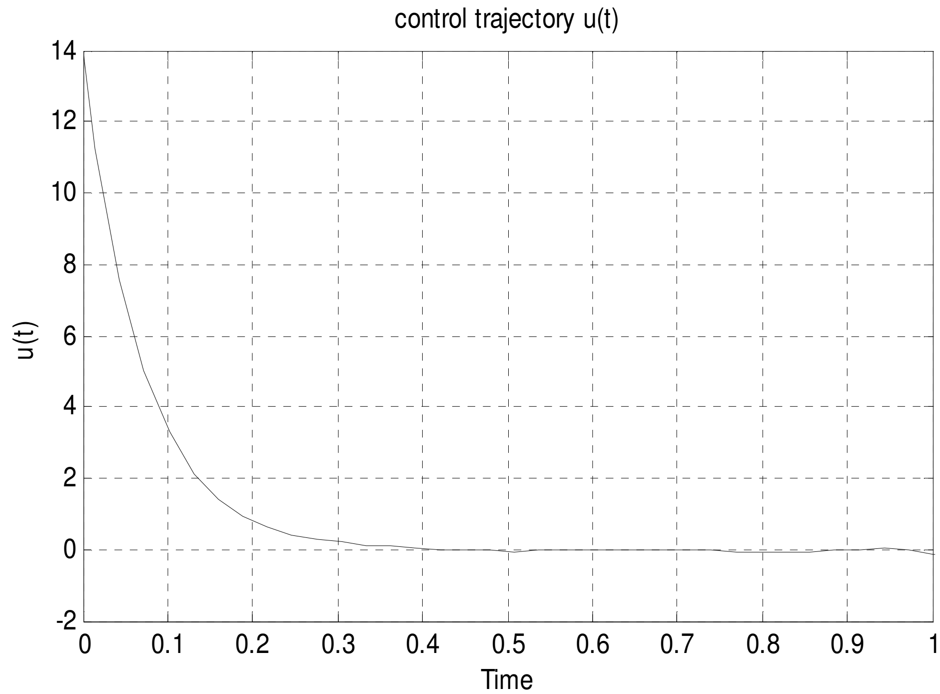


Figure (3.2) optimal control $u(t)$

3.6 Conclusion

In this chapter, we proposed a numerical method for solving linear quadratic optimal control problems. The proposed method is based on parameterizing the system state variables using a finite length Legendre polynomials. We also derived an explicit formula for the performance index. In addition a new Legendre polynomials property called the differentiation operational matrix D was derived and used to approximate the derivative of the state variables.

Compared with other methods and based on the simulation carried out in this work, our method gives better or comparable results with other methods. Using this method, the difficult linear quadratic optimal control problem is converted into a quadratic programming problem that is easy to solve. Using state parameterization, the dimension of the optimal control problem is reduced sharply compared with other methods that are based on control parameterization or control-state parameterization or even discretization.

Most of the ideas, equations and formulas used in this chapter will be used in the next chapter which handles the unconstrained nonlinear quadratic optimal control problems.

Chapter Four

Unconstrained Nonlinear Quadratic Optimal Control Problem

4.1 Introduction

In this chapter, we extend the ideas presented in the previous chapter to handle the nonlinear optimal control problem. The basic idea is to use the iteration technique developed by Banks [3-7] which replaces the original nonlinear dynamic state equations by an equivalent sequence of linear time-varying state equations. By this, the original nonlinear quadratic optimal control problem is converted into a sequence of quadratic linear time-varying optimal control problems which are much easier to solve.

Many papers that handle the nonlinear optimal control problems using parameterization methods have been published. For example, Sirisena [12] used the piecewise polynomials to parameterize the control variables. Vlassenbroeck and Van Doreen [13] used the control-state parameterization using Chebyshev polynomials to convert the nonlinear optimal control problem into a nonlinear mathematical programming problem. In its turn, the nonlinear mathematical programming problem can then be solved using different methods. One of the popular methods that are used to handle the nonlinear mathematical programming problem is the sequential quadratic programming method [46] which replaces the nonlinear mathematical programming problem by a sequence of quadratic programming problems.

Jaddu [8, 16, 24] proposed a method that is based on state parameterization via Chebyshev polynomials and quasilinearization to handle the nonlinear quadratic optimal control problems. By this, the original optimal control problem is converted directly into a quadratic programming problem.

As indicated earlier, our method is based on replacing the difficult nonlinear dynamic system by a sequence of linear time-varying dynamic system using iteration technique [3-7]. These sequences of linear time-varying systems are to be solved using the method proposed in the previous chapter and the method proposed by Jaddu [8, 40] to handle linear quadratic optimal control problems; which parameterize the state variables using Chebyshev polynomials.

4.2 Statement of the Unconstrained Nonlinear Quadratic Optimal Control Problem

The problem treated in this chapter can be stated as follows: Find an optimal control $u^*(t)$; where $t \in [0, t_f]$, that minimizes the following quadratic performance index:

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{4.1}$$

subject to the following system of nonlinear state equations and initial conditions

$$\dot{x} = f(x(t), u(t), t) \qquad x(0) = x_0 \tag{4.2}$$

where $x \in R^n, u \in R^m$, Q is $n \times n$ positive semidefinite matrix, R is $m \times m$ positive definite matrix, $x_0 \in R^n$ and f is assumed continuous differentiable function with respect to all its arguments.

The proposed method is based on using the iteration technique; in which the nonlinear dynamic system (4.1) is to be replaced by a sequence of linear time-varying dynamic system. By this, the original nonlinear quadratic optimal control problem described in (4.1)-(4.2) is replaced by a sequence of linear quadratic optimal control problems that are easier to solve. The resulted linear quadratic time-varying optimal control problems are then to be solved using two methods. The first method is based on extending the method described in the previous chapter to handle, as a special case, the linear time-varying optimal control problems. The second method proposed by Jaddu [8, 40] to handle linear time-varying optimal control problems using state parameterization via Chebyshev polynomials.

Through combination of iteration technique and state parameterization via Legendre or Chebyshev polynomials, the solution of the difficult nonlinear optimal control problem is reduced to a simple matrix-vector multiplication as will be seen in the next sections.

4.3 Iteration Technique

This technique was developed by Banks [3-7]. In this technique, the nonlinear system described in (4.2) can be replaced by an equivalent sequence of linear time-varying state equations. Mathematically, this technique is formulated as follows: The nonlinear system in (4.2) can be rewritten in pseudo-linear form [9, 37]:

$$\dot{x} = A(x)x + B(x)u \qquad , x(0) = x_0 \tag{4.3}$$

Then, the following sequence of linear time-varying state equations can replace the original nonlinear system described in (4.2) and (4.3):

$$\dot{x}^{[0]} = A(x_0)x^{[0]} + B(x_0)u^{[0]}, \qquad x^{[0]}(0) = x_0 \tag{4.4}$$

and for $i \geq 1$

$$\dot{x}^{[i]} = A\big(x^{[i-1]}(t)\big)x^{[i]} + B\big(x^{[i-1]}(t)\big)u^{[i]}\ ,\ x^{[i]}(0) = x_0 \tag{4.5}$$

It can be shown that the above sequence converges to the solution of the original nonlinear system if the Lipschitz condition $\|A(x) - A(y)\| \leq \alpha\|x - y\|$ is satisfied, where α is the Lipschitz constant. The proof for this convergence can be found in [3] or [4].

Applying the above technique to the optimal control problem described in (4.1)-(4.2), the following sequence of linear time-varying quadratic optimal control problems can replace the original problem in (4.1) and (4.2):

Minimize

$$J^{[0]} = \int_0^{tf} (x^{[0]^T} Q x^{[0]} + u^{[0]^T} R u^{[0]}) dt \tag{4.6}$$

subject to:

$$\dot{x}^{[0]} = A(x_0) x^{[0]} + B(x_0) u^{[0]}, \quad x^{[0]}(0) = x_0 \tag{4.7}$$

and for $k \geq 1$

Minimize

$$J^{[i]} = \int_0^{t_f} (x^{[i]^T} Q x^{[i]} + u^{[i]^T} R u^{[i]}) dt \tag{4.8}$$

subject to:

$$\dot{x}^{[i]} = A\big(x^{[i-1]}(t)\big) x^{[i]} + B\big(x^{[i-1]}(t)\big) u^{[i]}\,, \; x^{[i]}(0) = x_0 \tag{4.9}$$

4.4 Problem Reformulation

As indicated earlier, the proposed method in this work converts the optimal control problem under consideration directly into a quadratic programming problem. To convert the optimal control problem (4.6)-(4.9) into a quadratic programming problem, some state variables are approximated by a finite length Chebyshev or Legendre series with unknown parameters. The remaining state and control variables are determined as a function of the unknown parameters of the state variables from the state equations (4.9). These approximations are used to approximate the initial state conditions of the system, which will be treated as linear constraints.

Both Chebyshev and Legendre polynomials are defined on the interval $\tau \in [-1,1]$. Therefore, it is necessary to transform the time interval of the original problem $t \in [0, t_f]$ into $\tau \in [-1,1]$. This can be done using the transformations (3.21)-(3.22) presented in the previous chapter.

As a result of applying the transformation (3.21)-(3.22), the optimal control problem (4.6)-(4.9) can be reformulated and rewritten in terms of τ as follows:

Minimize

$$J^{[0]} = \frac{t_f}{2} \int_{-1}^{1} (x^{[0]^T} Q x^{[0]} + u^{[0]^T} R u^{[0]}) d\tau \tag{4.10}$$

subject to:

$$\frac{dx^{[0]}}{d\tau} = \frac{t_f}{2}(A(x_0)x^{[0]} + B(x_0)u^{[0]}), \quad x^{[0]}(-1) = x_0 \tag{4.11}$$

and for $i \geq 1$

Minimize

$$J^{[i]} = \frac{t_f}{2}\int_{-1}^{1}(x^{[i]^T}Qx^{[i]} + u^{[i]^T}Ru^{[i]})d\tau \tag{4.12}$$

subject to:

$$\frac{dx^{[i]}}{d\tau} = \frac{t_f}{2}[A\left(x^{[i-1]}(\tau)\right)x^{[i]} + B\left(x^{[i-1]}(\tau)\right)u^{[i]}], \; x^{[i]}(-1) = x_0 \tag{4.13}$$

To solve the optimal control problems (4.10)-(4.13), we will go into two tracks. The first track is to extend the method described in chapter three to handle the linear quadratic time-varying optimal control problems using state parameterization via Legendre polynomials. The second track is to use the method proposed by Jaddu [8, 40] to solve the linear quadratic time-varying optimal control problems using state parameterization via Chebyshev polynomials. We will then compare the results obtained by both methods (tracks).

4.4.1 Solution via Legendre Polynomials:

A look at the optimal control problems (4.10)-(4.13) shows that the 0^{th} iteration ($i = 0$) problem (4.10)-(4.11) is a time-invariant optimal control problem. The solution to this problem will be considered as the starting nominal trajectory to the sequence of optimal control problems (4.12)-(4.13). The solution to this particular problem was described in details in the previous chapter.

To solve the remaining linear time-varying optimal control problems using state parameterization via Legendre polynomials, we will extend the method described in the previous chapter.

Since the dynamic system (4.13) contains two matrices $A(x^{[i-1]}(\tau))$ and $B(x^{[i-1]}(\tau))$ that are a function of τ, and to simplify computation it is necessary to express every τ dependant element in both matrices in terms of a Legendre series of known parameters. To this end, let $A_{jl}(\tau) = g(x^{[i-1]}(\tau), \tau)$ be the (j, l) element of the matrix $A(x^{[i-1]}(\tau))$ where $x^{[i-1]}(\tau)$ is the nominal trajectory of the previous iteration. Then the term $A_{jl}(\tau)$ can be expressed in terms of a Legendre series of known parameters of the form [39, 41]

$$A_{jl}(\tau) = \sum_{i=0}^{M} W_i P_i(\tau) \tag{4.14}$$

where the coefficients W_i are given by [39, 41]

$$W_i = \frac{2i+1}{2}\int_{-1}^{1} g(\tau)P_i(\tau)d\tau \tag{4.15}$$

The same approximation can be done for the matrix $B(x^{[i-1]}(\tau))$.

Since the elements of $A(x^{[i-1]}(\tau))$, $B(x^{[i-1]}(\tau))$ are time-varying matrices expressed as a function of a finite length Legendre polynomials with known parameters of the previous iteration, and since the matrix $A(x^{[i-1]}(\tau))$ is multiplied by $x^{[i]}$; which is expressed in terms of a finite length Legendre series with unknown parameters, and the matrix $B(x^{[i-1]}(\tau))$ is multiplied by $u^{[i]}$, which is also expressed as a function of a finite length Legendre polynomials with unknown parameters, it is necessary to have a multiplication algorithm to multiply Legendre series. This algorithm is given by Lemma 1.

Lemma 1 *Given two Legendre series*

$$X = \sum_{i=0}^{n} x_i P_i \tag{4.16}$$

$$Y = \sum_{j=0}^{m} y_j P_j \tag{4.17}$$

Then the multiplication of these two Legendre series is a Legendre series of length $n + m$ *given by*

$$X * Y = \sum_{i=0}^{n} \sum_{j=0}^{m} x_i y_j P_i P_j \tag{4.18}$$

where $P_i P_j$ *is given by* [20]

$$P_i P_j = \sum_{r=0}^{j} \frac{A_r A_{i-r} A_{j-r}}{A_{i+j-r}} \frac{2i+2j-4r+1}{2i+2j-2r+1} P_{i+j-2r} \tag{4.19}$$

where $i \geq j$, $A_r = \frac{\left(\frac{1}{2}\right)r}{r!}$ *where* $(a)r = a(a+1)(a+2) \ldots (a+r-1)$, *and* $(a)_0 = 1$.

By this, the solution of the linear quadratic time-varying optimal control problem is reduced into a matrix-vector multiplication algorithm and as a result the proposed algorithm in chapter three can now be applied.

4.4.2 Solution via Chebyshev Polynomials:

In this section, we will describe briefly the method proposed by Jaddu [8, 40] to handle linear quadratic optimal control problems using state parameterization via Chebyshev polynomials of the first type. The steps required to solve the linear quadratic time-varying optimal control problems proposed by Jaddu are exactly the same compared to the method described in chapter three with differences in the formulas and equations.

As in chapter three, some state variables are to be approximated directly via a finite length Chebyshev series with unknown parameters. Mathematically, this can be expressed as follows

$$x_k(\tau) = \frac{a_0^{(k)}}{2} + \sum_{i=1}^{N} a_i^{(k)} T_i(\tau) \qquad k = 1,2,\ldots,n \tag{4.20}$$

where a_i's are the unknown parameters and $T_i(\tau)$ is the i-th order Chebyshev polynomial of the first type and is given by

$$T_0(\tau) = 1 \tag{4.21}$$

$$T_1(\tau) = \tau \tag{4.22}$$

The remaining Chebyshev polynomials can be obtained using the following recurrence formula

$$T_{r+1}(\tau) = 2\tau T_r(\tau) - T_{r-1}(\tau) \qquad r \geq 1 \tag{4.23}$$

To approximate the derivate of the state variables in (4.20), the following equation [8] is used

$$\dot{x}_k(\tau) = \frac{\acute{a}_0}{2} + \sum_{i=1}^{N-1} \acute{a}_\iota T_i(\tau) \tag{4.24}$$

where

$$\begin{cases} \acute{a}_{N-1} = 2Na_N \\ \acute{a}_{N-2} = 2(N-1)a_{N-1} \\ \acute{a}_{r-1} = \acute{a}_{r+1} + 2ra_r \qquad\qquad , r = 1,2,\ldots,N-2 \end{cases} \tag{4.25}$$

The control variables for the system can be obtained as a function of the unknown parameters of the state variables. The control variables can be expressed in terms of a Chebyshev series by

$$u_l(\tau) = \frac{b_0^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i(\tau) \qquad l = 1,2,\ldots,m \tag{4.26}$$

Since the dynamic system (4.13) contains two matrices $A(x^{[i-1]}(\tau))$ and $B(x^{[i-1]}(\tau))$ that are a function of τ, and to simplify computation it is necessary to express every τ dependant element in both matrices in terms of a Chebyshev series of known parameters. To this end, let $A_{jl}(\tau) = g(x^{[i-1]}(\tau), \tau)$ be the (j, l) element of the matrix $A(x^{[i-1]}(\tau))$ where $x^{[i-1]}(\tau)$ is the nominal trajectory of the previous iteration. Then the term $A_{jl}(\tau)$ can be expressed in terms of a Chebyshev series of known parameters of the form [18]

$$A_{jl}(\tau) = \frac{W_0}{2} + \sum_{i=1}^{M} W_i T_i(\tau) \tag{4.27}$$

where

$$W_j = \frac{2}{K} \sum_{i=1}^{K} g(\cos(\theta_i)) \cos(j\theta_i), \qquad\qquad j = 0,1,\ldots,N \tag{4.28}$$

and $\theta_i = \frac{2i-1}{2K}\pi, i = 1,2,\ldots,K$ and $K > N$

The same approximation can be done for the matrix $B(x^{[i-1]}(\tau))$. A look at (4.13) shows that $A(x^{[i-1]}(\tau))$, $B(x^{[i-1]}(\tau))$ are time-varying matrices expressed as a function of finite length Chebyshev series with known parameters of the previous iteration, and since the matrix $A(x^{[i-1]}(\tau))$ is multiplied by $x^{[i]}$, which is expressed in terms of Chebyshev series with unknown parameters, and the matrix $B(x^{[i-1]}(\tau))$ is multiplied by $u^{[i]}$, which is also expressed as a function of Chebyshev polynomials with unknown parameters, it is necessary to have a multiplication algorithm to multiply Chebyshev series. This algorithm is given by Lemma 2 [8].

Lemma 2 *Given two Chebyshev series*

$$X = \sum_{i=0}^{n} x_i T_i \tag{4.29}$$

$$Y = \sum_{j=0}^{m} y_j T_j \tag{4.30}$$

Then the multiplication of these two Chebyshev series is a Chebyshev series of length $n + m$ *given by*

$$X * Y = \sum_{i=0}^{n} \sum_{j=0}^{m} x_i y_j T_i T_j \tag{4.31}$$

where

$$T_i T_j = \frac{T_{|i-j|} + T_{i+j}}{2} \tag{4.32}$$

The next step is to approximate the initial condition vector $x(-1) = x_0$. This can be done using the following Chebyshev initial value property

$$T_n(-1) = (-1)^n \tag{4.33}$$

Then substituting $\tau = -1$ into (4.20) and using the property in (4.33), the initial condition vector is approximated as follows

$$\frac{a_0^{(k)}}{2} - a_1^{(k)} + a_2^{(k)} - a_3^{(k)} + \cdots + (-1)^N a_N^{(k)} - x_k(-1) = 0 \quad k = 1,2,\ldots,n \tag{4.34}$$

The last step is to approximate the performance index J. To do this, (4.20) and (4.26) are rewritten in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0.5a_0^{(1)} & a_1^{(1)} & \ldots & a_N^{(1)} \\ 0.5a_0^{(2)} & a_1^{(2)} & \ldots & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0.5a_0^{(n)} & a_1^{(n)} & \ldots & a_N^{(n)} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} \tag{4.35}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} 0.5b_0^{(1)} & b_1^{(1)} & \ldots & b_N^{(1)} \\ 0.5b_0^{(2)} & b_1^{(2)} & \ldots & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0.5b_0^{(m)} & b_1^{(m)} & \ldots & b_N^{(m)} \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_N \end{bmatrix} \tag{4.36}$$

or in compact form

$$x^{[k]} = \alpha T \qquad u^{[k]} = \beta T \tag{4.37}$$

Substituting (4.37) into (4.12)

$$\hat{J} = \frac{t_f}{2}\int_{-1}^{1}(T^T\alpha^T Q\alpha T + T^T\beta^T R\beta T)d\tau \tag{4.38}$$

where $\hat{J}$ is the approximated value of J. By letting $M = \alpha^T Q\alpha$ and $P = \beta^T R\beta$ and noting that both matrices M and P are symmetrical, Jaddu [8] derived an explicit formula for the approximated performance index $\hat{J}$

$$\hat{J} = t_f \sum_{i=1}^{N+1-k}\frac{1}{2}\left(\acute{p}_{i,i+k} + \acute{m}_{i,i+k}\right)\left(\frac{-2}{(2i-2+k)^2} + \frac{-2}{k^2-1}\right) \tag{4.39}$$

where

$$\acute{p}_{i,i+k} = \begin{cases} p_{i,i+k} & k \neq 0 \\ \frac{p_{ii}}{2} & k = 0 \end{cases} \tag{4.40}$$

$$\acute{m}_{i,i+k} = \begin{cases} m_{i,i+k} & k \neq 0 \\ \frac{m_{ii}}{2} & k = 0 \end{cases} \tag{4.41}$$

where $k = 0,2,4,\dots,N\ (N\ even) or\ N-1 (N\ \ odd)$ and $p_{i,j}$, $m_{i,j}$ are the elements of the symmetrical matrices P and M respectivly.

(4.39) can be expressed as,

$$\hat{J} = \frac{1}{2}a^T H a \tag{4.42}$$

where $a^T = [a_0^{(1)} a_1^{(1)} \dots a_N^{(1)}\ a_0^{(2)} a_1^{(2)} \dots a_N^{(2)} \dots a_0^{(q)} \dots a_N^{(q)}]$ is the unknown parameter vector and H is the positive definite [8] Hessian matrix which can be found using (3.42).

The nonlinear quadratic optimal control problem (4.1)–(4.2) is converted into parameters optimization problem which is quadratic in the unknown parameters and the new problem can be stated as,

$$min_a \quad \frac{1}{2}a^T H a \tag{4.43}$$

subject to the linear constraints

$$Fa - b = 0 \tag{4.44}$$

where the linear constraints are due to initial conditions and in some cases are due to some system state equations which are not satisfied. The optimal values of the unknown parameters vector a^* can be obtained from (3.45).

The steps required to solve the nonlinear quadratic optimal control problem can be summarized in the flow chart of figure (4.1).

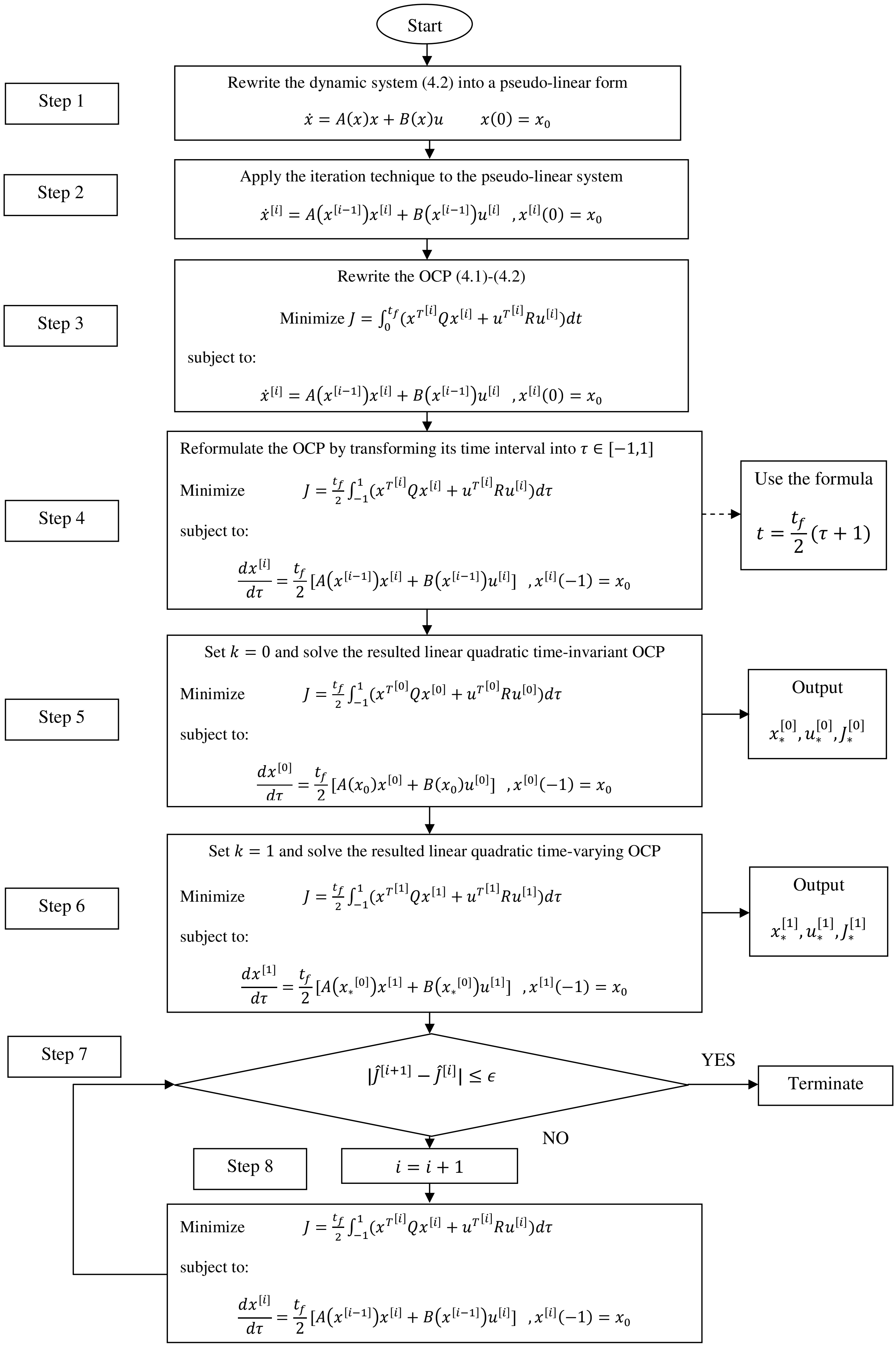


Figure (4.1) Flow chart for solving nonlinear quadratic OCP

4.5 Computation Results

In this section, the simulation results of a Van der Pol oscillator problem are illustrated. The simulation was carried out using MATLAB software package and the problem was solved using Chebyshev and Legendre polynomials. This problem is stated as follows:

Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt \tag{4.45}$$

subject to

$$\dot{x}_1 = x_2 \qquad , x_1(0) = 1 \tag{4.46}$$

$$\dot{x}_2 = -x_1 + x_2 - x_1^2 x_2 + u \quad , x_2(0) = 0 \tag{4.47}$$

Using the technique in section (4.3), the problem can be reformulated as

Minimize

$$J^{[i]} = \frac{1}{2}\int_0^5 (\left(x_1^{[i]}\right)^2 + \left(x_2^{[i]}\right)^2 + \left(u^{[i]}\right)^2)dt \tag{4.48}$$

subject to

$$\dot{x}_1{}^{[i]} = x_2{}^{[i]} \qquad , x_1{}^{[i]}(0) = 1 \tag{4.49}$$

$$\dot{x}_2{}^{[i]} = -x_1{}^{[i]} + \left(1 - \left(x_1^{[i-1]}\right)^2\right) x_2{}^{[i]} + u^{[i]} \quad , x_2{}^{[i]}(0) = 0 \tag{4.50}$$

and for $i = 0$

Minimize

$$J^{[0]} = \frac{1}{2}\int_0^5 (\left(x_1^{[0]}\right)^2 + \left(x_2^{[0]}\right)^2 + \left(u^{[0]}\right)^2)dt \tag{4.51}$$

subject to

$$\dot{x}_1{}^{[0]} = x_2{}^{[0]} \qquad , x_1{}^{[0]}(0) = 1 \tag{4.52}$$

$$\dot{x}_2{}^{[0]} = -x_1{}^{[0]} + u^{[0]} \quad , x_2{}^{[0]}(0) = 0 \tag{4.53}$$

After changing the time interval $t \in [0,5]$ into the time interval $\tau \in [-1,1]$, $x_1(\tau)$ is approximated by a 9th order Chebyshev (Legendre) series, $x_2(\tau)$ is determined from (4.49) while $u(\tau)$ is determined from (4.50). Starting from the linear quadratic time-invariant problem (4.51)-(4.53) which will be considered as the starting nominal trajectories, the linear quadratic time-varying optimal control problems (4.48)-(4.50) are solved for $i = 5$ iterations. Table (4.1) illustrates the results of the optimal values of the cost function J versus iteration i for Chebyshev and Legendre polynomial.

Table (4.1) J vs. i for Chebyshev and Legendre Polynomials

Iteration i	J			
	N = 9		N = 15	
	Chebyshev	Legendre	Chebyshev	Legendre
0	0.9533622676	0.9533638119	0.9533622624	0.9533622622
1	1.4516540238	1.4515573659	1.4515286014	1.4515286441
2	1.4497313124	1.4496686129	1.4496981112	1.4496983487
3	1.4493918353	1.4493656156	1.4493287644	1.4493288574
4	1.4494606241	1.4494054733	1.4494047276	1.4494048523
5	1.4494528889	1.4494004218	1.4493959719	1.4493960944

Note that the optimal values of J using Legendre polynomials are relatively smaller than that of using Chebyshev polynomials. But, the convergence speed using Chebyshev polynomials is much faster than that of Legendre polynomials.

This problem was solved by Jaddu [8] using quasilinearization and state parameterization via Chebyshev polynomials and J was found to be 1.433487. J was found by Bullock and Franklin [27] to be 1.433508 using the second variation method. Bashein and Enns [28] found J to 1.438097 using quasilinearization and discretization.

Figure (4.2) and (4.3) shows the optimal trajectories of the Van der Pol oscillator problem.

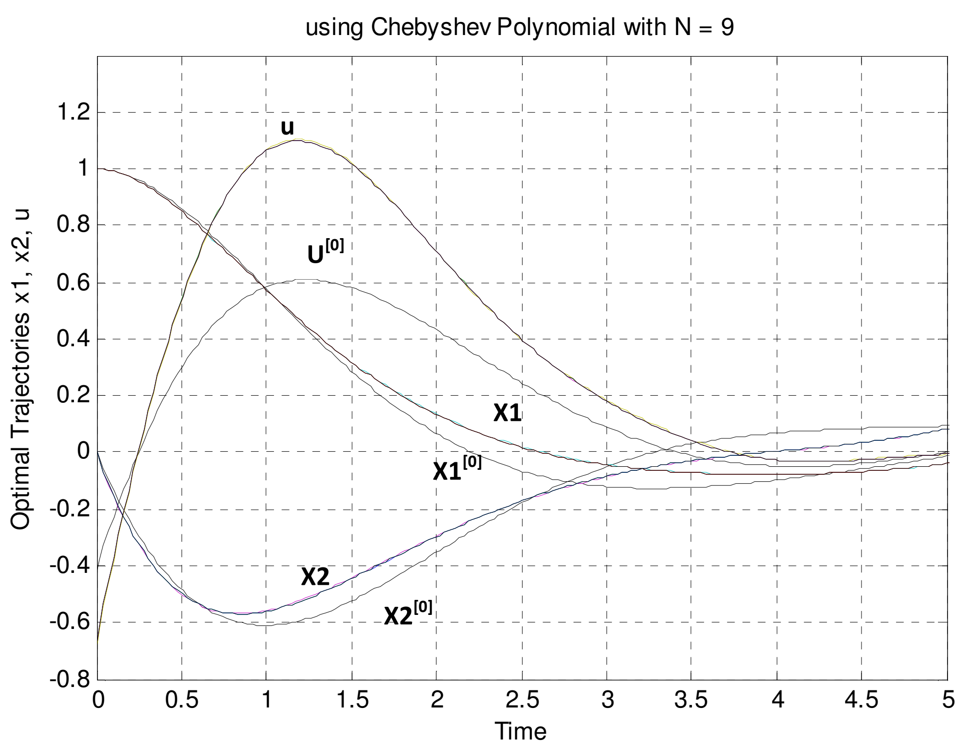


Figure (4.2) Optimal trajectories of Van der Pol problem using Chebyshev polynomials

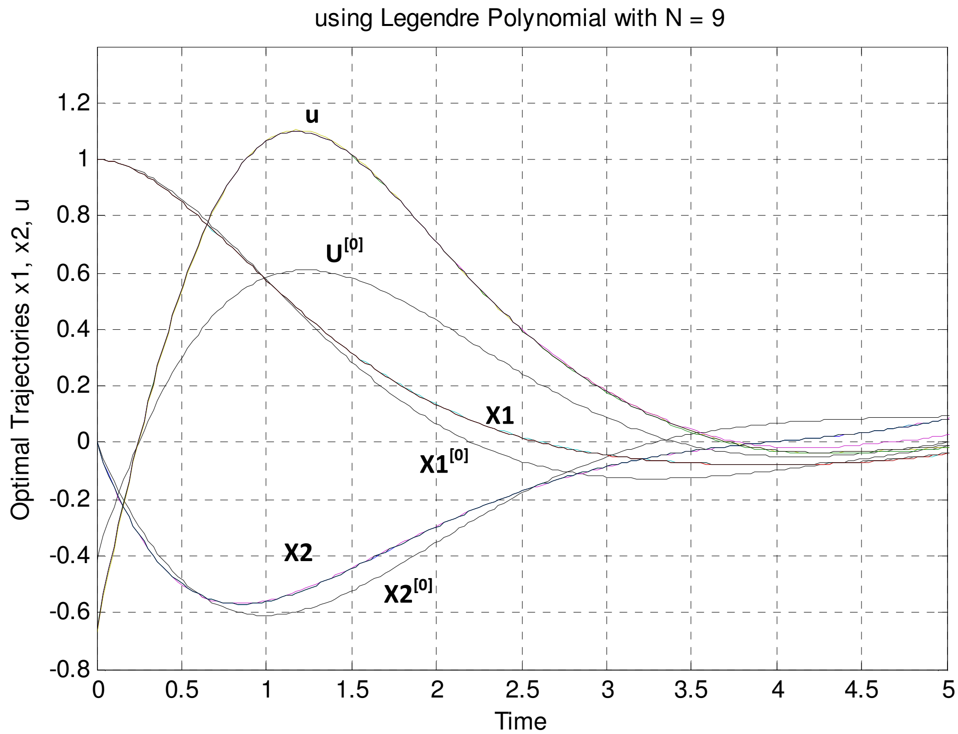


Figure (4.3) Optimal trajectories of Van der Pol problem using Legendre polynomials

4.6 Conclusion

In this chapter, a numerical technique for solving the unconstrained nonlinear quadratic optimal control problem was presented. This method is based on using the iteration technique in combination with state parameterization via Legendre or Chebyshev polynomials to convert the difficult nonlinear quadratic optimal control problem into a sequence of linear quadratic time-varying optimal control problems which are much easier to solve. The simulation results carried out using MATLAB software shows the effectiveness of the proposed method. The most important property of this method compared with other methods is its simplicity.

The method proposed in this chapter will be extended in the next chapter to handle different types of constraints imposed on the dynamic system under consideration.

Chapter Five

Constrained Nonlinear Quadratic Optimal Control Problem

5.1 Introduction

Up to this point, the optimal control problems treated in this work were free of constraints. However, in practical applications optimal control problems are usually subject to different types of constraints. Usually, optimal control problems constraints can be classified as follows:

1. Control saturation constraints

$$U_{min} \leq u(t) \leq U_{max} \tag{5.1}$$

2. State saturation constraints

$$X_{min} \leq x(t) \leq X_{max} \tag{5.2}$$

3. Terminal state constraints

$$\varphi\big(x(t_f), t_f\big) = 0 \tag{5.3}$$

4. Interior point constraints

$$\mu(x(t_i), t_i) = 0 \qquad 0 \leq t_i \leq t_f \tag{5.4}$$

5. Equality constraints on functions of the state and/or the control variables

$$F(x(t), u(t), t) = 0 \tag{5.5}$$

6. Inequality constraints on functions of the state and/or the control variables

$$F(x(t), u(t), t) \leq 0 \tag{5.6}$$

Existence of constraints imposed on the dynamic system usually complicate the optimal control problem. Many text books and papers were published to handle optimal control problems subject to different types of constraints. Examples of text books that theoretically treats constrained imposed on the dynamic system are [1-2]. On the other hand, [21] is survey paper that review most of the computational methods that were published to handle such problems.

Direct methods that can be employed by either discretization or parameterization are widely used to convert the constrained optimal control problem to a mathematical programming problem. Many direct methods have been proposed to handle the constrained nonlinear optimal control problems. For example, Vlassenbroeck [13] used the control-state parameterization using Chebyshev polynomials to handle the constrained nonlinear optimal control problem. Frick and Stech [22] used the Walsh functions to solve the nonlinear optimal control problems subject to saturation constraints. Goh and Teo [21], Troch *et al.* [23] proposed a method to handle the constrained nonlinear optimal control problems using control parameterization. Jaddu [8, 16, 24] proposed a method that is based on the second method of quasilinearization and the state parameterization via Chebyshev polynomials to handle the nonlinear optimal control problems subject to state and control saturation constraints.

In this chapter, we will extend the method proposed in the previous chapter to handle the constrained nonlinear quadratic optimal control problem. Different types of constrained imposed on the dynamic system will be considered. As in the previous chapter, the basic idea is to use the iteration technique and the state parameterization via Legendre or Chebyshev polynomials to replace the constrained nonlinear quadratic optimal control problem by a sequence of constrained linear quadratic time-varying optimal control problems, which are easier to solve.

5.2 Statement of the Constrained Nonlinear Quadratic Optimal Control Problem

The optimal control problem treated in this chapter can be stated as follows: Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = x(t_f)^T S x(t_f) + \int_0^{t_f} (x^T Q x + u^T R u) dt \tag{5.7}$$

subject to the following constraints:

- Dynamic system state equations and initial conditions

$$\dot{x} = f(x(t), u(t), t) \qquad x(0) = x_0 \tag{5.8}$$

- Terminal state constraints

$$\varphi(x(t_f), t_f) = 0 \tag{5.9}$$

- Saturation state and control constraints

$$x(t) \leq X_{max} \quad x(t) \geq X_{min} \quad , u(t) \leq U_{max} \quad u(t) \geq U_{min} \tag{5.10}$$

where Q, S are positive semidefinite matrices, R is a positive definite matrix, $x \in R^n$ is the state vector, R^m is the control vector, $x_0 \in R^n$ is the initial condition vector, f is a nonlinear continuous differentiable function with respect to all its arguments $(x(t), u(t), t)$. We will assume that: $m \leq n$, $X_{max}, X_{min}, U_{min}$ and U_{max} are scalar quantities and t_f is fixed.

This problem will be solved by converting the constrained nonlinear quadratic optimal control problem into a sequence of constrained linear quadratic programming problems, which are easier to solve. The solution is based on using the iteration technique; which will replace the nonlinear dynamic state equations into equivalent linear time-varying state equations. Then, each of these equations will be solved by converting it into a quadratic programming problem by using state parameterization via Chebyshev or Legendre polynomials.

5.3 Proposed Method

5.3.1 Iteration Technique:

Applying the iteration technique described in the previous chapter on the optimal control problem (5.7)-(5.10), the optimal control problem under consideration can be restated as follows:

Minimizes

$$J^{[0]} = x(t_f)^{[0]^T} S x(t_f)^{[0]} + \int_0^{t_f} \left(x^{[0]^T} Q x^{[0]} + u^{[0]^T} R u^{[0]} \right) dt \qquad (5.11)$$

subject to the state equations and initial conditions

$$\dot{x}^{[0]} = A(x_0) x^{[0]} + B(x_0) u^{[0]}, \quad x^{[0]}(0) = x_0 \qquad (5.12)$$

and to the following terminal constraints

$$\varphi\left(x(t_f)^{[0]}, t_f \right) = 0 \qquad (5.13)$$

and to the following control saturation constraints

$$x(t)^{[0]} \leq X\_max \quad x(t)^{[0]} \geq X\_min \quad , u(t)^{[0]} \leq U\_max \quad u(t)^{[0]} \geq U\_min \qquad (5.14)$$

And for $i \geq 1$

Minimizes

$$J^{[i]} = x(t_f)^{[i]^T} S x(t_f)^{[i]} + \int_0^{t_f} \left(x^{[i]^T} Q x^{[i]} + u^{[i]^T} R u^{[i]} \right) dt \qquad (5.15)$$

subject to the state equations and initial conditions

$$\dot{x}^{[i]} = A\left(x^{[i-1]}\right)x^{[i]} + B\left(x^{[i-1]}\right)u^{[i]}, \quad x^{[i]}(0) = x_0 \tag{5.16}$$

and to the following terminal constraints

$$\varphi\left(x\left(t_f\right)^{[i]}, t_f\right) = 0 \tag{5.17}$$

and to the following control saturation constraints

$$x(t)^{[i]} \leq X\_max \quad x(t)^{[i]} \geq X\_min \quad , u(t)^{[i]} \leq U\_max \quad u(t)^{[i]} \geq U\_min \tag{5.18}$$

5.3.2 State parameterization:

The constrained linear time-varying quadratic optimal control problems (5.11)-(5.14) and (5.15)-(5.18) can be solved by converting them into a quadratic programming problems and then solve them using an available software like MATLAB. To achieve this purpose, we will use Legendre polynomials of the first kind or Chebyshev polynomials of the first type described in previous chapters to parameterize the state variables. Then, the optimal control problem under consideration can be reformulated as follows:

- System state equation parameterization:

  Using the information presented in the previous chapters about state parameterization, the state and control variables can be approximated via Legendre polynomials of finite length N with unknown parameters as follows:

$$x_j^{[i]} = \sum_{i=0}^{N} a_i^{(j)} P_i(\tau) \quad j = 1,2,..,n \tag{5.19}$$

$$u_l^{[i]} = \sum_{i=0}^{N} b_i^{(l)} P_i(\tau) \quad l = 1,2,..,m \tag{5.20}$$

  where $i = 0,1,2, \ldots$ is the iteration sequence number, a's and b's are the unknown parameters, and $P(\tau)$ is a vector of Legendre polynomials of the first kind. (5.19)-(5.20) can be rewritten in matrix form as follows:

$$x^{[i]} = \alpha P(\tau) \tag{5.21}$$

$$u^{[i]} = \beta P(\tau) \tag{5.22}$$

  where:

$$\alpha = \begin{bmatrix} a_0^{(1)} & a_1^{(1)} & \ldots & a_N^{(1)} \\ a_0^{(2)} & a_1^{(2)} & \ldots & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_0^{(n)} & a_1^{(n)} & \ldots & a_N^{(n)} \end{bmatrix} \tag{5.23}$$

$$\beta = \begin{bmatrix} b_0^{(1)} & b_1^{(1)} & \dots & b_N^{(1)} \\ b_0^{(2)} & b_1^{(2)} & \dots & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ b_0^{(m)} & b_1^{(m)} & \dots & b_N^{(m)} \end{bmatrix} \tag{5.24}$$

and $P(\tau) = [P_0(\tau)\ P_1(\tau) \dots P_N(\tau)]^T$.

The same idea can be applied using Chebyshev polynomials of the first type, with (5.19)-(5.20) being replaced by

$$x_j^{[i]} = \frac{a_0{}^{(j)}}{2} + \sum_{i=1}^{N} a_i^{(j)} T_i(\tau) \quad j = 1,2,..,n \tag{5.25}$$

$$u_l^{[i]} = \frac{b_0{}^{(l)}}{2} + \sum_{i=1}^{N} b_i^{(l)} T_i(\tau) \quad l = 1,2,..,m \tag{5.26}$$

Again, (5.25)-(5.26) can be rewritten in matrix form as follows:

$$x^{[i]} = \alpha T(\tau) \tag{5.27}$$

$$u^{[i]} = \beta T(\tau) \tag{5.28}$$

where:

$$\alpha = \begin{bmatrix} 0.5a_0^{(1)} & a_1^{(1)} & \dots & a_N^{(1)} \\ 0.5a_0^{(2)} & a_1^{(2)} & \dots & a_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0.5a_0^{(n)} & a_1^{(n)} & \dots & a_N^{(n)} \end{bmatrix} \tag{5.29}$$

$$\beta = \begin{bmatrix} 0.5b_0^{(1)} & b_1^{(1)} & \dots & b_N^{(1)} \\ 0.5b_0^{(2)} & b_1^{(2)} & \dots & b_N^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ 0.5b_0^{(m)} & b_1^{(m)} & \dots & b_N^{(m)} \end{bmatrix} \tag{5.30}$$

where $T(\tau) = [T_0(\tau)\ T_1(\tau) \dots T_N(\tau)]^T$ is a vector of Chebyshev polynomials of the first type.

- Performance index parameterization:

The next step in converting the optimal control problem under consideration into a standard programming problem is to approximate the performance index (5.11) and (5.15). As indicated in previous chapters, both Legendre and Chebyshev polynomials are defined on the time interval $\tau \in [-1,1]$. For this, it is necessary first to change the time interval of the optimal control problem at hand $t \in [0, t_f]$ into $\tau \in [-1,1]$. This can be done using the transformation equation presented in previous chapters.

The resulted performance index after the time interval switching is:

$$J^{[0]} = x(1)^{[0]^T} S x(1)^{[0]} + \frac{t_f}{2} \int_{-1}^{1} \left(x^{[0]^T} Q x^{[0]} + u^{[0]^T} R u^{[0]} \right) d\tau \tag{5.31}$$

and for $i \geq 1$

$$J^{[i]} = x(1)^{[i]^T} S x(1)^{[i]} + \frac{t_f}{2} \int_{-1}^{1} \left(x^{[i]^T} Q x^{[i]} + u^{[i]^T} R u^{[i]} \right) d\tau \tag{5.32}$$

Substituting (5.21) and (5.22) in (5.32) yields

$$\hat{J}^{[i]} = [P^T(1)\alpha^T S \alpha P(1)] + \frac{t_f}{2} \int_{-1}^{1} (P^T(\tau)\alpha^T Q \alpha P(\tau) + P^T(\tau)\beta^T R \beta P(\tau)) d\tau \tag{5.33}$$

where $\hat{J}^{[i]}$ is the approximated performance index of $J^{[i]}$. Using the final value property of Legendre polynomials at $\tau = 1$, (5.33) can be rewritten as

$$\hat{J}^{[i]} = \alpha^T S \alpha + \frac{t_f}{2} \int_{-1}^{1} (P^T(\tau)\alpha^T Q \alpha P(\tau) + P^T(\tau)\beta^T R \beta P(\tau)) d\tau \tag{5.34}$$

The integration part of (5.34) can be obtained using theorem 1 of chapter three, and therefore (5.34) can be reformulated and written as

$$\hat{J}^{[i]} = \alpha^T S \alpha + \frac{t_f}{2} \sum_{i=1}^{N+1} \frac{2}{2i-1} (m_{ii} + z_{ii}) \tag{5.35}$$

where m_{ii} and z_{ii} are the diagonal elements of the symmetrical matrices $M = \alpha^T Q \alpha$ and $Z = \beta^T R \beta$ respectively.

The same procedure can be applied to approximate the performance index using Chebyshev polynomials of the first type. Substituting (5.27) and (5.28) into (5.32) yields

$$\hat{J}^{[i]} = [T^T(1)\alpha^T S \alpha T(1)] + \frac{t_f}{2} \int_{-1}^{1} (T^T(\tau)\alpha^T Q \alpha T(\tau) + T^T(\tau)\beta^T R \beta T(\tau)) d\tau \tag{5.36}$$

Again, using the final value property of Chebyshev polynomials at $\tau = 1$, (5.36) can be rewritten as follows

$$\hat{J}^{[i]} = \alpha^T S \alpha + \frac{t_f}{2} \int_{-1}^{1} (T^T(\tau)\alpha^T Q \alpha T(\tau) + T^T(\tau)\beta^T R \beta T(\tau)) d\tau \tag{5.37}$$

The integration part of (3.37) can be obtained using the result obtained by Jaddu [8], (5.37) can be rewritten as

$$\hat{J}^{[i]} = \alpha^T S \alpha + t_f \sum_{i=1}^{N+1-k} \frac{1}{2} \left(\acute{p}_{i,i+k} + \acute{m}_{i,i+k} \right) \left(\frac{-2}{(2i-2+k)^2} + \frac{-2}{k^2-1} \right) \tag{5.38}$$

where

$$\acute{p}_{i,i+k} = \begin{cases} p_{i,i+k} & k \neq 0 \\ \frac{p_{ii}}{2} & k = 0 \end{cases} \quad (5.39)$$

$$\acute{m}_{i,i+k} = \begin{cases} m_{i,i+k} & k \neq 0 \\ \frac{m_{ii}}{2} & k = 0 \end{cases} \quad (5.40)$$

where $k = 0,2,4,\dots,N\ (N\ even) or\ N-1 (N\ \ odd)$ and $p_{i,j}$, $m_{i,j}$ are the elements of the symmetrical matrices $P = \beta^T R\beta$ and $M = \alpha^T Q\alpha$ respectively.

- Initial and terminal state constraints approximation:

  Using the initial value property of Legendre polynomials at $\tau = -1$, the initial condition vector can be approximated as follows

$$a_0^{(j)} - a_1^{(j)} + a_2^{(j)} - a_3^{(j)} + \cdots + (-1)^N a_N^{(j)} - x_j(-1) = 0 \quad j = 1,2,\dots,n \quad (5.41)$$

  where $x_j(-1) = x_0$. Using the same property, it is possible to approximate the initial condition vector using Chebyshev polynomials as follows

$$\frac{a_0^{(j)}}{2} - a_1^{(j)} + a_2^{(j)} - a_3^{(j)} + \cdots + (-1)^N a_N^{(j)} - x_j(-1) = 0 \quad j = 1,2,\dots,n \quad (5.42)$$

  where $x_j(-1) = x_0$.

  The same procedure can be applied to approximate the terminal state vector. By using the final value property of Legendre polynomials at $\tau = 1$, the following approximation of the terminal state vector can be obtained

$$a_0^{(j)} + a_1^{(j)} + a_2^{(j)} + a_3^{(j)} + \cdots + a_N^{(j)} - x_j(1) = 0 \quad j = 1,2,\dots,n \quad (5.43)$$

  and using Chebyshev polynomials

$$\frac{a_0^{(j)}}{2} + a_1^{(j)} + a_2^{(j)} + a_3^{(j)} + \cdots + a_N^{(j)} - x_j(1) = 0 \quad j = 1,2,\dots,n \quad (5.44)$$

  where $x_j(1) = x(T)$.

- Saturation control and state constraints:

  Many methods have been proposed to handle saturation constraints on state or control variables. One method is to add a slack variable to the inequality constraint to convert them into equality constraints. This method was used by [13]. However, using this method would produce two drawbacks: The first is adding a slack variable would convert the linear problem into a nonlinear one, while the second drawback is the increase in the system dimension due the increase of the unknown parameters that is resulted from parameterizing the extra slack variable.

Another method used by [8, 25, 26] is to discretize the time interval $\tau \in [-1,1]$ with $r+1$ discrete points, and satisfy the constraints at each point. By this, every continuous constraint is replaced by $r+1$ constraints. To avoid the drawbacks of the slack variables method, we will adopt this method in this work.

Mathematically, the time interval $\tau \in [-1,1]$ is discretized as follows

$$-1 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_r = 1 \tag{5.45}$$

Therefore, each of the continuous control saturation constraints is replaced by $r+1$ finite dimension inequality constraints. Using Legendre polynomials, the $r+1$ constraints are given by

$$\sum_{i=0}^{N} b_i^{(l)} P_i(\tau_s) \leq U_{max} \tag{5.46}$$

$$-\sum_{i=0}^{N} b_i^{(l)} P_i(\tau_s) \leq -U_{min} \tag{5.47}$$

whereas the state saturation constraints are given by

$$\sum_{i=0}^{N} a_i^{(j)} P_i\,(\tau_s) \leq X_{max} \tag{5.48}$$

$$-\sum_{i=0}^{N} a_i^{(j)} P_i\,(\tau_s) \leq -X_{min} \tag{5.49}$$

Using Chebyshev polynomials, the control saturation constraints are given by

$$\frac{b_0^{(l)}}{2} + \sum_{i=0}^{N} b_i^{(l)} T_i(\tau_s) \leq U_{max} \tag{5.50}$$

$$-\frac{b_0^{(l)}}{2} - \sum_{i=0}^{N} b_i^{(l)} T_i(\tau_s) \leq -U_{min} \tag{5.51}$$

and the state saturation constraints are giving by

$$\frac{a_0^{(l)}}{2} + \sum_{i=0}^{N} a_i^{(l)} T_i(\tau_s) \leq X_{max} \tag{5.52}$$

$$-\frac{a_0^{(l)}}{2} - \sum_{i=0}^{N} a_i^{(l)} T_i(\tau_s) \leq -X_{min} \tag{5.53}$$

where $s = 0,1,2,\ldots,r$.

The difficult constrained nonlinear quadratic optimal control problem is converted into a standard constrained quadratic programming problem that can be restated as follows:

$$min_a \quad \frac{1}{2} a^T H a \tag{5.54}$$

subject to

$$F_1 a = b_1 \tag{5.55}$$

$$F_2 a \leq b_2 \tag{5.56}$$

where the equality constraints are due to initial conditions, terminal state constraints, and in some cases unsatisfied state equations. While the inequality constraints are due to saturation constraints of the control and/or the state variables.

The standard quadratic programming problem (5.55)-(5.56) can be solve using any available software package. In this work, we use the active set method [18] in MATLAB software to solve this problem.

5.4 Computation results

In this section, we will revisit the known Van der Pol oscillator problem solved in the previous chapter free of constraints.

The dynamic state equations are:

$$\dot{x}_1 = x_2 \qquad x_1(0) = 1 \tag{5.57}$$

$$\dot{x}_2 = -x_1 + (1 - x_1^2)x_2 + u \qquad x_2(0) = 0 \tag{5.58}$$

The performance index to be minimized is

$$J = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2)dt \tag{5.59}$$

This time, we will consider two cases imposed on this problem, namely: Terminal state constraints problem and saturation control constraints problem.

- First case problem: Terminal state constraints

$$x_1(5) - x_2(5) = -1 \tag{5.60}$$

Based on the proposed solution in this chapter and on the solution of the unconstrained problem given in the previous chapter, this problem can be reformulated and stated as:

Find an optimal control $u^*(t)$ that minimizes the following performance index:

$$J^{[i]} = \frac{5}{4}\int_{-1}^{1} ((x_1^{[i]})^2 + (x_2^{[i]})^2 + (u^{[i]})^2)d\tau \tag{5.61}$$

subject to:

$$\frac{dx_1{}^{[i]}}{d\tau} = \frac{5}{2}x_2{}^{[i]} \qquad , x_1{}^{[i]}(-1) = 1 \tag{5.62}$$

$$\frac{dx_2{}^{[i]}}{d\tau} = \frac{5}{2}[-x_1{}^{[i]} + \left(1 - \left(x_1^{[i-1]}\right)^2\right)x_2{}^{[i]} + u^{[i]}\ ] \ , x_2{}^{[i]}(-1) = 0 \tag{5.63}$$

$$x_1{}^{[i]}(1) - x_2{}^{[i]}(1) = -1 \tag{5.64}$$

and for $i = 0$

$$J^{[0]} = \frac{5}{4}\int_{-1}^{1}(\left(x_1^{[0]}\right)^2 + \left(x_2^{[0]}\right)^2 + \left(u^{[0]}\right)^2)d\tau \tag{5.65}$$

subject to:

$$\frac{dx_1{}^{[0]}}{d\tau} = \frac{5}{2}x_2{}^{[0]} \qquad , x_1{}^{[0]}(-1) = 1 \tag{5.66}$$

$$\frac{dx_2{}^{[0]}}{d\tau} = \frac{5}{2}[-x_1{}^{[0]} + u^{[0]}] \quad , x_2{}^{[0]}(-1) = 0 \tag{5.67}$$

$$x_1{}^{[0]}(1) - x_2{}^{[0]}(1) = -1 \tag{5.68}$$

This problem were treated by Bullock and Franklin [27] using the second order method and J was found to be 1.6857 in seven iteration, while Bashein and Enns [28] found J to be 1.6905756 in five iteration. Jaddu [8] solved this problem and J was found to be 1.6857113. We solve this problem by using Chebyshev and Legendre polynomials. Table (5.1) shows the optimal value of J obtained using both Legendre and Chebyshev polynomials versus the iteration number i.

Table (5.1) approximated optimal value J for the first case

Iteration i	J			
	N = 9		N = 15	
	Chebyshev	Legendre	Chebyshev	Legendre
0	1.351722166	1.351722165	1.351722143	1.351722141
1	1.694713687	1.69467671	1.694622084	1.694622067
2	1.702286013	1.702260267	1.702252424	1.702252529
3	1.700999316	1.700974892	1.700940308	1.700940306
4	1.70104079	1.701016636	1.70098762	1.700987637
5	1.701022721	1.70099843	1.700968718	1.700968733

The approximated optimal controller and state trajectories for the first case problem using Chebyshev and Legendre polynomials are shown in figures (5.1) and (5.2)

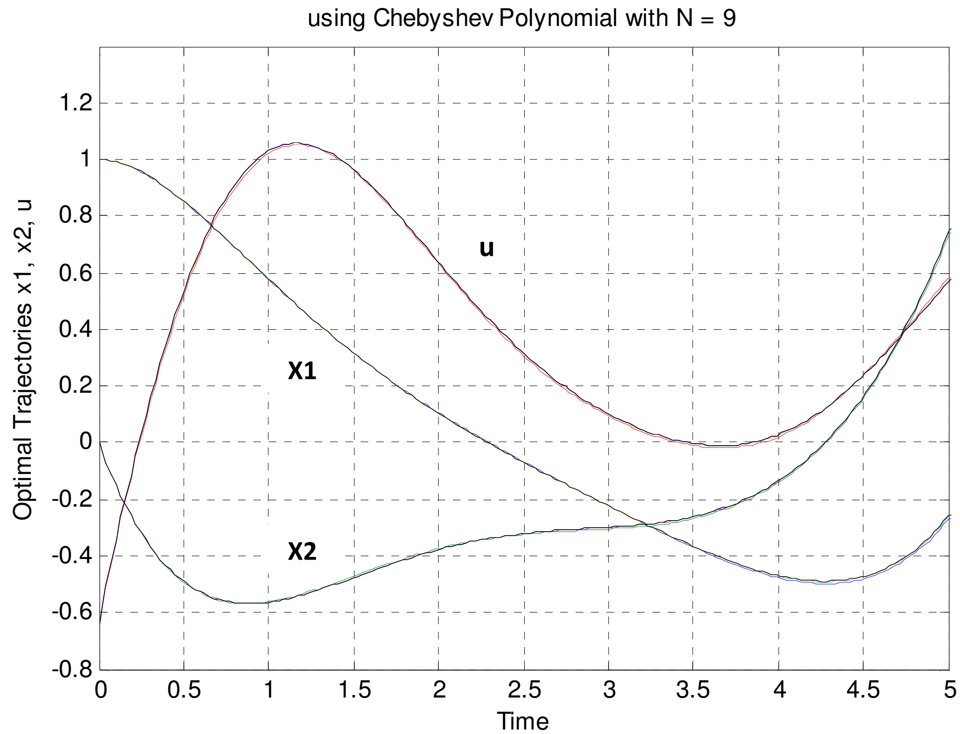


Figure (5.1) Optimal control and state trajectories using Chebyshev, case 1

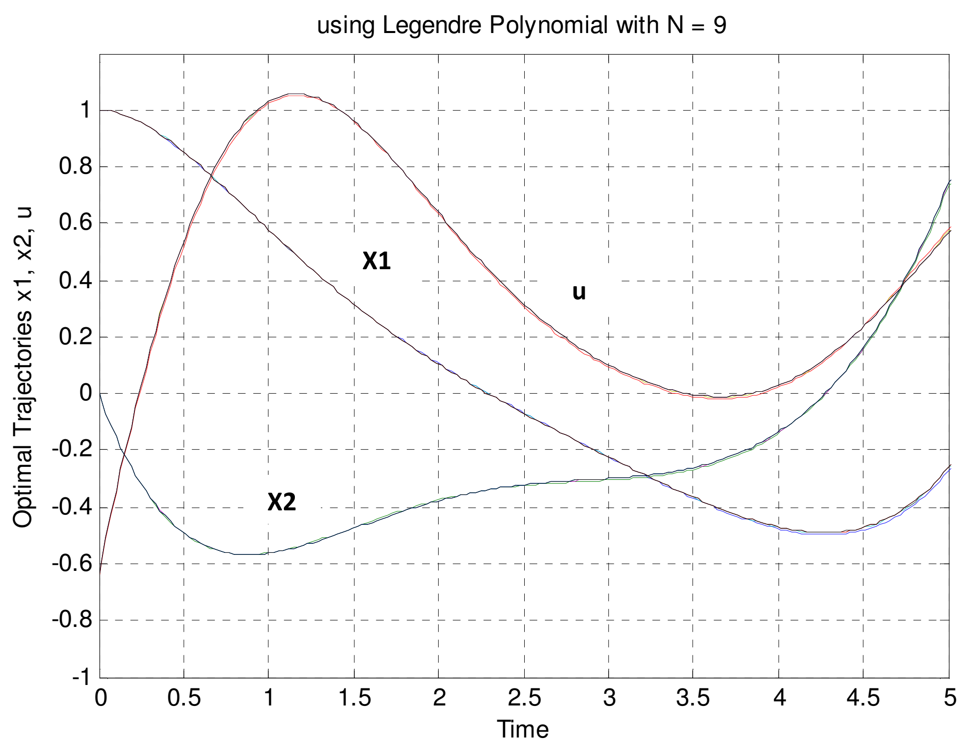


Figure (5.2) Optimal control and state trajectories using Legendre, case 1

- Second case problem: Terminal state constraints and saturation control constraints

$$x_1(5) = -1 \tag{5.69}$$

$$x_2(5) = 0 \tag{5.70}$$

$$|u(t)| \leq 0.75 \tag{5.71}$$

The second case problem can be reformulated and restated using the proposed method as follows:

Find an optimal control $u^*(t)$ that minimizes the following performance index:

$$J^{[i]} = \frac{5}{4}\int_{-1}^{1}(\left(x_1^{[i]}\right)^2 + \left(x_2^{[i]}\right)^2 + \left(u^{[i]}\right)^2)d\tau \tag{5.72}$$

subject to:

$$\frac{dx_1{}^{[i]}}{d\tau} = \frac{5}{2}x_2{}^{[i]} \qquad , x_1{}^{[i]}(-1) = 1 \tag{5.73}$$

$$\frac{dx_2{}^{[i]}}{d\tau} = \frac{5}{2}[-x_1{}^{[i]} + \left(1 - \left(x_1^{[i-1]}\right)^2\right)x_2{}^{[i]} + u^{[i]}\ ] \ \ , x_2{}^{[i]}(-1) = 0 \tag{5.74}$$

$$x_1{}^{[i]}(1) = -1 \tag{5.75}$$

$$x_2{}^{[i]}(1) = 0 \tag{5.76}$$

$$\left|u(\tau)^{[i]}\right| \leq 0.75 \tag{5.77}$$

and for $i = 0$

$$J^{[0]} = \frac{5}{4}\int_{-1}^{1}(\left(x_1^{[0]}\right)^2 + \left(x_2^{[0]}\right)^2 + \left(u^{[0]}\right)^2)d\tau \tag{5.78}$$

subject to:

$$\frac{dx_1{}^{[0]}}{d\tau} = \frac{5}{2}x_2{}^{[0]} \qquad , x_1{}^{[0]}(-1) = 1 \tag{5.79}$$

$$\frac{dx_2{}^{[0]}}{d\tau} = \frac{5}{2}[-x_1{}^{[0]} + u^{[0]}] \quad , x_2{}^{[0]}(-1) = 0 \tag{5.80}$$

$$x_1{}^{[0]}(1) = -1 \tag{5.81}$$

$$x_2{}^{[0]}(1) = 0 \tag{5.82}$$

$$\left|u(\tau)^{[0]}\right| \leq 0.75 \tag{5.83}$$

Again, Bashein and Enns [28] solved this problem, they obtained $J = 2.1439199$, after seven iteration. This problem also solved by Jaddu [8] using quasilinearization and state parameterization and J was found to be 2.1443893 after seven iteration. We solved this problem by the proposed method in which the saturation control constrains is satisfied at 21 equally spaced points in the interval $[-1,1]$, namely

$$\tau = -1, -0.9, -0.8, -0.7, \ldots, 0.7, 0.8, 0.9, 1 \tag{5.84}$$

Table (5.2) shows the results obtained for the approximated optimal performance index using Chebyshev and Legendre polynomials versus the iteration i.

Table (5.2) Optimal performance index J versus i

Iteration i	J			
	N = 9		N = 15	
	Chebyshev	Legendre	Chebyshev	Legendre
0	1.854040815	1.854040831	1.854044103	1.854040796
1	2.380623763	2.381162382	2.29577025	2.29574189
2	2.237530787	2.236824541	2.182965056	2.182873313
3	2.293293339	2.292189099	2.232657937	2.232578054
4	2.264423469	2.263384111	2.211405134	2.211388792
5	2.278922409	2.277806397	2.220946759	2.220823033

Figures (5.3) and (5.4) show the approximated optimal control and state trajectories for Chebyshev based method and Legendre based method for the second case problem.

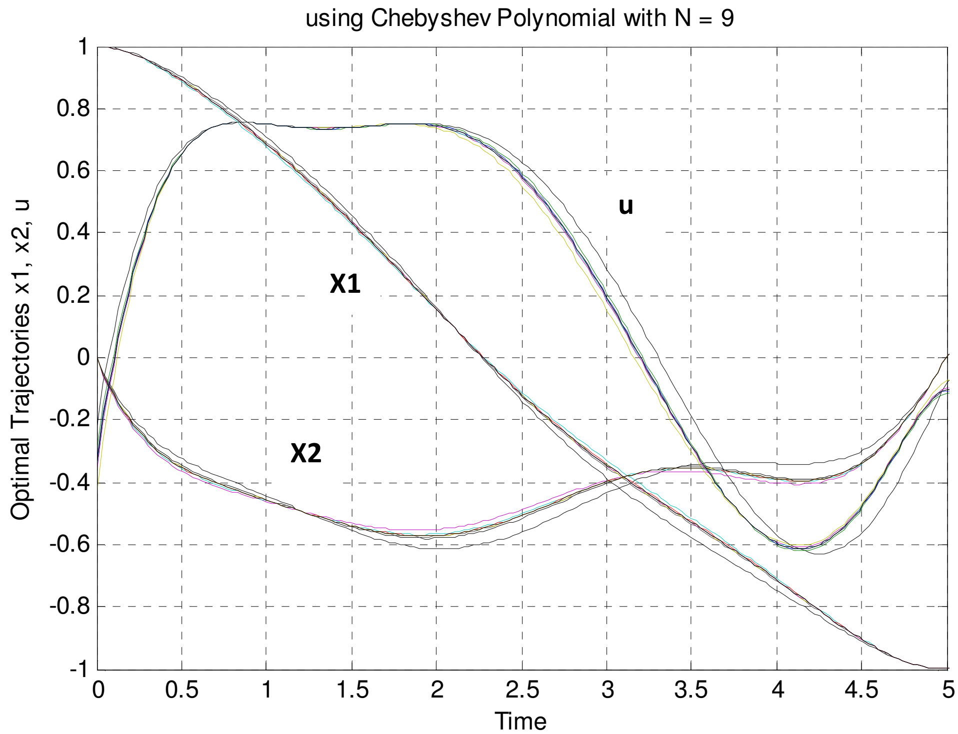


Figure (5.3) Optimal control and state trajectories using Chebyshev, case 2

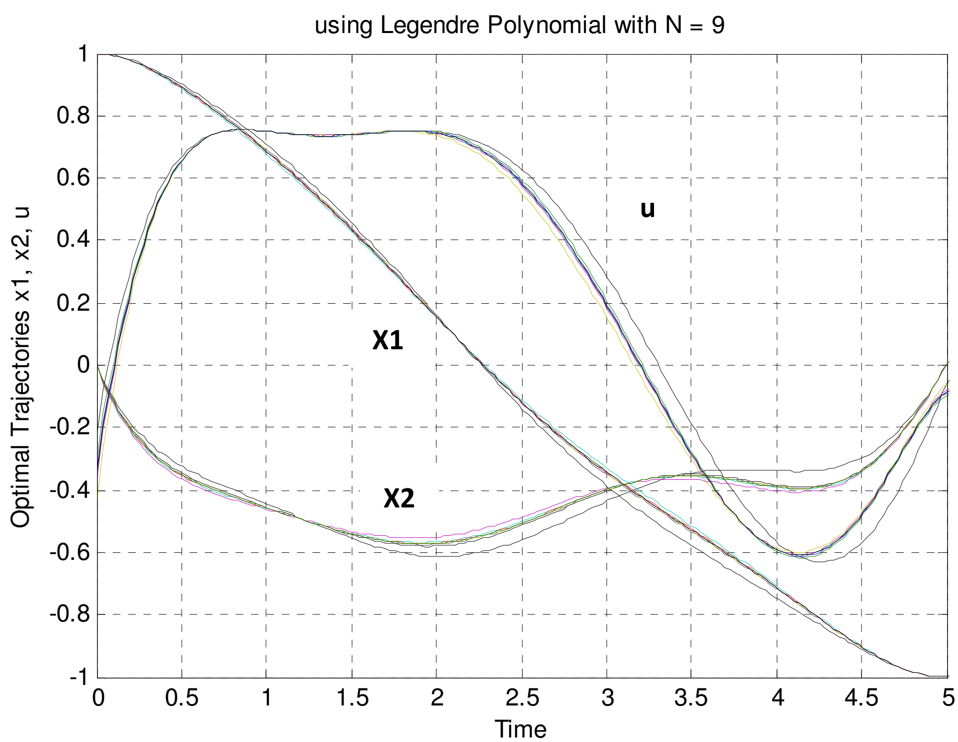


Figure (5.4) Optimal control and state trajectories using Legendre, case 2

5.5 Practical Application

In this section, we will apply the proposed method of this research to a practical and complex problem; the container crane. It is desired to transfer containers at the port of Kobe [29] from a ship to a cargo truck. For safety reasons, the objective is to minimize the swing during and at the end of the transfer operation.

Without going into the complex modeling aspects of this problem, which can be found in details in [29], this problem can be state as follows:

Find an optimal controller $u^*(t)$ that minimizes the following performance index

$$J = \frac{1}{2}\int_0^9 (x_3^2 + x_6^2)dt \qquad (5.85)$$

subject to the following state equations

$$\dot{x}_1 = x_4 \qquad (5.86)$$

$$\dot{x}_2 = x_5 \qquad (5.87)$$

$$\dot{x}_3 = x_6 \qquad (5.88)$$

$$\dot{x}_4 = u_1 + 17.2656x_3 \qquad (5.89)$$

$$\dot{x}_5 = u_2 \qquad (5.90)$$

$$\dot{x}_6 = -\frac{1}{x_2}(u_1 + 27.0756x_3 + 2x_5x_6) \qquad (5.91)$$

where

$$X(0) = [0,22,0,0,-1,0]^T \qquad (5.92)$$

$$X(9) = [10,14,0,2.5,0,0]^T \qquad (5.93)$$

and

$$|u_1(t)| \leq 2.83374 \ \ \forall t \in [0,9] \qquad (5.94)$$

$$-0.80865 \leq u_2(t) \leq 0.71265 \ \ \forall t \in [0,9] \qquad (5.95)$$

with continuous state inequality constraints

$$|x_4(t)| \leq 2.5 \ \ \forall t \in [0,9] \qquad (5.96)$$

$$|x_5(t)| \leq 1 \ \ \forall t \in [0,9] \qquad (5.97)$$

This problem was treated by Sakawa and Shindo [29], but no optimal value was reported. Goh and Teo [18] used a piecewise constant functions to parameterize the control variables and J was found to be 0.005361. They also used a piecewise linear functions to parameterize the control variables and found $J = 0.005412$. Jaddu [8, 24] solved this problem using the second method of quasilinearization and state parameterization using Chebyshev polynomials, and J was found to be 0.00562 after three iterations.

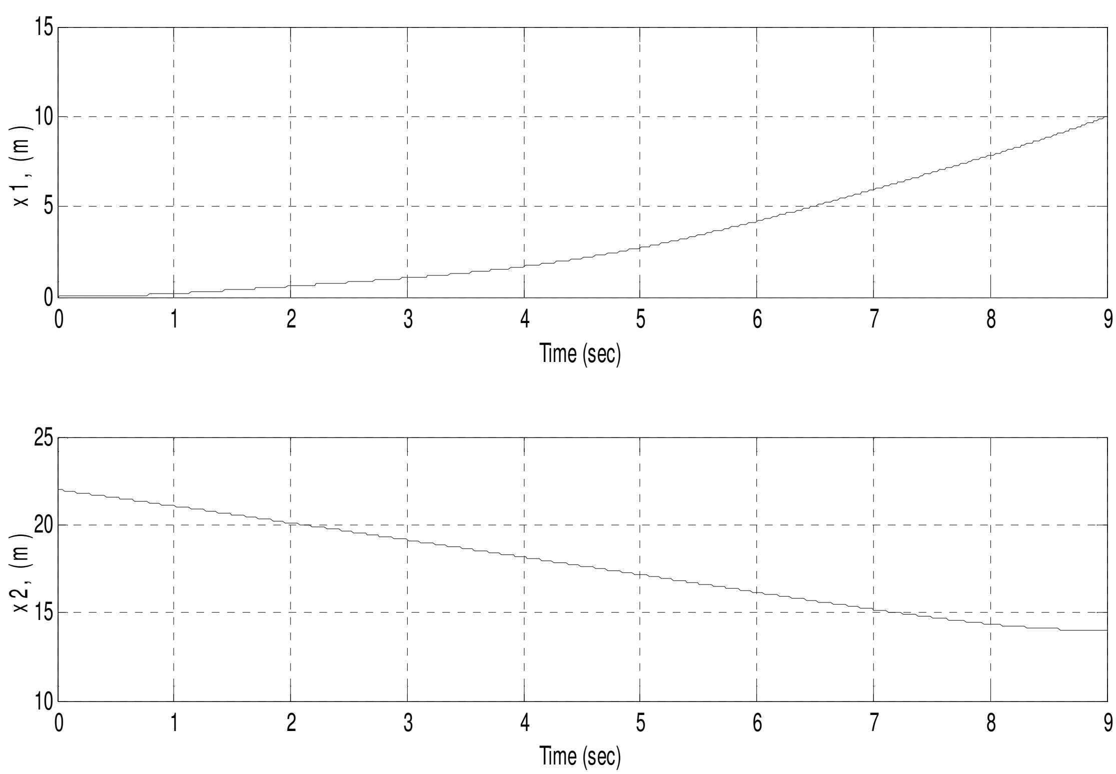
Using the proposed method, the iteration technique was applied and then the state variables x_1, x_2, x_3 were approximated by 9th order Chebyshev (Legendre) series with unknown parameters. The remaining state variables x_4, x_5, x_6 and control variables u_1, u_2 are obtained using the first five state equations. All state equations are directly satisfied except the last equation which will be replaced by $N + 1$ equality constraints.

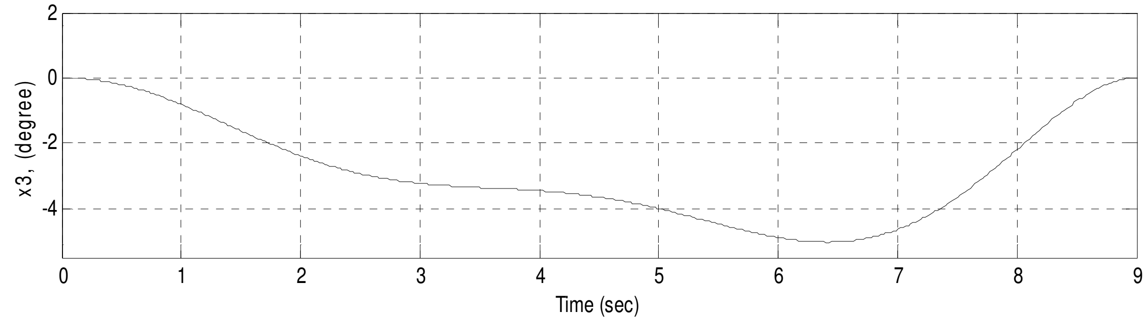
Table (5.3) illustrates the results of simulation carried out on the crane problem using both Chebyshev and Legendre polynomials

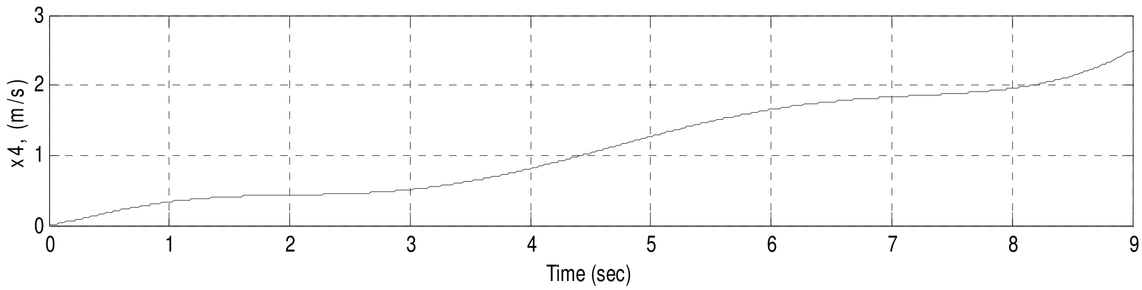
Table (5.3) App. Optimal values for the crane problem

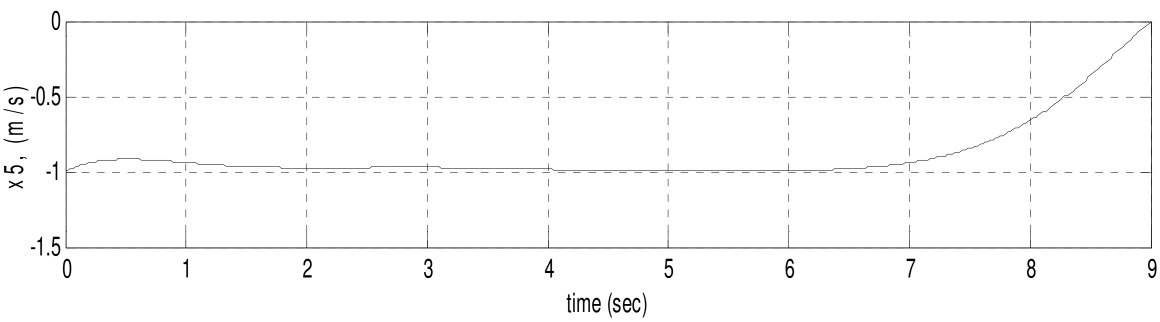
Iteration i	J	
	Chebyshev	Legendre
0	0.00520001274144672	0.005213296926173
1	0.00564480381208082	0.005647797358452
2	0.00564480200191086	0.005647752274546
3	0.00564480041441502	0.005647708914071
4	0.00564479900640596	0.005647667212744

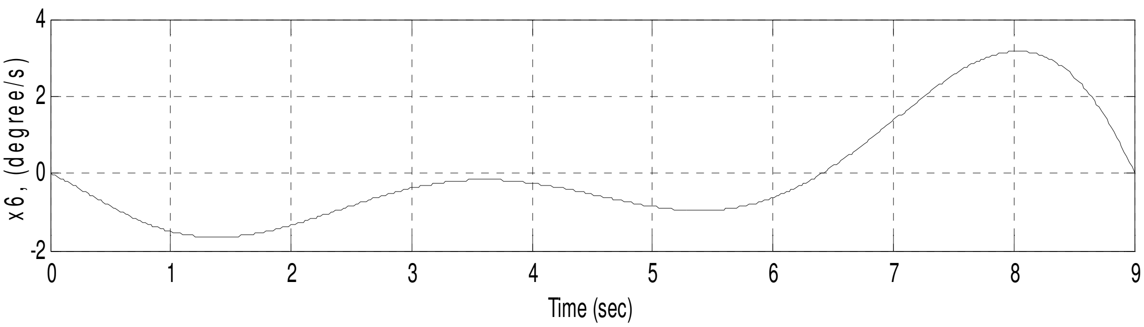
Figure (5.5) and (5.6) shows a set of state trajectories and approximate controls of the containers crane problem using Chebyshev and Legendre polynomials




x3, (degree)
Time (sec)


x4, (m/s)
Time (sec)


x 5, (m/s)
time (sec)


x 6, (degree/s)
Time (sec)

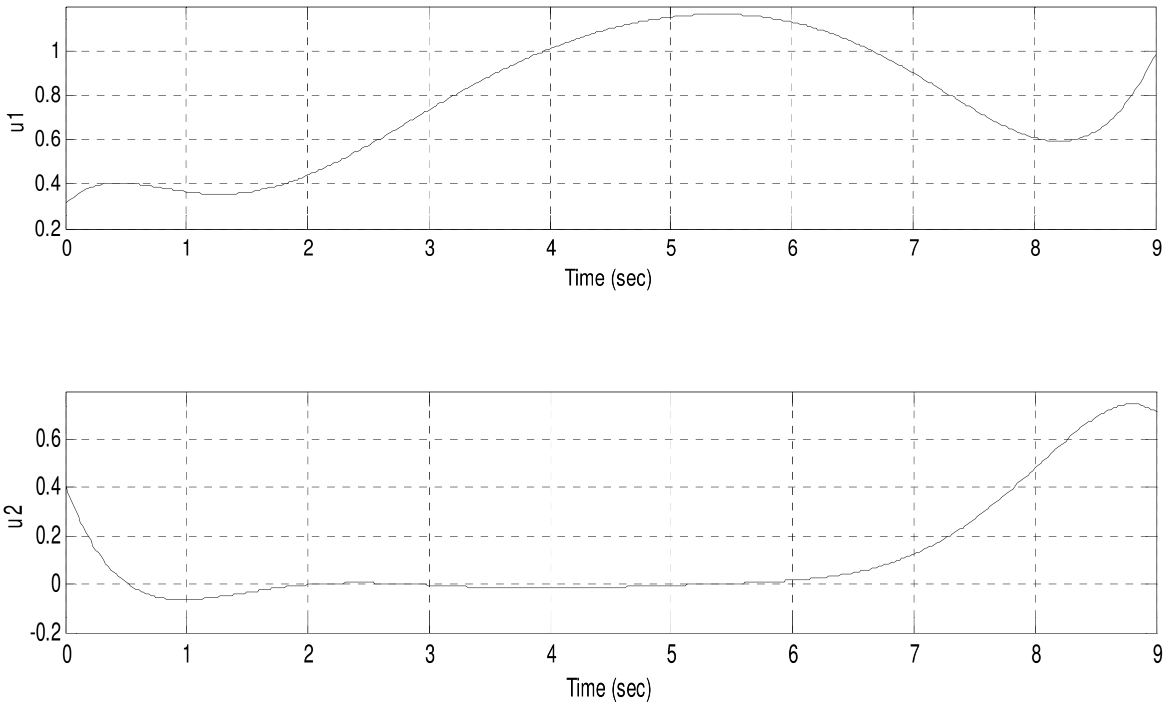
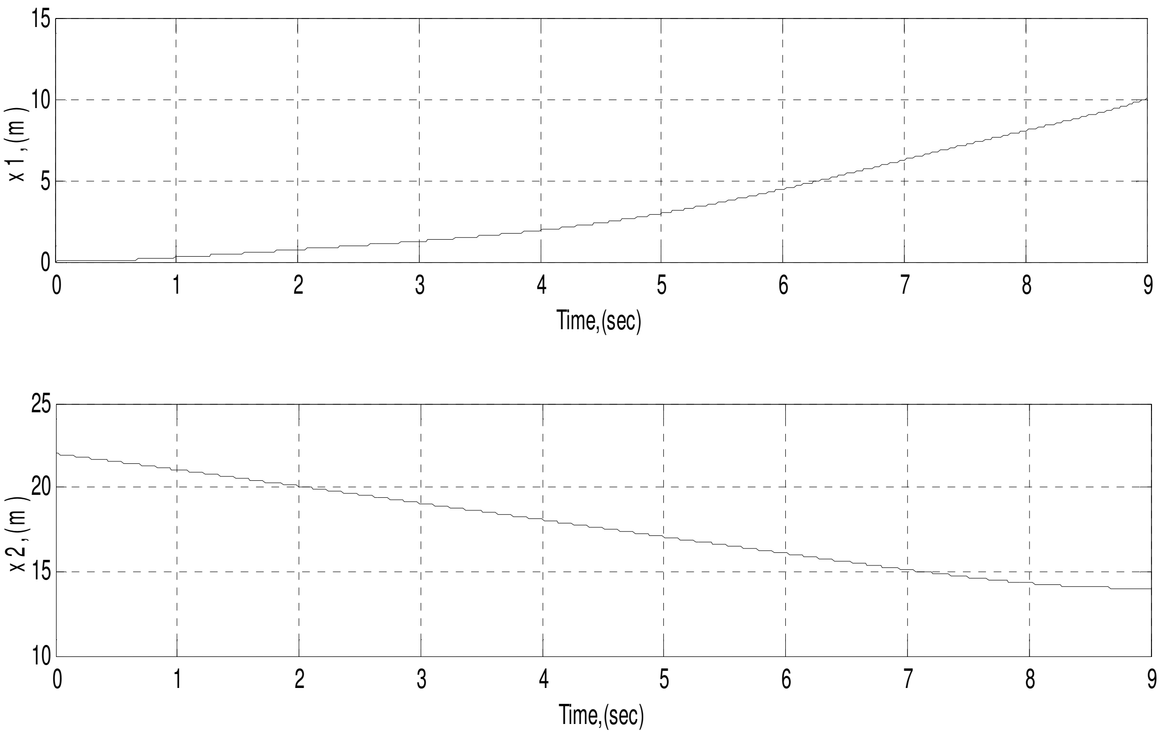
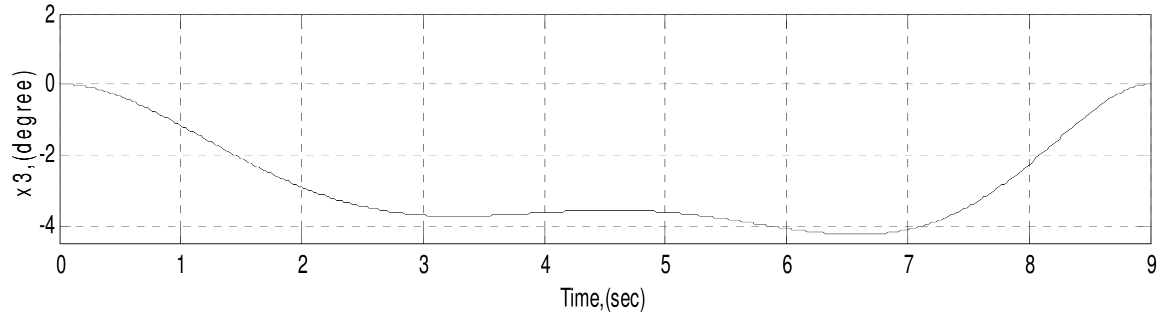
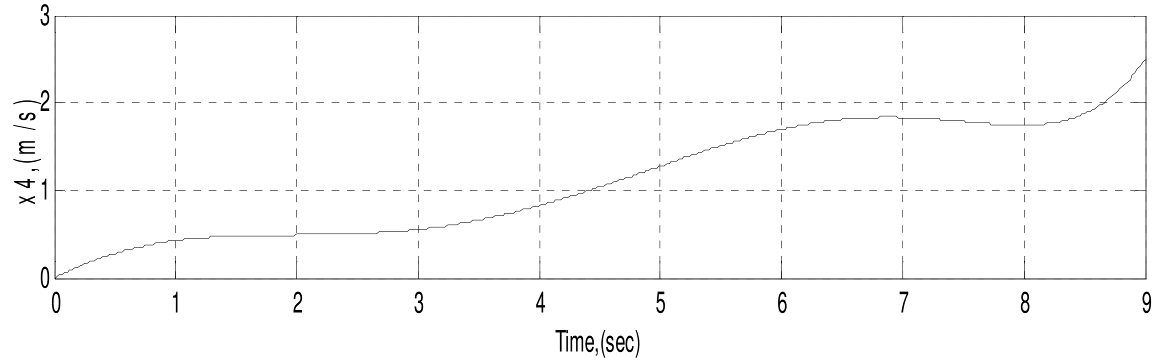
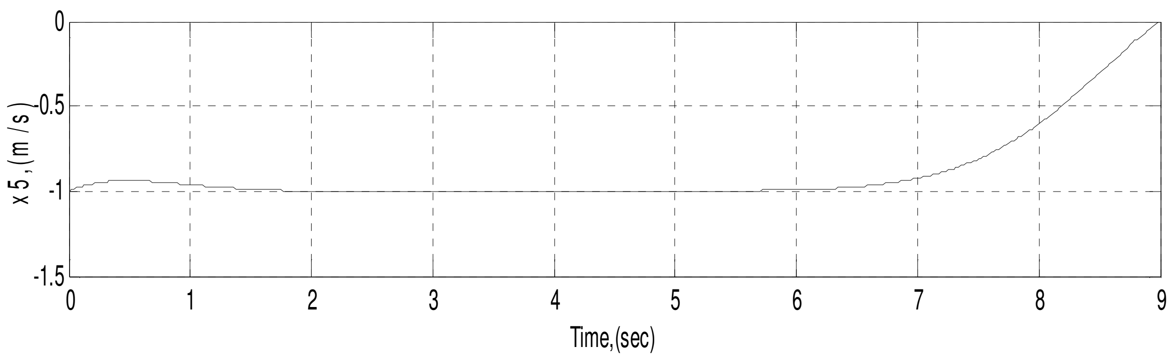


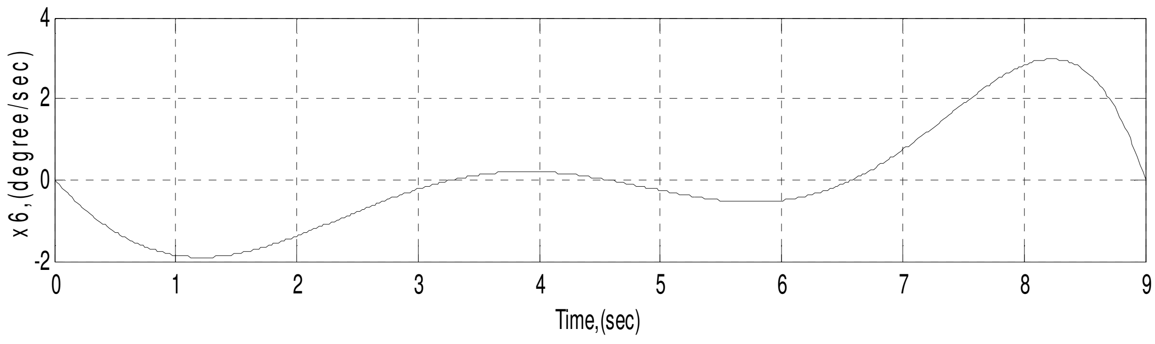
Figure (5.5) Optimal controls and state trajectories using Chebyshev polynomials




x 3 , (degree)
2
0
-2
-4
0
1
2
3
4
5
6
7
8
9
Time,(sec)


x 4 , (m / s)
3
2
1
0
0
1
2
3
4
5
6
7
8
9
Time,(sec)


x 5 , (m / s)
0
-0.5
-1
-1.5
0
1
2
3
4
5
6
7
8
9
Time,(sec)


x 6 , (degree/s e c)
4
2
0
-2
0
1
2
3
4
5
6
7
8
9
Time,(sec)

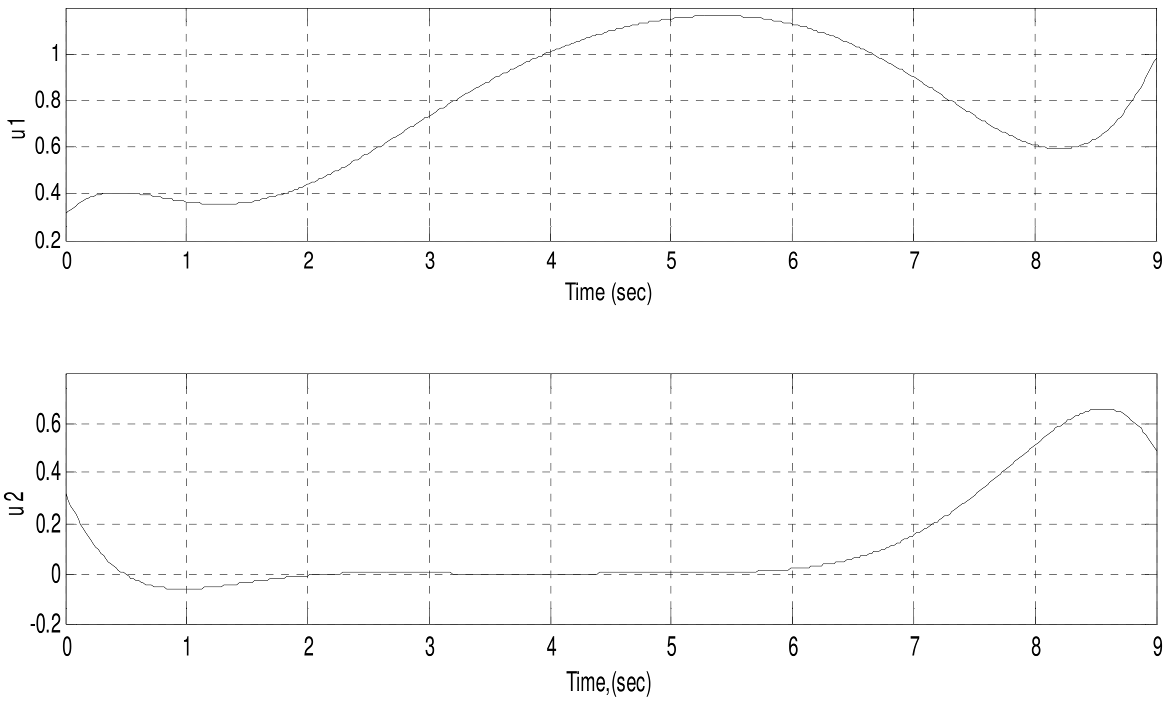


Figure (5.6) Optimal controls and state trajectories using Legendre polynomials

5.6 Conclusion

In this chapter, a method for solving the constrained nonlinear quadratic optimal control problem has been proposed. This method is based on replacing the nonlinear state equation by a sequence of linear time-varying state equations and then parameterizing the system state equations by a finite length series of Chebyshev or Legendre polynomials with unknown parameters. The inequality constraints of the system are replaced by finite dimensions inequality constrains through discretization of the time interval $\tau \in [-1,1]$ into an equally spaced point and satisfy the constraints at each point.

To show the effectiveness of the proposed method several examples that include different types of constraints were solved including a practical and complex problem; the containers crane problem.

Chapter Six

Conclusion and Future Work

6.1 Conclusion

In this work, we proposed a method to solve the nonlinear quadratic optimal control problem. In spite of this fact, the proposed method is also suitable for solving linear quadratic optimal control problems. The method is based on using the newly introduced iteration technique in which the nonlinear state equations are replaced by a sequence of linear time-varying state equations. This technique is combined with state parameterization in which some system state variables are approximated directly using a finite length series of Chebyshev or Legendre polynomials with unknown parameters. The remaining state and control variables are obtained from the system state equations. If any state equation remains unsatisfied, it will be replaced by equality constraints.

In comparison with other parameterization techniques, namely control-state and control parameterization, this method offer several advantages. Examples of such advantages are: Low dimension due to small number of unknown parameters required to approximate the state and control variables; unlike control parameterization, there is no need to integrate the system state equations to obtain the state variable. The optimal control problem under consideration is converted directly to a quadratic programming problem without the need to convert the problem to the intermediate nonlinear programming problem as the case with control parameterization, control-state parameterization and discretization.

Using the proposed method, the optimal control problem under consideration is directly converted into a quadratic programming problem. By this, the difficult nonlinear quadratic optimal control problem is converted to a sequence of quadratic linear optimal control problems that are much easier to solve.

The proposed method was applied to several examples free and subject to different types of constraints. The simulation results were good and the proposed method converges. To show the effectiveness of the proposed method, a complex practical containers crane problem was solved.

A new property for Legendre polynomials called the differentiation operational matrix was derived. This property was used to approximate the derivatives of the state variables when state parameterization via Legendre polynomials is used. This property can also be used to obtain a feedback control law via Legendre polynomials using the method proposed by Jaddu [8, 30].

Comparing the results of simulation obtained using Chebyshev and Legendre polynomials shows that the results are almost identical with relatively smaller optimal values of the performance index in the case of using Legendre polynomials. However, the results also show that the convergence speed when using Chebyshev polynomials is much faster than that of using Legendre polynomials.

6.2 Future Work

The work done in this thesis can be extended as follows:

- Modify the iteration to produce more accurate results.

- Develop the proposed method to handle the infinite horizon optimal control problem by selecting an orthogonal function that is defined orthogonally on the interval $[0, \infty[$.

- Develop the propose method to produce a closed loop feedback control law using Legendre polynomials. In this regard we propose using the method proposed by Jaddu [30] in which Chebyshev polynomials are used to obtain such a controller. The derivation of the differentiation operational matrix for Legendre polynomials in this work would facilitate this mission.

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طريقة تكرارية لحل مسألة التحكم اللاخطي الأمثل باستخدام الاقترانات المتعامدة

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ملخص:

على مدى السنوات السابقة طرحت الكثير من الطرق والأساليب لحل مسألة التحكم الأمثل. يمكن تصنيف هذه الطرق كطرق مباشرة وطرق غير مباشرة. ان الطريقة المقترحة في هذه الرسالة تصنف على انها طريقة مباشرة، والتي يتم من خلالها تحويل مسألة التحكم الامثل الى مسألة رياضية برمجية. من خلال اسمها، يتم تنفيذ الطرق المباشرة من خلال التعويض المباشر لمتغيرات التحكم والحالة في مؤشر الأداء.

يمكن تنفيذ الطرق المباشرة إما باستخدام التجزئة أو التقريب. بدوره، يتم تنفيذ التقريب باستخدام واحدة من الطرق الثلاث: (أ) تقريب التحكم (ب) تقريب الحالة–التحكم (ج) تقريب الحالة. ان الطريقة المقترحة في هذا البحث تعتمد على طريقة تقريب الحالة والتي من خلالها يتم تقريب متغيرات الحالة للنظام باستخدام سلسلة منتهية الطول باستخدام كثيرات الحدود تشيباشيب أو ليجندر وبمتغيرات مجهولة.

ان الطريقة المقترحة في هذا البحث تعتمد أيضا على طريقة التكرار والتي من خلالها يتم استبدال معادلات الحالة الغير الخطية بسلسلة من معادلات الحالة الخطية المعتمدة على الزمن. يتم بعد ذلك تطبيق تقريب الحالة على هذه السلسلة، وبذلك فان مسألة التحكم الغير خطي الأمثل تتحول الى مسائل خطية برمجية حلها أسهل بكثير من المسألة الأصلية.

وأخيرا، ومن أجل اظهار فعالية الطريقة المقترحة، تم حل العديد من مسائل التحكم الأمثل المتنوعة منها الخالى من القيود ومنها المعرض لقيود مختلفة الانواع. لقد أظهرت نتائج النمذجة ان الطريقة المقترحة جيدة وتعطي نتائج متقاربة مع العديد من الطرق الأخرى. ومن أهم المسائل التي تم تطبيق الطريقة المقترحة عليها، مسألة نقل الحاويات في الموانىء من السفن الى الشاحنات.