Deanship of Graduate Studies

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Analytic Solution of Linear Difference Equations with Applications

By

Jamela Hamdi Zamil Abu-Maria

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Declaration

I certify that this submitted for the degree of master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed …………………..

Jamela Hamdi Zamil Abu-Maria

Date: June 14, 2005

Dedication

To my parents. To my Family and to my colleagues.

Acknowledgment

I would like to express my thanks to all those who helped me to prepare and complete this work, specially and personally to my supervisor, Dr. Tahseen Mughrabi for his help and advice through the period of study.

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Abstract

The purpose of this work is to study and investigate topics related to the theory and applications of linear difference equations. After introducing the basic concepts related to the existence and uniqueness of the solution for the non-homogeneous equation [2], we present the notion of equilibrium points, stability, asymptotic stability and the phenomena of chaotic behavior of a solution [7], where we discuss the differences that arise about a strict definition of chaos which was initiated by Devany in 1986 and it is now the widely known and accepted definition. Moreover, we investigate the unpredictable behavior of a solution in the interval where the solution jumps randomly, this phenomena it explained by considering the logistic map and using simulation.

We further extend our study to higher order linear difference equations where the general solution of homogeneous equations is presented. For the non-homogeneous case the method of undetermined coefficients is employed.

Real –life problems whose behavior for large time can be described by difference equations can be solved using the theory and methods contained in this thesis. This is illustrated with specific models from different fields:
From communication we consider the transmission of information model, our next model comes from economics where we represent the analysis of national income of a certain country. Another important model comes from biological sciences where the concern is on the description of the number of plants in any desired generation during growth seasons.

These models which represent real- life phenomena in diverse fields where difference equations are the natural setting for such problems.

الخلاصة

الهدف من هذه الدراسة هو البحث في مواضيع نظرية و تطبيقية تتعلق بالمعادلات الفرقية الخطية حيث تم دراسة النظريات المتعلقة بوجود حل وحيد للمعادلات غير المتجانسة [2]. كذلك تم التطرق لخاصية الاستقرار في الحل بالاضافة الى ظاهرة الفوضى [7] حيث تم مناقشة الاختلافات التي ظهرت حول وضع تعريف محدد و متفق عليه من قبل الدارسين لهذه الظاهرة التي بدأ بمحاولة صياغة تعريف مقبول لها العالم ديفاني في عام 1986 و هذا التعريف هو الأوسع انتشارا و قبولا في الوقت الحالي, كذلك تم البحث في سلوك الحل خلال فترة زمنية معينة بحيث يقفز الحل عشوائيا , وهذه الظاهرة تم توضيحها عن طريق عرض الاقتران اللوجستي واستخدام الحاسوب في الحسابات.

كما تعدت الدراسة الى المعادلات الفرقية ذات درجات عليا و حلولها العامة, واستخدمت طريقة المعاملات غير المحددة لحل معادلات غير متجانسة.

هناك تطبيقات عملية من مجالات معرفية متنوعة: من هذه المجالات الاتصالات حيث تفسر المعادلات الفرقية نموذج نقل المعلومات, والمجال الذي يليه هو المجال الاقتصادي الذي يعنى بتقديم تحليل للدخل القومي في دولة معينة. أما المجال الأخير الذي تم التطرق إليه هو علم الأحياء الذي يهتم بتقديم وصف لأعداد النباتات المرغوب فيها خلال مواسم النمو.
هذه النماذج التي تعكس الواقع التطبيقي للظاهرة في المجالات المتنوعة من الحياة بحيث تشكل المعادلات الفرقية الوضع الطبيعي لحل و تفسير مسائل من هذا النوع.

Contents:

Introduction

The theory of difference equations, which has been almost hidden up to now, begins to open in all its richness. Difference equations, which reflect one of the most essential properties of the real-world, rightfully occupy a fundamental role in our understanding of nonlinear phenomena in various and drastically different systems. To a certain extent, the growing interest in difference equations may be attributed to their simplicity. Although only quite simple computational and graphical representation tools are necessary to study the behavior of the solutions of difference equations and their bifurcations for changing parameters, it is possible to appreciate the complicated and surprisingly diverse dynamics of difference equations.

In this work our goal is the study of the basic theory of linear difference equations and analyzes some applications which arise in some branches of the natural sciences and applied mathematics. In chapter one we will be concerned with first order difference equations where we introduce all basic ideas related to the linear class of such equations. Also, some key notions are investigated; namely, the equilibrium points, stability and asymptotic stability of equilibrium points. Avery important phenomena related to the solution of difference equations that is the chaotic behavior of the solution concludes this chapter.

Chapter two is devoted to study higher order linear difference equations where key theorems are presented [4], the idea of a fundamental set of solutions is also defined which are necessary to develop some concepts used to prove basic theorems concerning the general solution of a linear difference equation[4]. Moreover, some analytic methods for solving linear non-homogeneous difference equations are discussed.

The final main feature of this work is the inclusion of chapter three where we employ the theory and discussions in the previous material to show only some of the too many applications of difference equations that resulted from modeling some real-life phenomena in economics, biology, statistics, communications, where problems whose solutions behavior for large time can be described only in terms of difference equations.

Chapter One

First Order Difference Equations

1.1. Introduction

In this section we define the concept of a difference equation and its solution together with some examples.

Definition 1.1.1:[4]

A difference equation is an equation relating the values of a function $x = x(n)$ and one or more of its differences $\Delta x, \Delta^2 x, \ldots, \Delta^k x \ldots$ for each k-value in some set **S** of integers where $\Delta x(n) = x(n+h) - x(n)$

$\Delta^2 x = \Delta(\Delta x)$ And $\Delta^n x = \Delta(\Delta^{n-1} x)$ $n = 1, 2, \ldots$.

A difference equation usually describes the evolution of certain phenomena over the course of time. For example if a certain population has a discrete generation then the size of the $n+1$st generation $x(n+1)$ is a function of the n-th generation $x(n)$ such that:

$$x(n+1) = f(x(n)). \tag{1.1.1}$$

We can look at this problem from another point of view, starting from a point x_0 we may generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots$$

For convenience we adopt the notation

$$f^2(x_0) = f(f(x_0)) \quad , \quad f^3(x_0) = f(f(f(x_0)))$$

So the sequence becomes

$f(x_0)$, $f^2(x_0)$, $f^3(x_0)$ $\cdots$

$f(x_0)$ is called the first iterate of x_0 under f

$f^2(x_0)$ is called the second iterate of x_0 under f

$\vdots$

$f^n(x_0)$ in general , is the nth iterate of x_0 under f

This iterative procedure is an example of a discrete dynamical system.

Hence we may conclude correctly that difference equations and discrete dynamical systems represent two sides of the same coin .

Now if the function *f* in equation (1.1.1) is replaced by a function *g* of two variables that is $g : Z^+ \times \Re \to \Re$ where $\Re$ is the set of real numbers, then we have

$$x(n+1) = g(n, x(n)) . \qquad (1.1.2)$$

Equation (1.1.2) is called non-autonomous or (time-variant) ,while equation (1.1.1) is called autonomous (time-invariant) difference equation.

1.2. Linear First Order Difference Equations

In this section we study two special cases of equations (1.1.1) and (1.1.2) .

A typical linear homogeneous first order equation is given by :

$$x(n+1) = a(n)\,x(n) \qquad x(n_0) = x_0 \qquad \text{for} \quad n \geq n_0 \geq 0 \qquad (1.2.1)$$

And the non-homogeneous equation is given by :

$$x(n+1) = a(n)\,x(n) + g(n) \qquad x(n_0) = x_0 \qquad \text{for} \quad n \geq n_0 \geq 0 \qquad (1.2.2)$$

where $a(n) \neq 0$ and $g(n)$ are real –valued functions defined for $n \geq n_0 \geq 0$.

We can obtain the solution of equation (1.2.1) by a simple iteration:

$$x(n_0+1) = a(n_0)\,x(n_0) = a(n_0)x_0$$

$$x(n_0+2) = a(n_0+1)\,x(n_0+1) = a(n_0+1)a(n_0)x_0$$

$$x(n_0+3) = a(n_0+2)\,x(n_0+2) = a(n_0+2)a(n_0+1)a(n_0)x_0$$

And inductively, it is easy to see that

$$x(n) = x(n_0+n-n_0) = a(n-1)a(n-2)....a(n_0)x_0$$

$$= \left[\prod_{i=n_0}^{n-1} a(i)\right] x_0$$

Theorem 1.2.1:[3]

The non-homogenous equation

$$x(n+1) = a(n)x(n) + g(n) \quad , \quad x(n_0) = x_0 \quad , \; n \geq n_0 \geq 0$$

has a unique solution given by:

$$x(n) = \left[\prod_{i=n_0}^{n-1} a(i)\right] x_0 + \sum_{r=n_0}^{n-1}\left[\prod_{i=r+1}^{n-1} a(i)\right] g(r) \;\; \text{for all } n \in Z^+ \qquad (1.2.3)$$

Proof:

Starting with the equation

$x(n_0+1)=a(n_0)\,x(n_0)+g(n_0)$

We find

$x(n_0+2)=a(n_0+1)\,a(n_0)x_0+a(n_0+1)g(n_0)+g(n_0+1)$

Now we use mathematical induction to show that for all $n\in\mathbf{Z}^+$ equation (1.2.3)holds, and to establish this , we assume that formula (1.2.3) holds for $n = k$, then from equation (1.2.2)

$x(k+1) = a(k)x(k) + g(k)$

which by formula (1.2.3) yields

$$x(k+1)=a(k)\left[\prod_{i=n_0}^{k-1}a(i)\right]x_0+\sum_{r=n_0}^{k-1}\left[a(k)\prod_{i=r+1}^{k-1}a(i)\right]g(r)+g(k)$$
$$=\left[\prod_{i=n_0}^{k}a(i)\right]x_0+\sum_{r=n_0}^{k-1}\left(\prod_{i=r+1}^{k}a(i)\right)g(r)+\left(\prod_{i=k+1}^{k}a(i)\right)g(k)$$
$$=\left[\prod_{i=n_0}^{k}a(i)\right]x_0+\sum_{r=n_0}^{k}\left(\prod_{i=r+1}^{k}a(i)\right)g(r)$$

(Note that $\prod_{i=k+1}^{k}a(i)=1$ and $\sum_{i=k+1}^{k}a(i)=0$)

Hence formula (1.2.3) holds for all $n \in Z^+$.

Now, we consider the two special cases:

1) First, the equation given by

$x(n+1) = ax(n)+ g(n)$, $x(0) = x_0$ (1.2.4)

Formula (1.2.3) implies:

$$x(n)=a^n x_0+\sum_{k=0}^{n-1}a^{n-k-1}g(k) \quad (1.2.5)$$

Which establishes the first case.

2) Second equation given by

$$x(n+1) = ax(n) + b, \qquad x(0) = x_0 \tag{1.2.6}$$

by using formula (1.2.5) we obtain

$$x(n) = \begin{cases} a^n x_0 + b\left[\dfrac{a^n - 1}{a-1}\right] & \text{if } a \neq 1 \\ x_0 + bn & \text{if } a = 1 \end{cases} \tag{1.2.7}$$

which yields the second case, that we prove next (theorem 1.2.3).

Example 1.2.2 :

Consider the equation :

$$x(n+1) = 2\,x(n) + 3^n, \qquad x(0) = 0.25.$$

By using Equation (1.2.5) we have

$$x(n) = \left(\frac{1}{2}\right)2^{n-1} + \sum_{k=0}^{n-1} 2^{n-k-1}\, 3^k$$

$$= 2^{n-2} + 2^{n-1}\sum_{k=0}^{n-1}\left(\frac{3}{2}\right)^k$$

$$= 2^{n-2} + 2^{n-1}\left(\frac{\left(\frac{3}{2}\right)^n - 1}{\frac{3}{2} - 1}\right) = 3^n + 2^{n-2} - 2^n$$

$$x(n) = 3^n - \tfrac{3}{4}2^n$$

Theorem 1.2.3 :[3]

The function x given by (1.1.7) is the only solution of the difference equation

$$x(n+1) = ax(n) + b \qquad , n = 0, 1, 2, \ldots$$

Where $x(0) = x_0$, x_0 is prescribed .

Proof :

We have already noted that the solution of (1.2.7) does in fact reduce to $x(0)$ when $n = 0$, that we have a solution can be shown by direct substitution into the difference equation :

From (1.2.7) with $a \neq 1$, we have

$$x(n+1) = a^{n+1}x(0) + b\frac{1-a^{n+1}}{1-a}$$

So we must show that

$$a^{n+1}x(0) + b\frac{1-a^{n+1}}{1-a} = a\left(a^n x(0) + b\frac{1-a^n}{1-a}\right) + b$$

is an identity , i.e. true for $n = 0$,1,2,… . It suffices to simplify the right hand side .

Now the right hand side is equal to

$$a^{n+1}x(0) + b\left[1 + a\frac{(1-a^n)}{1-a}\right]$$

we have only to observe that

$$1 + \frac{a(1-a^n)}{1-a} = \frac{1-a+a(1-a^{n)}}{1-a} = \frac{1-a^{n+1}}{1-a}$$

$$so$$

$$a^{n+1}x_0 + b\left[\frac{1-a^{n+1}}{1-a}\right] \quad if \quad a \neq 1$$

Now if $a = 1$

We have already noted that the solution (1.2.7) does in fact reduce to $x(0)$ when $n = 0$. That we have a solution can be shown by direct substitution into the difference equation from (1.2.7) with $a = 1$ and replacing n by $(n+1)$, we have

$x(n+1) = x(0) + b(n+1)$, or

$x(n+1) = x_0 + b(n+1) = (x(0) + bn) + b = x(n) + b$

Which is an identity i.e. true for $n = 0, 1, 2\ldots$

Corollary1.2.4 :[4]

let x be a solution of the difference equation $x(n+1) = a\,x(n) + b \quad , \quad n = 0,1,2,\ldots$

Then there is a constant c for which

$$x(n) = \begin{cases} ca^n + b\dfrac{1-a^n}{1-a} & if \quad a \neq 1 \\ c + bn & if \quad a = 1 \end{cases} \qquad n = 1,2,\ldots \tag{1.2.8}$$

Proof :

We have only to equate the constant c with the value of the solution x at $n = 0$ and use the previous theorem (1.2.3) (i.e., take $c = x(0) = x_0$).

Remark: Formula (2.1.8) gives all the solutions of the difference equation .

$x(n+1) = ax(n) + b$, $n = 0,1,2\ldots$

One solution for each value of the arbitrary constant c.

Example1.2.5:

Suppose we are given the difference equation

$x(n+1)=2x(n) +1$, $n = 0,1,2,\ldots$ with the initial condition $x(0) = 5$

from (2.1.8) we find the unique solution

now $a = 2$, $b = 1$

$$x(n) = 5(2^n) + 1\left(\frac{1-2^n}{1-2}\right)$$

or

$x(n) = 6(2^n) - 1 \quad , n = 0,1,2,\ldots$

if we write the consecutive value of x serially starting with $x(0) = 5$ we have the sequence of increasing values 5, 11 ,23 , 47 , 95 , 191 … .

1.3. Equilibrium Points

This section gives the definition of an equilibrium point and also it introduces the basic definitions of stability together with some illustrations.

Definition 1.3.1: **[3]**

A point x^* in the domain of f is said to be an equilibrium point of (1.1.1) [$x(n+1) = f(x(n))$] if it is a fixed point of f ,i. e , $f(x^*) = x^*$.

Note that x^* is a constant solution of equation (1.1.1) since if $x(0) = x^*$ is an initial point, then we have.

$$x(1) = f(x^*) = x^*$$

$$x(2) = f(x(1)) = f(x^*) = x^*$$

$$\vdots$$

$$x(n) = x^*$$

Graphically: the equilibrium point is the x coordinate of the point where the graph of f intersects the diagonal line $y = x$.

Definition 1.3.2: [3]

Let x be a point in the domain of f, if there exists a positive integer k and an equilibrium point x^* of equation (1.1.1).such that $f^k(x) = x^*$, $f^{k-1}(x) \neq x^*$ then x is called an eventually equilibrium point.

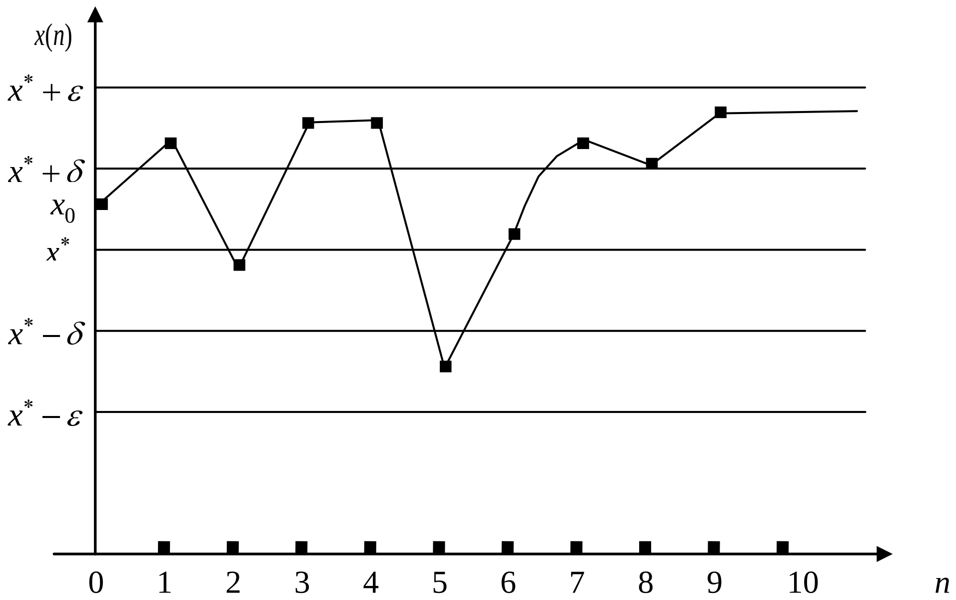
Example 1.3.3:

Let $x(n+1) = f(x(n))$

Where $f(x)=\begin{cases}2x & for \quad 0\le x\le \frac{1}{2}\\ 2(1-x) & for \quad \frac{1}{2}<x\le 1\end{cases}$ There are two equilibrium points 0 and $\frac{2}{3}$. If $x(0)$ = $\frac{1}{4}$,then $x(1)$ = $\frac{1}{2}$, $x(2)$ = 1 and $x(3)$ = 0 . Thus $\frac{1}{4}$ is an eventually equilibrium point.

Definition 1.3.4: [3]

The equilibrium point x^* of equation (1.1.1) is said to be stable if given $\varepsilon>0$ there exists $\delta>0$ such that $\left|x_0-x^*\right|<\delta$ implies $\left|f^n(x_0)-x^*\right|<\varepsilon$ for all $n>0$ (see figure 1.3.1).If x^* is not stable then it is called unstable (figure1.3.2)



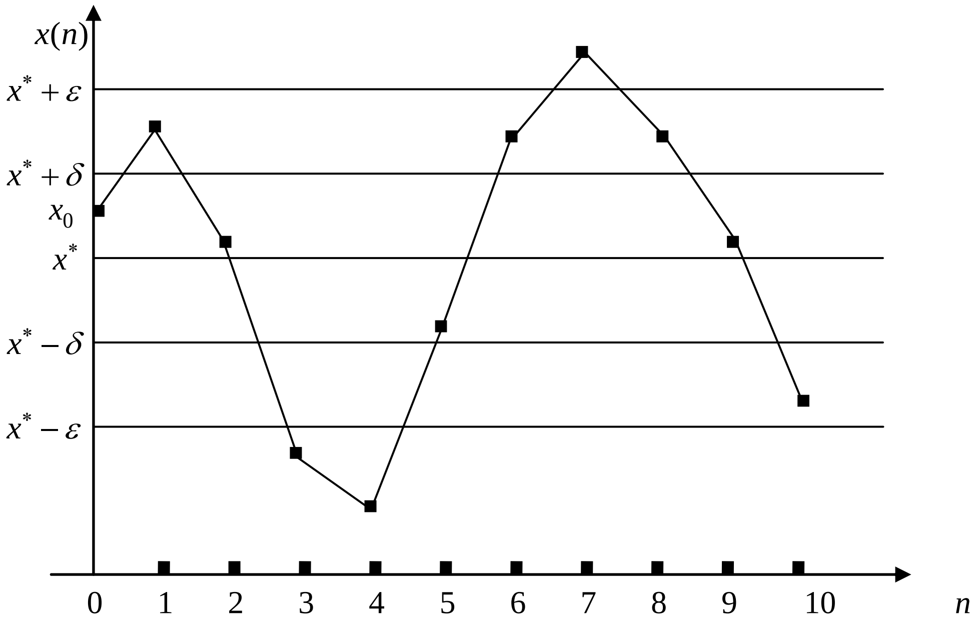
(figure1.3.1 .Stable x*)

Example 1.3.5:

$x(n+1)=-x^3(n)-x\ (n)$

Since $f'(0)=-1<1$

So by theorem (1.4.1) $x^*=0$ is a stable equilibrium point.



(Figure 1.3.2.Unstble x^*)

Example 1.3.6:

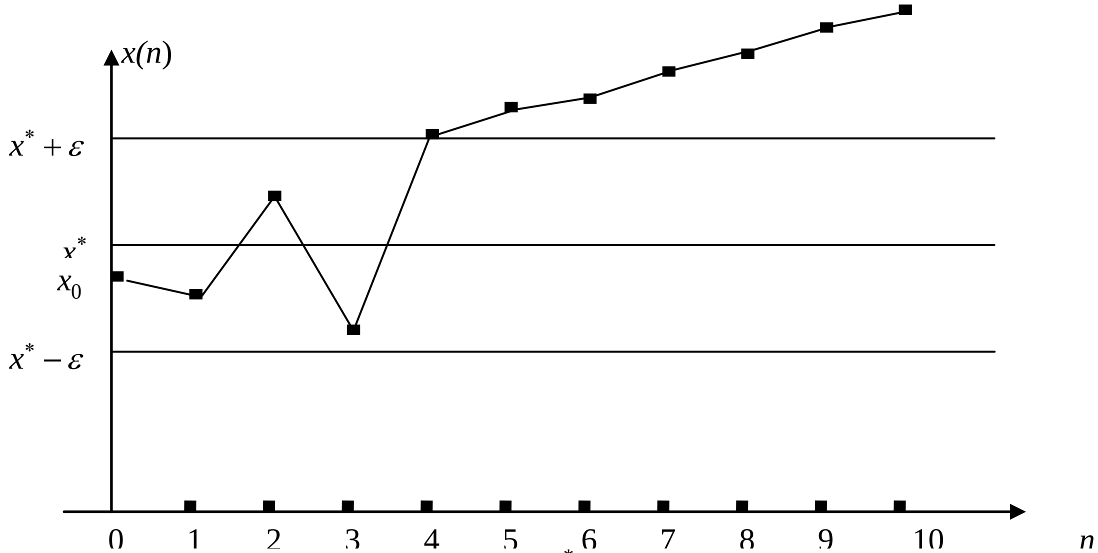
$x(n+1) = x^2(n) + 3x(n)$

Since $f'(0) = 3 > 1$

So by theorem (1.4.1) $x^* = 0$ is an unstable equilibrium point.

Definition 1.3.7: [3]

The point x^* is said to be a repelling (source) equilibrium point if there exists $\varepsilon > 0$ such that $0 < |x_0 - x^*| < \varepsilon$ implies $|f(x_0) - x^*| > |x_0 - x^*|$ (see figure1.3.3).



(Figure 1.3.3 Repelling x^*)

Example 1.3.8:

$x(n+1) = x^2(n)$

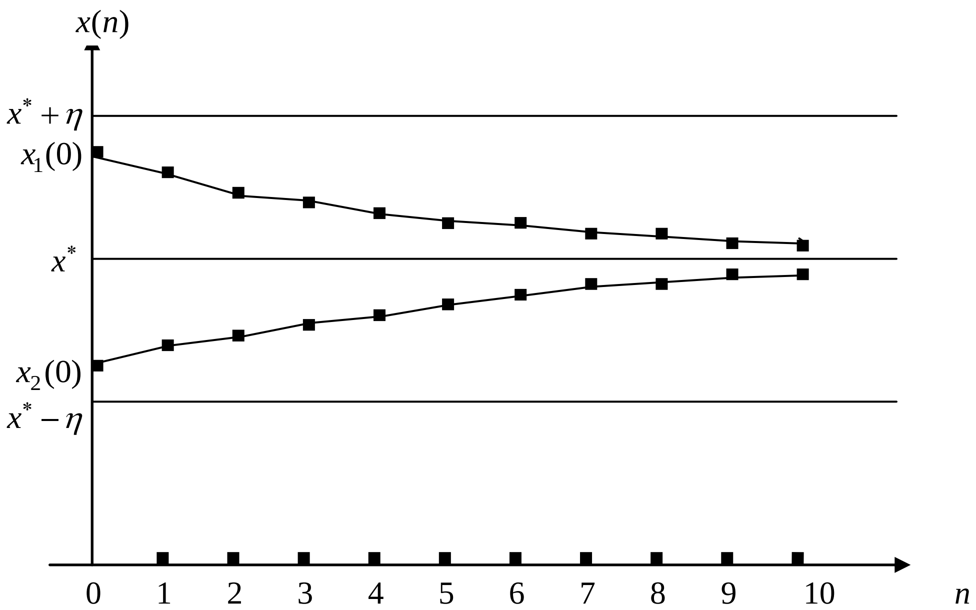
Since $f'(1) = 2 > 1$

So by theorem (1.4.1) $x^* = 1$ is a repelling equilibrium point.

Definition 1.3.9 :[3]

The point x^* is said to be an asymptotically stable (attracting) equilibrium point if it is stable and there exists $\eta > 0$ such that $0 < |x_0 - x^*| < \eta$ implies $\lim_{n\to\infty} x(n) = x^*$ (figure 1.3.4).

If $\eta = \infty$, x^* is said to be globally asymptotically stable (figure 1.3.6)



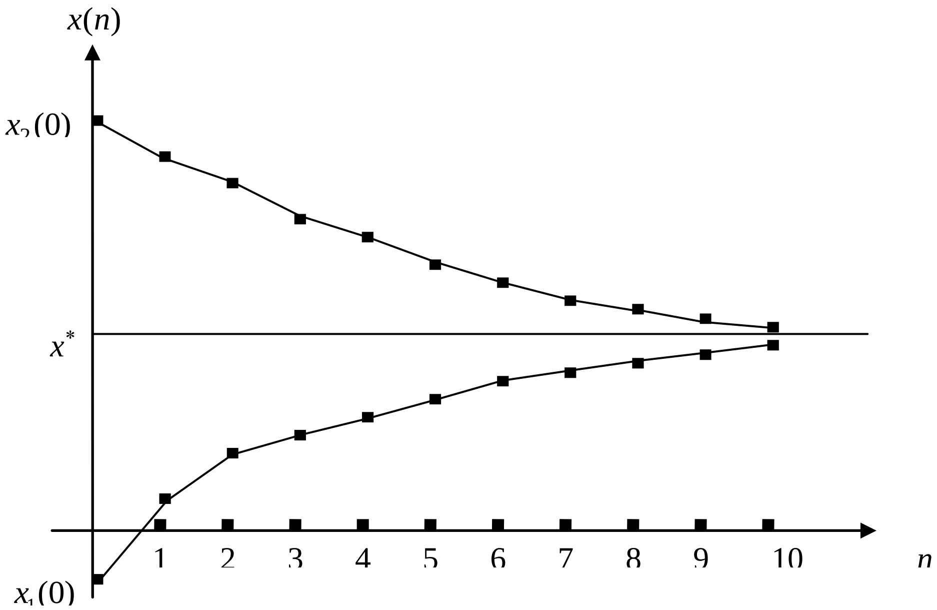
(Figure 1.3.4 Asymptotically stable x^*).

Example 1.3.10:

$x(n+1) = 2x(n) - 2x^2(n)$, $x(0) \in [0, 1]$

Since $f'(\frac{1}{2}) = 0 < 1$

So by theorem (1.4.1) $x^* = \frac{1}{2}$ is an asymptotically stable equilibrium point.



(Figure 1.3.5 .Globally asymptotically stable x^*).

Example 1.3.11:

$x(n+1) = \frac{1}{2}x(n)$, $x(0) = x_0$

Since $f'(0) = 0 < 1$, so by theorem (1.4.1) $x^* = 0$ is a stable equilibrium point , and

$\lim_{n\to\infty} x(n+1) = 0$ for all $x(0)$.

So $x^* = 0$ is globally asymptotically stable equilibrium point.

1.4. Asymptotic stability of Equilibrium Points

This section discusses sufficient conditions needed to guarantee the asymptotic stability of equilibrium points of equation in the form (1.4.1), some key theorem will be established.

Theorem 1.4.1: [3]

Let x^* be an equilibrium point of the difference equation

$$x(n+1)=f(x(n)) \tag{1.4.1}$$

where f is continuously differentiable at x^*.

The following statements are true:

i. If $\left|f'(x^*)\right|<1$,then x^* is an asymptotically stable (attracting)equilibrium point .

ii. If $\left|f'(x^*)\right|>1$,then $x^*$is not stable ,in fact x^* is a repelling equilibrium point.

Proof:

i. Suppose that $\left|f'(x^*)\right| \le M < 1$. Then there is an interval

$J=(x^*-\gamma, x^*+\gamma)$ containing x^* such that $\left|f'(x^*)\right| \le M < 1$ for all $x \in$ J

Now, for $x(0) \in$ J, we have

$$\left|x(1)-x^*\right| = \left|f(x(0))-f(x^*)\right|$$

By the Mean Value Theorem ,there exists ξ between $x(0)$ and x^* such that

$$\left|f(x(0))-f(x^*)\right| = \left|f'(\xi)\right|\left|x(0)-x^*\right| \quad \text{thus}$$

$$\left| f(x(0)) - x^* \right| \le M \left| x(0) - x^* \right|$$

Hence

$$\left| (x(1)) - x^* \right| \le M \left| x(0) - x^* \right| \qquad (1.4.2)$$

Since M<1, inequality (1.4.2) shows that x (1) is closer to x^* than x (0), consequently, $x(1) \in$ J

By induction we conclude that

$$\left| x(n) - x^* \right| \le M^n \left| x(0) - x^* \right|.$$

For $\varepsilon > 0$, let $\delta = \dfrac{\varepsilon}{2M}$.

Thus $\left| x(0) - x^* \right| < \delta$ implies that $\left| x(n) - x^* \right| < \varepsilon$ for all $n \ge 0$. So this conclusion suggests stability.

Furthermore, $\lim_{n\to\infty} \left| x(n) - x^* \right| = 0$ and thus $\lim_{n\to\infty} x(n) = x^*$, which implies asymptotic stability.

ii. Suppose that $\left| f'(x^*) \right| > 1$, choose $M > 1$ and so $\left| f'(x^*) \right| \ge M > 1$,there is an interval $I = (x^* - \gamma, x^* + \gamma)$ for some $\gamma > 0$ so that by Mean Value Theorem there exists ξ between $x(0)$ and x^* such that

$$\left| f(x(0)) - x^* \right| = \left| f'(\xi) \right| \left| x(0) - x^* \right|$$

$$\left| f(x(0)) - x^* \right| \ge M \left| x(0) - x^* \right|$$

for all $x(0)$ and x^* in I , by induction we have :

$$\left| f^n(x(0)) - x^* \right| \ge M^n \left| x(0) - x^* \right|$$

As long as $f^n(x)$ is in I, it follows that all solutions of the equation (1.1.1) which originate in I except for the constant solution $x(n) = x^*$ must leave I for sufficiently large n, then x^* is unstable .

Remark: since the previous theorem does not address the case where $|f'(x^*)| = 1$, we will discuss the case where $f'(x^*) = 1$, in the next theorem.

Theorem 1.4.2:[3]

Suppose that for an equilibrium point x^* of the difference equation.

$$x(n+1) = f(x(n)) ,$$

$f'(x^*) = 1$, the following statements then hold :

i. If $f''(x^*) \neq 0$ then x^* is unstable.

ii. If $f''(x^*) = 0$ and $f'''(x^*) > 0$ then x^* is unstable.

iii. If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable .

Proof:

i. If $f''(x^*) \neq 0$, then the curve is either concave upward where $f''(x^*) > 0$ or concave downward where $f''(x^*) < 0$, as in figures (1.4.1),(1.4.2) . If $f''(x^*) > 0$,then $f'(x) > 1$ for all x in a small interval $I = (x^*, x^* + \varepsilon)$ where $\varepsilon > 0$. By theorem (1.4.1)) , x^* is unstable . On the other hand if $f''(x^*) < 0$, then $f'(x) > 1$ for all in a small interval I = $(x^* - \varepsilon, x^*)$. Hence x^* is again unstable.

ii. If $f''(x^*)=0$, then x^* is an inflection point .And if $f'''(x^*)>0$,then there exists an $\varepsilon>0$,such that for all x in a small interval I = $(x^*, x^*+\varepsilon)$, $f''(x)>0$ .By theorem (1.4.1) x^* is unstable from the left . On the other hand, the right side of x^* $f''(x)<0$, which implies $f'(x)>1$for all x in a small interval I = $(x^*-\varepsilon, x^*)$. Hence x^* is unstable.

iii. If $f''(x^*)=0$, x^* is an inflection point , since $f'''(x^*)<0$, then $\exists\varepsilon>0$such that for all x in a small interval $(x^*, x^*+\varepsilon)$ $f''(x)<0$,which implies $f'(x)<1$ so by theorem (1.12) x^* is asymptotically stable .For the other side $f''(x)>0$ implies $f'(x)<1$ for all x in a small interval I = $(x^*-\varepsilon, x^*)$. Hence x^* is asymptotically stable.

We shall use geometrical figures to investigate the type of stability of x^*, these are called stair step (cobweb) diagrams that are constructed as follows :

a) Consider the equation :

$$x(n+1) = f(x(n))$$

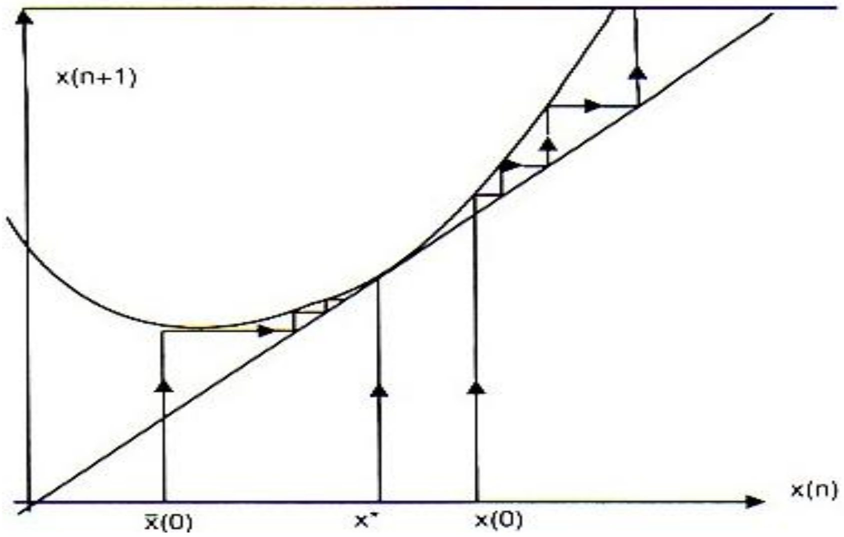
b) Draw the graph of f in the ($x(n)$, $x(n+1)$) plane.

c) Given $x(0) = x_0$, we pinpoint $x(1)$ by drawing a vertical line from x_0 which intersects the graph of f at (x_0 , $x(1)$) .

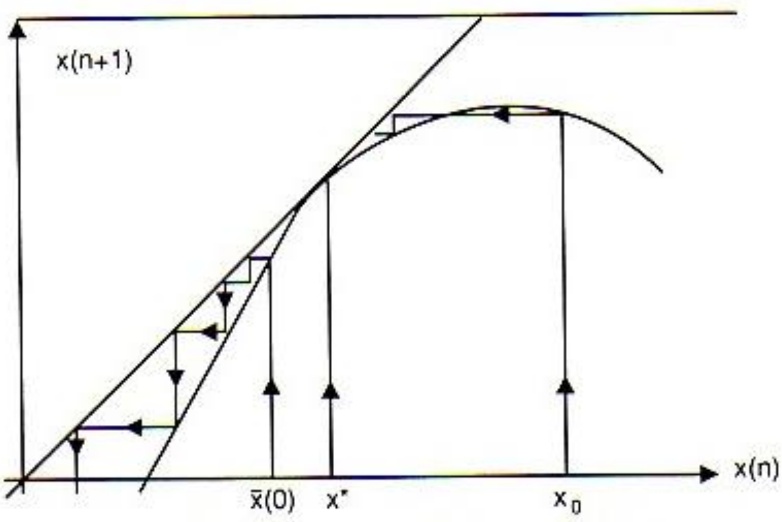
d) Draw a horizontal line from (x_0 , $x(1)$) to meet the diagonal line $y = x$ at the point ($x(1)$,$x(1)$) on the curve of f.

e) A vertical line from ($x(1)$,$x(1)$) will meet the graph of f at ($x(1)$,$x(1)$) .

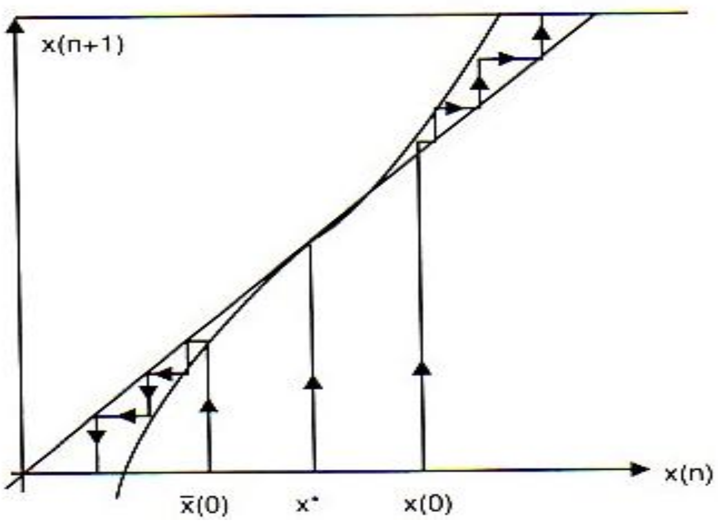
f) Continue by the same way .



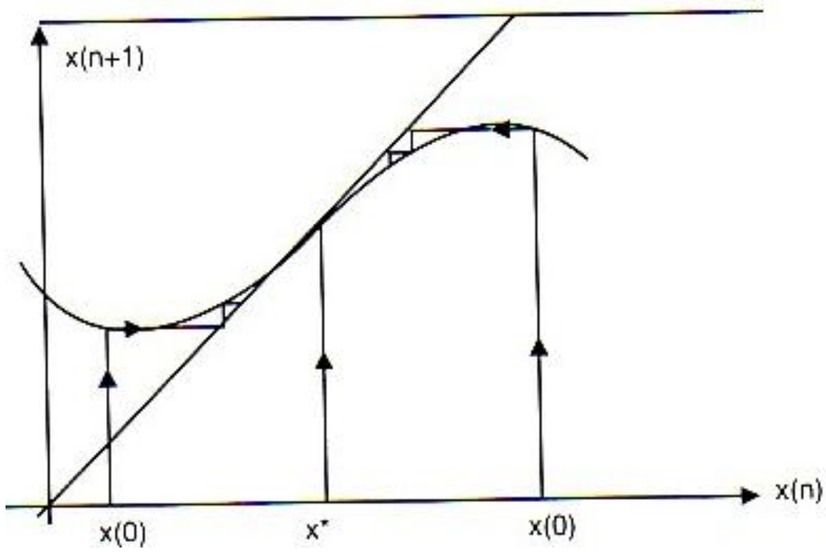
(Figure 1.4.1 Unstable $f''(x^*) > 0$)



(Figure 1.4.2 Unstable $f''(x^*) < 0$)



(Figure 1.4.3 Unstable $f'(x^*)=1, f''(x^*)=0$ *and* $f'''(x^*)>0$)



(Figure1.4.4 asymptotically stable $f'(x^*)=1, f''(x^*)=0$ *and* $f'''(x^*)<0$)

When $f'(x^*) = -1$ we use the following result:

Theorem 1.4.3: [3]

Suppose that $f'(x^*) = -1$, for the equilibrium point x^*, of equation (1.1.1)

the following statements then hold:

i. If $-2\, f'''(x^*) - 3[f''(x^*)]^2 < 0$ then x^* is asymptotically stable.

ii. If $-2\, f'''(x^*) - 3[f''(x^*)]^2 > 0$, then x^* is unstable.

Proof:

Contemplate the equation.

$x(n+1) = g(y(n))$. Where $g(y) = f^2(y)$ (1.4.4)

Two observations about equation (1.4.4) worth mentioning.

First the equilibrium point x^* of equation (1.1.1) is also an equilibrium point of equation (1.4.4).

Second, if x^* is asymptotically stable (unstable) with respect to equation (1.4.4), then it is so with respect to equation (1.1.1).

$$\frac{d}{dy}(g(y) = \frac{d}{dy} f(f(y)) = f'(f(y))\, f'(y)$$

Now

$$\frac{d}{dy} g(x^*) = [f'(x^*)]^2 = 1$$

Hence theorem (1.4.2) applies to this situation , we need to evaluate

$$\frac{d^2}{dy^2} g(x^*)$$

$$\frac{d^2}{dy^2} g(y) = \frac{d^2}{dy^2} f(f(y))$$

$$= \frac{d}{dy^2}[f'(f(y)\ f'(y)]$$

$$= (f'(y))^2\ f''(f(y)) + f'(f(y))\ f''(y).$$

$$\frac{d^2}{dy^2}\ g(x^*) = 0$$

Now theorem (1.4.2) [parts (ii) and (iii)] tell us that the asymptotic stability of x^* is determined by the sign of $[g(x^*)]'''$

Now by using again the chain rule, it can be shown that

$$\frac{d^3}{dy^3}g(y) = \frac{d}{dy}\ (f'(y))^2\ f''(f(y)) + f'(f(y))\ f''(y).$$

$$= (f'(y))^2\ f'''(f(y))f'(y) + f''(f(y))\ 2\ f'(y)f''(y)f'''(f(y))$$

$$+ f'''(y)f'(f(y)) + f''(f(y))f'(y)f''(y)(f'(y))^3\ f'''(f(y)) +$$

$$2(f''(f(y))^2\ f''(y)f'(y) + f'''(y)f'(f(y)) + f''(f(y))f''(y)f'(y)$$

$$(g(x^*))''' = -1 \times f'''(x^*) + 2(f''(x^*))^2 \times -1 + f'''(x^*) \times -1 + (f''(x^*))^2 \times -1$$

Hence

$$[g(x^*)]''' = -2f'''(x^*) - 3[f''(x^*)]^2$$

Example1.4.4:

Consider the difference equation

$$x\ (n+1) = -x^2(n) - x(n)$$

The equilibrium points are $x = 0$ and $x = -2$.

For $x^* = -2$, we have

$$f'(-2) = 3 > 1$$

By theorem (1.4.1) $x^* = -2$ is an unstable equilibrium point.

For $x^* = 0$, we have $f'(0) = -1$ so theorem (1.4.3) applies ,we obtain

$$-2f'''(x^*) - 3[f''(x^*)]^2 = -12 < 0$$

Theorem (1.4.3) then declares that the equilibrium point $x^* = 0$ is asymptotically stable.

1.5. Chaotic Behavior [3, 6, 11]

Introduction:

The roots of chaos theory date back to about 1900, in the studies of Henri Poincare on the problem of the motion of three objects in mutual gravitational attraction, the so-called three-body problem. Poincare found that there can be orbits which are non-periodic, and yet not forever increasing nor approaching a fixed point. Later studies, also on the topic of nonlinear differential equations, were carried out by G.D.Birkhoff, A.N.Kolmogorov, M.L.Cartwright, J.E.Littlewood, and Stephen Smale. Except for Smale, who was perhaps the first pure mathematician to study nonlinear dynamics, these studies were all directly inspired by physics: the three-body problem in the case of Birkhoff, turbulence and astronomical problems in the case Klomogorov, and radio engineering in the case of Catwright and Littlewood. Although chaotic planetary motion had not been observed, experimentalists had encountered turbulence in fluid motion and non-periodic oscillation in radio circuits without the benefit of a theory to explain what they were seeing.

Chaos theory progressed more rapidly after mid-century, when it first became evident for some scientists that linear theory, the prevailing system theory at that time, simply could not explain the observed behavior of certain experiments like that of the logistic map. The main catalyst for the development of chaos theory was the electronic computer. Much of the mathematics of chaos theory involves the repeated iteration of simple mathematical formulas, which would be impractical to do by hand. Electronic computers made these repeated calculations practical. One of the earliest electronic digital computers was used to run simple weather forecasting models.

An early pioneer of the theory was Edward Lorenz whose interest in chaos came about accidentally through his work on weather prediction in 1961.Lorenz was using computer to run his weather simulation.

The term chaos as used in mathematics was coined by the applied mathematician James A.Yorke.

The study of chaos is a new and rapidly evolving branch of mathematics, hence there is no general agreement on a strict definition of chaos, but Devany's definition of chaos (in 1986) is the most widely known and accepted.

Definition 1.5.1(Devaney) :[11]

We say that $f: X \rightarrow X$ is chaotic on X if

1. f is transitive.
2. f has sensitive dependence on initial conditions.
3. The set of periodic points of f is dense in X.

Definition1.5.2: [11]

Given $f: X \rightarrow X$, we say that f is transitive if for all non-empty open subsets U and V of X there exists a non-negative integer k that $f^k(U) \cap V \neq \phi$

Let us shed some light on this claim by considering the difference equation

$$x(n+1) = f(x(n)) \tag{1.5.1}$$

where f is a real-valued function of a real variable.

Definition1.5.1:[6]

The solution of equation (1.5.1) is said to have "sensitive dependence on initial conditions " if there is a $d > 0$, so that for each x_0 in an interval I and every open interval J containing x_0, there is a y_0 in J so that the solutions x and y of equation (1.5.1) with $x(0) = x_0$ and $y(0) = y_0$ satisfy

$|x(n) - y(n)| > d$ for some n .

This property is sometimes called "the butterfly effect" because if the laws of meteorology have this property, then the motion of a butterfly's wings can have large scale effects on the weather.

Example1.5.2:

Consider the logistic equation (nonlinear)

$$x(n+1) = ax(n)(1-x(n)) \qquad n = 1,2,3,\ldots \tag{1.5.2}$$

This difference equation is known in biology as the discrete logistic model. Such models occur in the study of populations which reproduce at discrete intervals, (such as once a year).

which can be transformed into a linear equation after letting $a = 2$, we get

$x(n+1) = 2x(n)(1-x(n))$

Now, let

$$x(n) = \frac{1}{2}(1 - y(n))$$

We obtain

$$y(n+1)-y^2(n)=0$$

Let $z(n)=\ln y(n)$

$$\ln y(n+1)=\ln y(n)^2$$
$$\ln y(n+1)=2\ln y(n)$$
$$z(n+1)-2z(n)=0$$

From the characteristic equation $m = 2$,

so the general solution is

$$z(n)=z(0)\,2^n$$
$$\ln y(n)=\ln x(0)\,2^n$$
$$e^{\ln y(n)}=e^{\ln y(0)}\,e^{2^n}$$
$$y(n)=y(0)\,e^{2^n}$$
$$x(n)=\frac{1}{2}(1-(1-2x(0))e^{2^n}$$

Similarly for $a = 4$, then the equation (1.5.2) become

$x(n+1) = 4x(n)(1-x(n))$

and substitute $x(n)=\sin^2(\theta(n))$

$$\sin^2\theta(n+1)=4\sin^2\vartheta(n)(1-\sin^2\theta(n)$$
$$=4\sin^2\theta(n)\cos^2\theta(n)$$
$$=\sin^2 2\theta(n)$$

So $$\theta(n+1)=2\theta(n)$$

The general solution is:

$$\theta(n)=2^n\theta_0$$

Now

$$x(n)=\sin^2\theta(n)=\sin^2(2^n\theta_0)$$

$$= \sin^2(2^{n-1}2\theta_0)$$

We need to evaluate $2\theta_0$

$$\begin{aligned}
x_0 &= \sin^2\theta_0 \\
-2x_0 &= -2\sin^2\theta_0 \\
1-2x_0 &= 1-2\sin^2\theta_0 \\
1-2x_0 &= \cos 2\theta_0 \\
\cos^{-1}(1-2x_0) &= 2\theta_0
\end{aligned}$$

Hence

$$x(n) = \sin^2\left[2^{n-1}\cos^{-1}(1-2x(0))\right] \qquad (1.5.3)$$

where $x(0) \in [0,1]$,the behavior of these solutions is different, so for a = 2 each

solution $x(n)$ of (1.5.1) with $0 < x(0) < 1$ converges asymptotically to the stable

equilibrium point $x = \frac{1}{2}$, but when a = 4, most solutions jump randomly about the

interval (0,1).

So there are many periodic points such as $x = \frac{3}{4}$ (period 1)

and $x = \frac{1}{2}(1-\cos 2\frac{\pi}{5})$ (period 2).

Now we investigate the nature of the solutions for intermediate values of a. To obtain

the fixed points, solve

$$f(x) = x \Rightarrow x = \frac{a-1}{a}, \qquad x = 0$$

By theorem (1.4.1), we have

$$f'(0) = a > 2 \quad and \quad f'\left(\frac{a-1}{a}\right) = 2-a$$

So $x^* = 0$ is unstable, while $x^* = \frac{a-1}{a}$ is asymptotically stable for $2 \le a < 3$

Now

$$f^2(x) = f(f(x)) = a^2 x(1-x)(1-ax+ax^2)$$

we set $f^2(x) = x$ then we have four values for x,

$$x_1 = 0, \quad x_2 = \frac{a-1}{a}, \quad x_3 = \frac{a+1+\sqrt{(a+1)(a-3)}}{2a}, \quad x_4 = \frac{a+1-\sqrt{(a+1)(a-3)}}{2a}$$

x_1 and x_2 are fixed points, but x_3 and $x_4$are periodic points of period 2.These points of period 2 are asymptotically stable as $3 < a < 3.45$ and when the forth period starts these points become unstable. At approximately $a = 3.57$ all points whose periods are powers of 2 are unstable, (we can see that in the following table (1.5.1)).

n	Cycle (2^n)	a_n
1	2	3
2	4	3.449490
3	8	3.544090
4	16	3.564407
5	32	3.568750
6	64	3.56969
7	128	3.56989
8	256	3.569934
9	512	3.569943
10	1024	3.5699451
11	2048	3.569945557
∞	Accumulation point	3.569945672

(Table 1.5.1)

For some values of $a > 3.57$, the motion of solutions appears random, while there are small a intervals on which asymptotically stable periodic solutions control the behavior of solutions. Near $a = 4$ most solutions bounce erratically around the interval (0,1)(see figures (1.5.1) and (1.5.2)).

Now when $a = 4$ let

$$\begin{aligned}\theta(n) &= 2^{n-1}\cos^{-1}(1-2x(0)) \bmod \pi \\ &= 2^{n-1}\theta(1) \bmod \pi\end{aligned}$$

then $\theta(n)$ is in the interval $[0,\pi\ )$ and $2^{n-1}\theta(1)-\theta(n)$ is an integral multiple of π .From equation (1.5.3) $x(n)=\sin^2\theta(n)$.

Since $x(n)=\sin^2\theta(n)$ so the probability density function p for $x(n)$ is given by

$$\int p(x(n))dx(n) = \tfrac{2}{\pi}\int d\theta(n)$$

Thus

$$\begin{aligned}p(x(n)) &= \frac{2}{\pi\dfrac{dx(n)}{d\theta(n)}} \\ &= \frac{2}{\pi 2\sin\theta(n)\cos\theta(n)} \\ &= \frac{1}{\pi\sqrt{x(n)(1-x(n))}}\end{aligned}$$

If we use p to subdivide [0,1] into subintervals of equal probability then we expect to find, after many iterations, the same number of values of a solution in each subinterval. For example to obtain four subintervals of equal probability, set

$$\int_0^b \frac{1}{\pi\sqrt{x(n)(1-x(n))}} = 0.25$$

then

$b = \sin^2\left(\frac{0.25\pi}{2}\right) \cong 0.14645$

the subintervals are [0, 0.14645], [0.14645, 0.5], [0.5, 0.85355], and [0.85355, 1]. As a rough check, 600 iterations was computed of equation (1.5.3) with a = 4 and x(0) = 0.2 and was obtained 158, 143, 155, and 144 points in the first, second, third, and fourth subintervals respectively**[6]**.

Hence the most fundamental characteristic of chaotic motion is the property of sensitive dependence on initial conditions, and this property implies that any small error in the initial conditions can lead to much larger errors in the values of a solution as n increases.

As we have seen in the previous example the solutions of equation (1.5.1) with a = 4 have sensitive dependence on initial conditions because of the angle doubling that occurs in equation (1.5.2)

To be more specific, consider the initial values x(0) and y(0) which lie near each other in (0,1).The corresponding angles

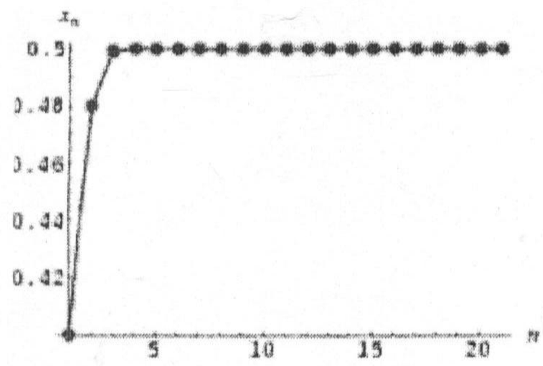
$\theta(n) = \cos^{-1}(1-2x(0)) \bmod \pi \quad and \quad \varphi(1) = \cos^{-1}(1-2y(0)) \bmod \pi$

For $n \geq 1$ $\theta(n) - \varphi(n) = 2^{n-1}(\theta(n) - \varphi(1)) \bmod \pi$

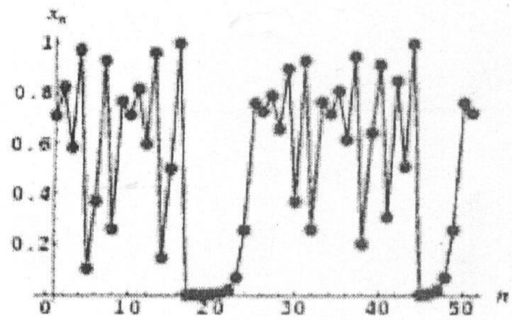
So the difference doubles (mod π)with each iteration ,so the solutions

$x(n) = \sin^2 \theta(n)$ and $y(n) = \sin^2 \varphi(n)$

will not be near each other for most values of n .



(Figure 1.5.1, With *a* = 2, the logistic map becomes $x(n+1) = 2\, x(n)\, (1- x(n))$)



(Figure 1.5.2, With a = 4, the logistic map becomes $x(n+1) = 4x(n)\, (1- x(n))$)

Another characteristic of chaotic motion is the existence of many unstable periodic points

Definition 1.5.3:[3]

Let *c* be a point in the domain of *f*, then *c* is called a periodic point of the equation

$x(n+1) = f(x(n))$

if for some positive integer *k* we have .

$f^k(c) = c$.

Hence a point is k-periodic if it is a fixed point of f^k ,that is , if it is an equilibrium point of the difference equation

$x(n+1) = g(x(n))$

where $g = f^k$.

Example1.5.4:

From the previous example(1.5.2)consider again the equation (1.5.3) written in the form $x(n) = \sin^2(2^{n-1}\theta(1))$

If there is an integer m such that

$2^{n-1}\theta(1) = \theta(1) + m\pi$ for some n, then $\theta(1)$ yields a periodic point

$$\theta(1) = \frac{m\pi}{2^{n-1} - 1}$$

for all integers $n \geq 2$ all $m \in Z$, so that $0 \leq m \leq 2^{n-1} - 1$.

Remark: There are dense periodic points in the interval [0,1] all of them are unstable because of the butterfly effect.

Chapter Two

Higher Order Linear Difference Equations

In this chapter linear difference equations of high order is studied in more than one direction: first basic theorems are discussed together with underlying definitions and notations. Moreover some widely used methods are presented with illustrations. It is relevant to say that high order equations arise in almost every field of scientific research like population dynamics, economics and the study of the motion of a single body (physics), some of which will be given in chapter three.

2.1 Definitions and Basic Theory

This section gives some definitions and theorems for linear difference equations.

Definition 2.1.1:[4]

A difference equation of order k over a set $\boldsymbol{S}$= $\{1,2,\ldots,k\}$ is said to be linear over $\boldsymbol{S}$ if it can be written in the form:

$$f_0(n)x(n+k)+f_1(n)x(n+k-1)+\cdots+f_{k-1}(n)x(n+1)+f_k(n)x(n)=g(n) \quad (2.1.1)$$

where $f_0, f_1, \ldots, f_k$ and g are functions defined for all values of n in the set $\boldsymbol{S}$.

If the coefficients $f_0(n), \ldots, f_k(n)$ are constant functions, the difference equation is called linear with constant coefficients. [The function g is arbitrary and need not be constant].

In general if (2.1.1) is of order k , then $f_0(n)$ is different from zero and we may divide by f_0 and rename our coefficients

$$a_1(n) = \frac{f_1(n)}{f_0(n)},\ a_2(n) = \frac{f_2(n)}{f_0(n)}, \ldots, a_k(n) = \frac{f_k(n)}{f_0(n)},\ r(n) = \frac{g(n)}{f_0(n)}$$

Hence we obtain

$$x(n+k) + a_1(n)x(n+k-1) + \cdots + a_{k-1}(n)x(n+1) + a_k(n)x(n) = r(n) \qquad (2.1.2)$$

Since $f_k(n)$is also different from zero , we know $a_k(n) \neq 0$.

Theorem 2.1.2: [3]

The initial value problem of k-th order

$$x(n+k) + f_1(n)x(n+k-1) + \cdots + f_{k-1}(n)x(n+1) + f_k(n)x(n) = g(n) \qquad (2.1.3)$$

over a set of consecutive integer values of n has one and only one solution $x(n)$ where $f_i(n)$ and $g(n)$ are real valued functions defined for $n \geq n_0$ and $f_i(n) \neq 0$ for all $n \geq n_0$, $x(n_0) = x_0$.

Proof:

The equation (2.1.3) can be written in the form

$$x(n+k) = -f_1(n)x(n+k-1) - f_2(0)x(n+k-2) - \cdots - f_k(n)x(n) + g(n) \qquad (2.1.4)$$

By letting $n = 0$ in equation (2.1.4), we obtain $x(k)$ in terms of $x(k\text{-}1)$, $x(k\text{-}2)$, …, $x(0)$. Explicitly, we have

$$x(k) = -f_1(0)x(k-1) - f_2(0)x(k-2) - \ldots - f_k(0)x(0) + g(0)$$

hence $x(k)$ is computed.

We can go to the next step and evaluate $x(k+1)$ by letting $n = 1$ in equation (2.1.4) which yields:

$$x(k+1) = -f_1(1)x(k) - f_2(1)x(k-1) - \cdots - f_k(1)x(1) + g(1)$$

By repeating the above process, it is possible to evaluate all values of $x(n)$ $n \geq k$

By using equation (2.1.4) for $n = n_0, n_0+1, n_0+2, \ldots$.

We prove using mathematical induction, that the value of x at each point of **S** is uniquely determined, thus proving that there is a unique solution.

The prescribed values determine the value $x(n_0+k)$uniquely for with $n = n_0$ in equation (2.1.3)

$$x(n_0+k) = g(n_0) - f_1(n_0)x(n_0+k-1) - \cdots - f_k(n_0)x(n_0)$$

from which $x(n_0+k)$ is determined..

Now let us make the induction hypothesis that x is known for all n-values in **S** up to and including $x(n_0+j)$ where $j \geq k$.

We complete the proof by showing that the following value $x(n_0+j+1)$ is uniquely determined.

In the difference equation (2.1.3) replaces $n = n_1$ by $n_0+j+1-k$ to obtain

$$x(n_0+j+1) = g(n_1) - f_1(n_1)x(n_0+j) - f_2(n_1)x(n_0+j) - \cdots - f_k(n_1)x(n_0+j-k+1) \qquad (2.1.5)$$

By our induction assumption, the values of x appearing on the right –hand side of (2.1.5) are known, therefore we can find $x(n_0+j+1)$ from this equation .

So we proved that x is uniquely determined for $n = n_0, n_0+1, n_0+2, \ldots, n_0+k-1$.

Now we need to show if $m > n_0$ such that $n = m-1, m-2, \ldots, n_0+1, n_0$ then $x(m-1)$, $x(m-2), \ldots, x(n_0+1)$, $x(n_0)$ are uniquely determined.

writing the difference equation with $n = m-1$ we obtain

$$f_k(m-1)x(m-1) = g(m-1) - f_1(m-1)x(m+k-2) - \ldots - f_{k-1}(m-1)x(m) \qquad (2.1.6)$$

f_k is never zero so we may divide by $f_k(m-1)$ to determine $x(m-1)$.

We have a unique value of x determined for every n – value in $\boldsymbol{S}$ and therefore a unique function which satisfies the difference equation , is obtained.

Next, we consider the case where $a_1,\ldots, a_k$ are constant functions over $\boldsymbol{S}$.

Theorem 2.1.3: [4]

If $x^{(1)}$ and $x^{(2)}$ are any two solutions of the linear homogeneous difference equation

$$x(n+k) + a_1x(n+k-1) + \cdots + a_{k-1}x(n+1) + a_kx(n) = 0 \qquad (2.2.7)$$

Then $C_1\, x^{(1)} + C_2\, x^{(2)}$ is also a solution for arbitrary constants $C_1$and C_2 .

Proof:(for more details see [4] page 123)

Theorem 2.1.4:[4]

If x is a solution of the homogeneous equation (2.1.7) and $\tilde{x}$ is a solution of the non-homogeneous equation

$$x(n+k) + a_1x(n+k-1) + \ldots + a_{k-1}x(n+1) + a_k\, x(n) = r(n)$$

then $x + \tilde{x}$ is a solution of (2.1.2)

Proof :(for more details see [4] page 124)

Theorem 2.1.5:[4]

Consider the linear first – order difference equation with constant coefficients :

$$x(n+1) + a_1x(n) = r(n) \ .$$

(a) The function x given by :

$x(n) = C(-a_1)^n$,

with C an arbitrary constant , is the general solution of the corresponding homogeneous equation :

$$x(n+1) + a_1 x(n) = 0$$

(b) If $\tilde{x} = \tilde{x}(n)$ is any particular solution of the complete equation, then $x + \tilde{x}$ is the general solution of the non-homogeneous equation.

$$x(n) = C(-a_1)^n + \tilde{x}(n) .$$

Proof : (for more details see [4] page 125)

Example 2.1.6 :

Consider the difference equation :

$$x(n+1) - 2x(n) = 5 \qquad (2.1.8)$$

The homogeneous equation $x(n+1) - 2x(n) = 0$

has the general solution x given by

$$x(n) = C.2^n$$

Since we need any particular solution of the complete equation let us try a constant function.

Suppose that $\tilde{x}(n) = A$ and try to determine the constant A so that $\tilde{x}$ will satisfy the complete equation .We must have

$A - 2A = 5$

So that $\tilde{x} = A = -5$

Thus the general solution of the given difference equation is $x + \tilde{x}$ or

$x(n) = C.2^n - 5$ (2.1.9)

If we are asked to find that solution of (2.1.9) for which

$$x(0) = 4 \tag{2.1.10}$$

We determine the particular value of C in (2.1.9) for which this initial condition is satisfied put $n = 0$ in (2.1.9) to find

$$x(0)= C - 5 \tag{2.1.11}$$

In view of (2.1.11) we take $C = 9$ .thus if $x(n) = 9.\ 2^n - 5$

x is the solution satisfying both the difference equation (2.1.8) and the initial condition (2.1.11).

2.2. Fundamental Set of Solutions

This section introduces the notion of a fundamental set of solutions.

Moreover, some important properties are proved together with some illustrations on the second order equations.

Definition 2.2.1:[3]

The functions $f_1(n), f_2(n), \ldots, f_k(n)$ are said to be linearly dependent for $n \geq n_0$ if there are constants $a_1, a_2, \ldots, a_k$, not all zero, such that

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_k f_k(n) = 0 \qquad n \geq n_0 \qquad (2.2.1)$$

If $a_j \neq 0$, then we may divide equation (2.2.1) by a_j to obtain

$$\begin{aligned} f_j(n) &= \frac{-a_1}{a_j} f_1(n) - \frac{a_2}{a_j} f_2(n) - \cdots - \frac{a_k}{a_j} f_k(n) \\ &= -\sum_{i \neq j} \frac{a_i}{a_j} f_i(n). \end{aligned}$$

Definition 2.2.2:[3]

The functions $f_1(n), f_2(n), \ldots, f_k(n)$ are said to be linearly independent for $n \geq n_0$ if whenever

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_k f_k(n) = 0 \qquad n \geq n_0.$$

then we must have $a_1 = a_2 = \cdots = a_k = 0$

Now, consider the problem of solving the linear second – order difference equation with constant coefficients .

$$x(n+2) + a_1x(n+1) + a_2x(n) = g(n) \qquad (2.2.2)$$

then the corresponding homogeneous equation is the following:

$$x(n+2) + a_1x(n+1) + a_2x(n) = 0 \qquad (2.2.3)$$

Theorem 2.2.3:[4]

let $x^{(1)}$ and $x^{(2)}$ be two solutions of the homogeneous difference equation (2.2.3) and let $X(n) = C_1x^{(1)}(n) + C_2\, x^{(2)}(n)$

Where C_1 and C_2 are arbitrary constants ,if the determinant of the fundamental matrix

$\begin{bmatrix} x^{(1)}(0) & x^{(2)}(0) \\ x^{(1)}(1) & x^{(2)}(1) \end{bmatrix}$ is not equal to zero

then X is the general solution of (2.2.3)

Proof :

By theorem (2.1.3) , X is a solution of (2.2.3) we need only show that x is any solution of (2.2.3). C_1 and C_2 may be determined so that X and x are identical .

By the uniqueness Theorem (2.1.2), it suffices to show that X and x are equal at $n = 0$ and $n = 1$, that is, we must determine C_1 and C_2 so that $X(0) = x(0)$ and $X(1) = x(1)$ for any choice of $x(0)$ and $x(1)$ .But

$$X(0) = C_1x^{(1)}(0) + C_2\, x^{(2)}(0)$$

$$X(1) = C_1x^{(1)}(1) + C_2\, x^{(2)}(1)$$

So C_1 and C_2 must satisfy the system

$x^{(1)}(0)\, C_1 + x^{(2)}(0)\, C_2 = x(0)$

$x^{(1)}(1)\, C_1 + x^{(2)}(1)\, C_2 = x(1)$ (2.2.4)

this is a system of two linear equations for the unknowns $C_1$and C_2 . By hypothesis the determinant formed by the coefficients of $C_1$and C_2 is different from zero . Which means that we can solve to find a unique pair of values of $C_1$and C_2 for each choice of $x(0)$ and $x(1)$.

Definition 2.2.4: [4]

Two solutions $x^{(1)}$ and $x^{(2)}$, of (2.2.3) which satisfy condition (2.2.4) are said to form a fundamental set (fundamental system) of solutions of (2.2.3).

Theorem 2.2.5: [4]

If $\tilde{x}$ is a particular solution of (2.2.3) and $x^{(1)}$ and $x^{(2)}$ form a fundamental set of solutions of (2.2.3) , then the general solution of (2.2.3) is given by

$X = C_1 x^{(1)} + C_2\, x^{(2)} + \tilde{x}$, where C_1 and C_2 are constants.

Proof:(for more details see[2] page132)

Example 2.2.6:

The difference equation

$x(n+2) - 3x(n+1) + 2x(n) = -1$ (2.2.5) has a particular solutions given by $x^*(n) = n$, as may be verified by direct substitution , similarly , it may be shown that the homogeneous equation

$$x(n+2) - 3\,x(n+1) + 2\,x(n) = 0 \qquad (2.2.6)$$

has the two solutions $x^{(1)}$ and $x^{(2)}$ given by $x^{(1)}(n) = 1$ and $x^{(2)}(n) = 2^n$.

these two solutions form a fundamental set since

$$\begin{vmatrix} x^{(1)}(0) & x^{(2)}(0) \\ x^{(1)}(1) & x^{(2)}(1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$$

hence , by the two previous theorems we get

$$X(n) = C_1 + C_2\, 2^n$$

Which is the general solution of the homogeneous equation (2.2.6) and hence

$$X(n) = C_1 + C_2\, 2^n + n \qquad (2.2.7)$$

If we want a solution of (2.2.5) (for which $x(0) = 0$ and $x(1) = 3$ (2.2.8)

We use these initial data to find $C_1 = -2$, $C_2 = 2$ and thus the solution is given by

$$X(n) = -2 + 2^{n+1} + n \,.$$

2.3. General Solution of Homogeneous Equations

We seek the general solution of the homogeneous equation

$$x(n+2) + a_1x(n+1) + a_2x(n) = 0 \qquad (2.3.1)$$

it is easy to find solutions of the difference equation for which

$$x(n) = m^n \qquad (2.3.2)$$

where m is some suitably chosen constant different from zero .

If this trial solution is substituted in (2.3.1) , we obtain , after division by the common factor m^n ,

$$m^2 + a_1m + a_2 = 0 \qquad (2.3.3)$$

this is called the characteristic equation of the difference equation (2.3.1)

The characteristic equation (2.3.3) is a quadratic algebraic equation and therefore has two nonzero roots , say m_1 and m_2 provided that $a_2 \neq 0$.

To these roots there correspond the solutions $x^{(1)}$and $x^{(2)}$ given by

$$x^{(1)}(n) = m_1{}^n \text{ and } \quad x^{(2)}(n) = m_2{}^n \qquad (2.3.4)$$

Now, we consider three cases :

1. The roots m_1 and m_2 are real numbers and unequal .
2. The two roots are real and equal .
3. The two roots are complex conjugate numbers .

Case 1: Roots Real and Unequal

In this case, the solutions in (2.3.4) form a fundamental set.

To see this, we calculate the determinant as given in theorem (2.3.2) .We obtain

$$\begin{vmatrix} x^{(1)}(0) & x^{(2)}(0) \\ x^{(1)}(1) & x^{(2)}(1) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ m_1 & m_2 \end{vmatrix} = m_2 - m_1 \qquad (2.3.5)$$

Which is different from zero since $m_1 \neq m_2$. Hence, by theorem (2.3.2) the general solution of the homogeneous difference equation (2.3.1) is given by

$x(n) = C_1 m_1^n + C_2 m_2^n$

Example2.3.1:

The difference equation

$x(n+2) + 3x(n+1) + x(n) = 0$

has the characteristic equation :

$m^2 + 3m + 1 = 0$

and has the roots $\dfrac{-3 \pm \sqrt{5}}{2}$,

which are real and unequal roots , so the general solution is given by

$$x(n) = C_1 \left(\frac{-3+\sqrt{5}}{2} \right)^n + C_2 \left(\frac{-3-\sqrt{5}}{2} \right)^n$$

Case 2: Real and Equal Roots:

The solutions $x^{(1)}$and $x^{(2)}$ in (2.3.4) no longer form a fundamental set since the determinant (2.3.5) is zero .

We keep $x^{(1)}$, but must find a new function $x^{(2)}$ which together will form a fundamental set . Such a solution is given by

$$x^{(2)}(n) = n\, m_1^n \qquad (2..3.6)$$

to prove that if $m_1 = m_2$ then (2.3.6) actually is a solution , we substitute , $x^{(2)}$ for x in the difference equation (2.3.1) and check that the equation is satisfied .

$x^{(2)}(n+2) + a_1 x^{(2)}(n+1) + a_2 x^{(2)}(n)$

$= (n+2)\, m_1^{n+2} + a_1 (n+1)\, m_1^{n+1} + a_2\, n\, m_1^{n}$

$$= n\, m_1^{n} (m_1^2 + a_1 m + a_2) + m_1^{n+1}(2m_1 + a_1) \qquad (2.3.7)$$

but the terms in the first parenthesis add to 0 since m_1 is a root of the auxiliary equation . Furthermore the sum of the roots of the auxiliary equation is $-a_1$. But if $m_1 = m_2$, this makes (2.3.7) identically zero so (2.3.6) is indeed a solution of the difference equation.

Moreover, the two solutions

$x^{(1)}(n) = m_1^{\,n}$ and $x^{(2)}(n) = n\, m_1^{\,n}$

from a fundamental set, since

$$\begin{vmatrix} x^{(1)}(0) & x^{(2)}(0) \\ x^{(1)}(1) & x^{(2)}(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ m_1 & m_1 \end{vmatrix} = m_1$$

which is different from zero since no root of the auxiliary equation is zero .

Thus, when the two roots m_1 and m_2 are equal so the general solution is given by

$X(n) = C_1 m_1^{n} + C_2 n m_1^{n}$

Or

$X(n) = (C_1 + C_2 n)\, m_1^{n}$

The difference equation

$4x(n+2) + 4x(n+1) + x(n) = 0$ has the auxiliary equation

$4m^2 + 4m + 1 = 0$ which implies

$m^2 + m + \frac{1}{4} = (2m + 1)^2 = 0$

so that roots $m_1 = m_2 = -\frac{1}{2}$ .Hence the general solution is given by

$X(n) = (C_1 + C_2 n)(-\frac{1}{2})^n$

Case 3: Complex Roots :

The complex roots of the quadratic equation always occur in conjugate pairs , therefore , if m_1 and m_2 are complex roots of the auxiliary equation , then $m_1 \neq m_2$ and the calculation in (2.3.5) shows that the two solutions

$x^{(1)}(n) = m_1{}^n$ and $x^{(2)}(n) = m_1{}^n$, form a fundamental set , hence the general solution is given by

$X(n) = C_1 m_1{}^n + C_2 m_1{}^n$

The only difficulty with this general solution may be a complex number if m_1 and m_2 are themselves complex . We require solutions which have real number values for all n – values for which they are defined.

It is possible to show that if C_1 and C_2 are complex conjugates, $X(n)$ is always a real number . To prove this we write all complex numbers in polar form

The roots of the auxiliary equation are complex conjugates and hence have polar form .

$m_1 = r(cos\theta + isin\,\theta)$

$m_2 = r(cos\theta - isin\,\theta)$

we are assume that C_1 and C_2 are complex conjugates :

$C_1 = a(cosB + isin\,B)$, $C_2 = a(cosB - isin\,B)$

By de Moiver's formula ,

$m_1{}^n = r^n(cos\,(n\theta) + i\,sin\,(n\theta))$, $m_2{}^n = r^n(cos(n\theta) - i\,sin\,(n\theta))$ (2.3.8)

Thus we have,

$X(n) = C_1 m_1{}^n + C_2 n m_1{}^n$

$= ar^n\,(cos(n\theta+B)+i\,sin(n\theta+B))+ar^n\,(cos(n\theta+B) - i\,sin(n\theta+B))$

$= 2\,ar^n\,cos(n\theta + B)$ (2.3.9)

and this is a real number as required.

The numbers r and θ are determined from (2.3.8) by writing the roots of the auxiliary equation in polar form. The constants a and B in (2.3.9) take the role of $C_1$and C_2 . If we denote the constants 2a by A , the general solution of the homogeneous difference equation when the auxiliary equation has complex roots can be written in a simpler form .

$$X(n) = Ar^n cos(n\theta + B) \qquad (2.3\ .10)$$

where A and B are arbitrary constants

Example 2.3.1 :

The difference equation

$x(n+2) – 2x(n+1) + 2x(n) = 0$

has the auxiliary equation

$m^2 – 2m + 2 = 0$

from the quadratic formula , we find the roots $m_1 = 1 + i$ and $m_2 = 1 – i$. In polar form :

$$m_1 = 1 + i = \sqrt{2}\ (cos\ \frac{\pi}{4} + isin\frac{\pi}{4}\ )$$

therefore , the general solution is given by

$$X(n) = A(\sqrt{2}\ )^n cos(n\frac{\pi}{4} + B)\ .$$

2.4. Undetermined Coefficients Method for solving Linear non-homogeneous Difference Equations

This section introduces the method of undetermined coefficients for solving the *k*- th order linear non-homogeneous equation

$$x(n+k)+f_1(n)x(n+k-1)+...+f_k(n)x(n)=g(n) \qquad (2.4.1)$$

where $f_k(n)\neq 0$ for all $n\geq n_0$.The function $g(n)$ is called the forcing term , the external force ,the control ,or the input of the system in which $g(n)$ is the input and $x(n)$ is the output .

However, we shall only be concerned with the case where the functions $f_i(n)$ are constants for $n\in Z^+$

Now turn to methods for finding solutions of the second order of the non-homogeneous equation

$$x(n+2)+a_1x(n+1)+a_2x(n)=g(n) \qquad (2.4.2)$$

Having already found the general solution of the corresponding homogeneous equation, if we add to it any solution of (2.4.2), the sum, will be the general solution of the complete equation.

A number of special techniques exist for finding particular solutions of (2.4.2).The most useful of these is the method of undetermined coefficients.

The method of undetermined coefficients can be applied successfully when the function $g(n)$ is a linear combination of sums and products of functions with the following values: $a^n, \sin bn$ $, \cos bn$ $, n^k$ where a and b are any constants and *n* is any nonnegative integer .

The following table indicates the form of the trial solution to be used corresponding to some simple function $g(n)$

In equation (2.4.2) $g(n)$ is the sum of several different functions , each function should be treated separately .If the trial solution $\tilde{x}$ includes a function which is a solution of the homogeneous equation , this $\tilde{x}$ should first be multiplied by k and the new trial function used . If this function also contains a term which satisfies the reduced equation, then multiply again by k .In each case, the constant coefficients must be determined so that $\tilde{x}$ satisfies the complete difference equation

$g(n)$ in (2.5.2)	Trial solution $\tilde{x}$ (n)
a^n	$A a^n$
$\sin bn$ *or* $\cos bn$	$A\sin bn + B\cos bn$
n^k	$A_0 + A_1 n + A_2 n^2 + ... + A_k n^k$
$n^k \ a^n$	$a^n (A_0 + A_1 n + A_2 n^2 + ... + A_k n^k)$
$a^n \sin bn$ *or* $a^n \cos bn$	$a^n (A\sin bn + B\cos bn)$

Example 2.4.1:

Consider the second order equation

$$x(n+2) - 4x(n+1) + 4x(n) = 3n + 2^n$$

Which has two equal roots $m_1 = m_2 = 2$

The general solution of the reduced equation is given by:

$$x(n) = (C_1 + C_2 n)2^n$$

Now, we need to find a particular solution of the non-homogeneous equation. The right hand term 3*n* suggests the trial solution $\tilde{x}(n) = A_0 + A_1 n$, and the term 2^n suggests the trial solution $A\,2^n$. But 2^n is a solution of the reduced equation therefore, we multiply by *n* and try $An2^n$, but since this is also a solution of the reduced equation so we multiply again by *n* and we arrive at the trial solution $An^2 2^n$.

Thus the trial solution has the form:

$x(n) = A_0 + A_1 n + An^2 2^n$

By algebraic manipulations, we find the coefficients $A_0 = 6$, $A_1 = 3$, $A_2 = 1/8$.

The general solution of the non-homogeneous equation is therefore given by

$x(n) = (C_1 + C_2 n)2^k + 6 + 3n + \frac{1}{8} n^2 2^n$,

Where C_1 and C_2 are arbitrary constants.

2.5 The General Case of Order k

The linear kth- order difference equation with constant coefficients is given by :

$$x(n+k) + a_1 x(n+k-1) + \cdots + a_{k-1} x(n+1) + a_k x(n) = g(n) \qquad (2.5.1)$$

We study first the homogeneous equation :

$$x(n+k) + a_1 x(n+k-1) + \cdots + a_{k-1} x(n+1) + a_k x(n) = 0$$

$$m^k + a_1 m^{k-1} + \cdots + a_{k-1} m + a_k = 0$$

which is an algebraic equation of degree n , it has exactly n roots , denoted by

m_1 , m_2 , ..., m_k .These may be real or complex numbers , and any root may be distinct from the others or may be a repeated root .

A fundamental set of solutions $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$, for which the k-th order determinant is different from zero .

$$\begin{vmatrix} x^{(1)}(0) & x^{(2)}(0) & \cdots & x^{(k)}(0) \\ x^{(1)}(1) & x^{(2)}(1) & \cdots & x^{(k)}(1) \\ \vdots & \vdots & & \\ x^{(1)}(k-1) & x^{(2)}(k-1) & \cdots & x^{(k)}(k-1) \end{vmatrix} \neq 0$$

such a fundamental set of solutions is obtained as follows :

1) For each real unrepeated root m , write the solution (containing one arbitrary constant).

$$C_1 m^n$$

2) If a real root m is repeated p times , write the solution (containing p arbitrary constant)

$$(C_1 + C_2 n + C_3 n^2 + \cdots + C_p n^{p-1}) m^n$$

3) For each pair of unrepeated complex conjugate roots with modulus r and amplitude θ (containing two arbitrary constant).

$$A\, r^n \cos (n\theta + B)$$

4) If a pair of complex conjugate roots is repeated p times , write the solution (containing $2p$ arbitrary constants).

$$r^n (A_1 \cos (n\theta + B_1) + A_2\, n \cos (n\theta + B_2) + \cdots + A_p\, n^{p-1} \cos (n\theta + B_p))$$

The sum of the solutions obtained in this way contains k arbitrary constants and is the general solution of the homogeneous equation .

The method of undetermined coefficients , may be used to obtain particular solutions of the complete equation , and the sum of the general solution of the homogeneous equation and any particular solution of the complete equation is the general solution of the equation (2.5.1).

Example 2.5.1:

The third – order difference equation :

$x(n+3) - 9\, x(n+2) + 26\, x(n+1) - 24\, x(n) = 3$

Has the auxiliary equation :

$$m^3 - 9m^2 + 26m - 24 = 0$$

and this equation can be factored into

$(m - 2)(m - 3)(m - 4) = 0$

so that the three roots are 2, 3 , and 4.

The general solution of the homogeneous equation is given by :

$$X(n) = c_1\, 2^n + c_2\, 3^n + c_3\, 4^n$$

We try a constant function as a trial solution for the complete equation .If $y^*(k) = c$ is to satisfy the equation .

$$C \ -9\ c + 26\ c - 24\ c = 3$$

Or $c = -\frac{1}{2}$. Hence the general solution of the complete difference equation is given by :

$$x(n) = c_1\, 2^n + c_2\, 3^n + c_3\, 4^n - \tfrac{1}{2}$$

Chapter Three

Mathematical Models and Applications

3.1. The Transmission of Information [3, 4, 9]

The theory of difference equations is rich in applications in many branches of the natural sciences , applied mathematics, mathematical economics and classes of problems with chaotic dynamics.

In this chapter four diverse applications are studied together with relevant theorems needed to establish these results.

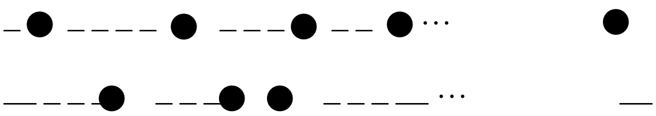
The problem and its model:

Suppose that a signaling has two signals S_1, and S_2 such as dots and dashes in telegraphy. Messages are transmitted over some channel by first encoding them into sequences of these two signals .Suppose that S_1 requires exactly t_1 units of time and S_2 exactly t_2 units of time to be transmitted . Let $N(t)$ be the number of possible message sequences of duration t time units. Now a signal of duration time t either ends with an S_1 or with an S_2 signal. (See figure 3.1.1).

If the message ends with S_1, the last signal must start at $t - t_1$ (since S_1 takes t_1 units of time). Hence there are $N(t - t_1)$ possible messages to which the last S_1 may be appended. Thus there are $N(t - t_2)$ messages of duration t that ends with S_1.

By a similar argument, we may conclude that there are $N(t - t_1)$ messages of duration t that ends with S_2 .Consequently the total number of messages $N(t)$ of duration t may be given by:

$$N(t) = N(t - t_1) + N(t - t_2) \tag{3.1.1}$$



; $N(t - t_1)$ possible message

; $N(t - t_2)$ possible message

(Figure 3.1 Two signals , one ends with S_1 and the other with S_2)

We study two cases :

i. If $t_2 \geq t_1$, then the above equation may be written in the familiar form of an t_2-th order equation

$$N(t+t_2) - N(t+t_2-t_1) - N(t) = 0$$

ii. On the other hand, if $t_1 \geq t_2$,we obtain the t_1-th order equation

$$N(t+t_1) - N(t+t_1-t_2) - N(t) = 0$$

These cases establish the model for our problem.

Let us consider the special case $t_1 = 1$ and $t_2 = 2$ where one signal takes twice as long to be transmitted over the channel as the other , then we have

$$N(t+2) = N(t+1) + N(t)$$

Which can be rewritten in the standard form :

$$N(t+2) - N(t+1) - N(t) = 0 \qquad t = 0,1,2,\ldots$$

This is a linear homogeneous second - order difference equation with constant coefficients.

To solve this equation we proceed as follows:

The auxiliary equation is $m^2 - m - 1 = 0$, with roots

$\frac{1 \pm \sqrt{5}}{2}$, which are real, distinct roots, hence the general solution is given by

$$N(t) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^t + c_2\left(\frac{1-\sqrt{5}}{2}\right)^t \tag{3.1.2}$$

Where c_1 and c_2 are arbitrary constants.

Now, if $N(0) = 0$ and $N(1) = 1$, then by equation (3.1.2) we get

$$c_1 = \frac{1}{\sqrt{5}} \quad , \quad c_2 = \frac{-1}{\sqrt{5}}$$

and the solution of our problem is given by:

$$N(t) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^t - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^t \tag{3.1.3}$$

Now let us introduce the notion of capacity of the channel, denoted by C, by the following definition:

$$C \equiv \lim_{t\to\infty} \frac{\log_2 N(t)}{t}$$

From equation (3.1.3) and this definition we have

$$C = \lim_{t\to\infty} \frac{\log_2 \frac{1}{\sqrt{5}}}{t} + \lim_{t\to\infty} \frac{1}{t}\log_2\left[\left(\frac{1+\sqrt{5}}{2}\right)^t - \left(\frac{1-\sqrt{5}}{2}\right)^t\right] \tag{3.1.4}$$

Now since $\left|\frac{1-\sqrt{5}}{2}\right| \approx 0.6 < 1$, $\left(\frac{1-\sqrt{5}}{2}\right)^t \to 0 \quad as \quad t \to \infty$, and

$\frac{1+\sqrt{5}}{2} \approx 1.6$ which is predominant ,we observe also that the first term in the right hand side of equation (3.1.4) goes to zero as $t \to \infty$.

Thus $C = \lim_{t\to\infty} \frac{1}{t} \log_2 \left(\frac{1+\sqrt{5}}{2}\right)^t$.

$$C = \log_2\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.7$$

The capacity of a channel measures how much the channel can allow the transmission of more complicated message sequences.

As a sequence of solving such problems, one can get an idea about deciding how much a specific channel can allow transmission by determining its capacity.

3.2. National Income [3]

In this example we analyze a problem in economics that is related to the national income of a certain country.

From an economics background, it is known that in capitalist countries the national income, $X(t)$, in a given period of time t may be written as

$$X(t) = C(t) + I(t) + G(t) \tag{3.2.1}$$

where

$C(t)$ represents the consumer expenditure for purchase of consumer goods.

$I(t)$ represents the Induced private investment for buying capital equipment, and

$G(t)$ represents the Government expenditure.

where t is usually measured in years.

The following assumptions are widely made and accepted by economists:

a) Consumer expenditure $C(t)$ is proportional to the national income $X(t\text{-}1)$ in the proceeding year $t-1$, mathematically this translates to:

$$C(t) = \alpha X(t-1) \tag{3.2.2}$$

where $\alpha > 0$ is commonly called marginal propensity to consume.

b) Induced private investment $I(t)$ is proportional to increase in consumption $C(t) - C(t\text{-}1)$, or mathematically:

$$I(t) = \beta[C(t) - C(t-1)] \tag{3.2.3}$$

where $\beta > 0$ is called the relation.

c) Finally , the government expenditure $G(t)$ is considered to be constant over the years and we may choose our units such that

$$G(t) = 1 \tag{3.2.4}$$

Employing formulas (3.2.2) , (3.2.3) and (3.2.4) in formula (3.2.1) produces the second order difference equation .

$$X(t+2) - \alpha(1+\beta)X(t+1) + \alpha\beta X(t) = 1 \qquad , t \in Z^{+} \tag{3.2.5}$$

Or

$$X(t+2) - p_1 X(t+1) + p_2 X(t) = 1 \tag{3.2.6}$$

here $p_1 = \alpha(1+\beta)$

$p_2 = \alpha\beta$

This models the national income in a certain country.

To find the equilibrium state of the national income let $X(n) = X^*$

$$X^* - \alpha(1+\beta)X^* + \alpha\beta X^* = 1$$
$$X^* = \frac{1}{1-\alpha} \qquad , \quad \alpha \neq 1$$

To find the conditions under which the solutions of the equation (3.2.5) converge to the equilibrium point X^* we use the following theorem:

Theorem 3.2.1:[1]

If the equilibrium state of the national income $X^* = \dfrac{1}{1-\alpha}$ is stable then the following conditions hold:

$$1 + p_1 + p_2 > 0 \quad , \quad 1 - p_1 + p_2 > 0 \quad , \quad 1 - p_2 > 0$$

these conditions are necessary and sufficient for the equilibrium point (solution) of the equation (3.2.6) to be asymptotically stable (i.e. ,all solutions converges to X^*)

Proof:

Assume that equilibrium point of the equation (3.2.6) is asymptotically stable .

The roots λ_1, λ_2 of the characteristic equation $\lambda_1^{\ 2} + p_1\lambda + p_2 = 0$ lie inside the unit disk i.e. , $|\lambda_1| < 1$ and $|\lambda_2| < 1$

$$\lambda_1 = \frac{p_1 + \sqrt{p_1^{\ 2} - 4p_2}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{p_1 - \sqrt{p_1^{\ 2} - 4p_2}}{2} \qquad (3.2.7)$$

Then there are two cases to consider:

Case 1:

First, λ_1, λ_2 are real roots ,i.e., $p_1 - 4p_2 \geq 0$

from formula (3.2.7) we have $-2 < -p_1 + \sqrt{p_1^{\ 2} - 4p_2} < 2$

$$-2 + p_1 < \sqrt{p_1^{\ 2} - 4p_2} < 2 + p_1 \qquad (3.2.8)$$

Squaring the second inequality in (3.2.8) then you get $1 + p_1 + p_2 > 0$

Similarly, if we square the first inequality in (3.2.8) $1 - p_1 + p_2 > 0$

Case 2:

Next, if λ_1, λ_2 are complex conjugates i.e. , $p_1 - 4p_2 < 0$ in this case we have

$$\lambda_1 = \frac{-p_1 + i\sqrt{4p_2 - p_1^{\ 2}}}{2} \quad , \quad \lambda_2 = \frac{-p_1 - i\sqrt{4p_2 - p_1^{\ 2}}}{2}$$

Since $-1 < \lambda_1 < 1$ and $-1 < \lambda_2 < 1$ we have $\lambda_1 \lambda_2 = p_2 < 1$ or $1 - p_2 > 0$

This completes the proof of the necessary conditions.

By applying the previous conditions on our equation we have

$\alpha < 1 \quad , \quad 1+\alpha+2\alpha\beta > 0 \quad , \quad \alpha\beta < 1$

Clearly the second inequality $1+\alpha+2\alpha\beta > 0$ is always satisfied since

α , β are both positive numbers.

The solutions are oscillatory about X^* if either λ_1, λ_2 are negative real numbers

or complex conjugates.

In the first case we have

$\alpha^2 (1+\beta)^2 > 4\alpha\beta$ or $\alpha > \dfrac{4\beta}{(1+\beta)^2}$

and $\alpha(1+\beta) < 0$ which is impossible

Thus if $\alpha > \dfrac{4\beta}{(1+\beta)^2}$ we have no oscillatory solutions

Now if λ_1, λ_2 are complex conjugates if $\alpha^2 (1+\beta)^2 < 4\alpha\beta$ or $\alpha < \dfrac{4\beta}{(1+\beta)^2}$

Hence all solutions are oscillatory if $\alpha < \dfrac{4\beta}{(1+\beta)^2}$

Now consider the difference equation in (3.2.5) in the special case when

$$\begin{aligned} &\alpha = 0.5 \quad and \quad \beta = 1, \\ &X(t+2) - X(t+1) + \tfrac{1}{2} X(t) = 1 \end{aligned} \tag{3.2.9}$$

We get the equation which has as its solution

$$X(t) = A\left(\frac{1}{\sqrt{2}}\right)^t \cos\left(\frac{t\pi}{4} + B\right)$$

To find a particular solution we assume a trial solution of the form $X^*(t) = 2$

The general solution of (3.2.9) is given by:

$$X(t) = A\left(\frac{1}{\sqrt{2}}\right)^t \cos\left(\frac{t\pi}{4} + B\right) + 2 \qquad (3.2.10)$$

let the initial conditions $X(0) = 2$ and $X(1) = 3$ be given ,then we have

$$A\cos B = 0 \qquad , \qquad A\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{4} + B\right) = 1$$

For which we find $A = -2$, and $B = \frac{\pi}{2}$ and the solution is

$$X(t) = -2\left(\frac{1}{\sqrt{2}}\right)^t \cos\left(\frac{t\pi}{4} + \frac{\pi}{2}\right) + 2$$

This may be simplified by noting that $\cos\left[x + \frac{\pi}{2}\right] = -\sin x$, therefore

$$X(t) = 2\left(\frac{1}{\sqrt{2}}\right)^t \sin\frac{t\,\pi}{4} + 2 \qquad (3.2.11)$$

The presence of the sine term makes X an oscillating function of time . Since

$r = \frac{1}{\sqrt{2}} < 1$,the oscillations are damped and the first term on (3.2.8) approaches zero as t increases , the constant term 2 remains and the sequence $\{X(t)\}$ converges to this limits undergoing damped oscillations above and below the level $X(t) = 2$

Thus , a constant containing level of government expenditure will result in damped oscillatory movements of national income , gradually approaching the asymptote $\frac{1}{1-\alpha}$ times the constant level of government expenditure.

Now let the case $\alpha = 0.8$, $\beta = 2$, which is iterated in much the same way . The equation (3.2.5) becomes

$$X(t+2) - 2.4X(t+1) + 1.6X(t) = 1 \qquad (3.2.12)$$

The auxiliary equation $m^2 - 2.4m + 1.6 = 0$

has the two roots $m_1 = 1.2 + 0.4i \quad and \quad m_2 = 1.2 - 0.4i$ since the roots are complex we are certain of oscillatory movements of national income , to find out if these oscillations are damped , steady , or explosive , we calculate the modulus

$$r = \sqrt{(1.2)^2 + (0.4)^2} = \sqrt{1.6}$$

Since $r > 1$, the oscillations are unbounded.

These results are essential to an economist in his investigation and planning for the economy of a certain country.

3.3. Gambler's Ruin **[3]**

Here is an interesting application which is relevant to probability theory.

Statement and model of the problem:

A gambler plays a sequence of games against an adversary in which the probability that the gambler wins $1.00 in any given game is a known value q , and the probability of his losing $1.00 is $(1-q)$, where $0 \le q \le 1$. He quits gambling if he either loses all his money or if he reaches his goal of acquiring N dollars . If the gambler runs out of his money first , we say that the gambler has been ruined .

Let $p(n)$ denote the probability that the gambler will be ruined if he possesses n dollars. He may be ruined in two ways :

First , by winning the next game ; the probability of this event is q ; then his fortune will be $(n+1)$ and the probability of being ruined will become $p(n+1)$.

Secondly , by losing the next game , the probability of this event is $(1-q)$and the probability of being ruined is $p(n-1)$.

Now the equation of the total probabilities is

$$p(n) = q\, p(n+1) + (1-q)\, p(n-1)$$

Replacing n by $n+1$, we get

$$p(n+2) - \frac{1}{q}\, p(n+1) + \frac{(1-q)}{q}\, p(n) = 0 \quad n = 0,1,2,\ldots,N \qquad (3.3.1)$$

with $p(0) = 1$ and $p(N) = 0$ .The characteristic equation of the difference equation is given by

$$m^2 - \frac{1}{q}m + \frac{1-q}{q} = 0$$

and hence the characteristic roots are :

$$m_1 = \frac{1}{2q} + \frac{1-2q}{2q} = \frac{1-q}{q} \quad , \quad m_2 = \frac{1}{2q} - \frac{1-2q}{2q} = 1$$

and thus the general solution may be written as

$$p(n) = c_1 + c_2\left(\frac{1-q}{q}\right)^n, if \quad q \neq \frac{1}{2}$$

Now using the initial conditions $p(0) = 1$and $p(N) = 0$, we obtain

$$c_1 + c_2 = 1$$
$$c_1 + c_2\left(\frac{1-q}{q}\right)^N = 0$$

which gives $$c_1 = -\frac{\left(\frac{1-q}{q}\right)^N}{1-\left(\frac{1-q}{q}\right)^N} \quad , \quad c_2 = \frac{1}{1-\left(\frac{1-q}{q}\right)^N}.$$

Thus $$p(n) = \frac{\left(\frac{1-q}{q}\right)^n - \left(\frac{1-q}{q}\right)^N}{1-\left(\frac{1-q}{q}\right)^N} \tag{3.3.2}$$

The special case when $q = \frac{1}{2}$ must be treated separately, since in this case we have repeated roots $m_1 = m_2 = 1$.This is certainly the case when we have a fair game . The general solution in this case may be given by

$$p(n) = a_1 + a_2 n$$

Which when using the initial conditions takes the value

$$p(n) = 1-\frac{n}{N} = \frac{N-n}{N} \tag{3.3.3}$$

For example , if you start with \$4 ,the probability that you win a dollar is 0.3 and you quit if you run out of money or have a total of \$10, then n = 4, q = 0.3 and N = 10, and the probability of being ruined is given by

$$p(4)=\frac{\left(\frac{7}{3}\right)^4-\left(\frac{7}{3}\right)^{10}}{1-\left(\frac{7}{3}\right)^{10}}=0.994$$

Hence the probability that the gambler wins is $\tilde{p}(4)=0.006$,which is too small.

On the other hand if q = 0.5 , N = \$100.00 and n = 20 , then from formula (3.3.3) we

have $$p(4)=1-\frac{20}{100}=0.8$$

And $\tilde{p}(4)=0.2$ which is small.

So we observe that if $q\le 0.5$ *and* $N\to\infty$, $p(n)$ tends to 1 in both formulas (3.3.2) and (3.3.3) and the gamblers ruin is certain .

The probability that the gambler wins is given by :

$$\tilde{p}(n)=1-p(n)\quad=\begin{cases}\dfrac{1-\left(\frac{1-q}{q}\right)^n}{1-\left(\frac{1-q}{q}\right)^N} & \text{if}\quad q\ne 0.5\\[2ex] \dfrac{n}{N} & \text{if}\quad q=0.5\end{cases}$$

we conclude that the probability that the gambler wins is too small as N grows up.

3.4 . Propagation of Annual Plants [3]

Our last example will be from the field of biological sciences where we start by developing a mathematical model that describes the number of plants in any desired generation. It is known that plants produce seeds at the end of their growth season (say August) , after which they die . Furthermore, only a fraction of these seeds survive in winter and those who germinate at the beginning of the season (May) giving rise to the new germination of plants.

A mathematical model:

Let us adopt the following variables and notations:

γ = number of seeds produced per plant in August .

α =fraction of one - year -old seeds that germinate in May .

β =fraction of tow - year - old seeds that germinate in May .

δ = fraction of seeds that survive a given winter .

If $p(n)$ denotes the number of plants in generation n , then

$$p(n) = \begin{pmatrix} plants\ \ from \\ one-year-old\ \ seeds \end{pmatrix} + \begin{pmatrix} plants\ \ from \\ tow-year-old\ \ seeds \end{pmatrix}$$

$$p(n) = \alpha s_1(n) + \beta s_2(n) \tag{3.4.1}$$

where $s_1(n)$, $s_2(n)$ is the number of one - year, two - year old seeds in April (before germination) respectively. The seeds left after germination may be written as:

$$\text{Seeds left} = \begin{pmatrix} fraction \\ not\ \ ger\min ated \end{pmatrix} \times \begin{pmatrix} original\ \ number \\ of\ \ seeds\ \ in\ \ April \end{pmatrix}$$

And this gives rise to the following two equations

$$\tilde{s}_1(n) = (1-\alpha)\, s_1(n) \tag{3.4.2}$$

$$\tilde{s}_2(n) = (1-\beta)\, s_2(n) \tag{3.4.3}$$

where $\tilde{s}_1(n), \tilde{s}_2(n)$ are the number of one - year , two - year seeds left in May after some have germinated, respectively . New seeds $s_0(n)$ (0 - year - old) are produced in August at the rate of γ per plant ,

$$s_0(n) = \gamma\, p(n) \tag{3.4.4}$$

After winter , seeds $s_0(n)$ that were new in generation n will be one year old in the next generation (n+1) and a fraction $\delta\, s_0(n)$ of them will survive .

Hence

$$s_1(n+1) = \delta\, s_0(n)$$

or by using (3.4.4) we have

$$s_1(n+1) = \delta\, \gamma\, p(n) \tag{3.4.5}$$

similarly

$$s_2(n+1) = \delta\, \tilde{s}_1(n)$$

by using formula (3.4.2)

$$\begin{aligned} s_2(n+1) &= \delta\,(1-\alpha)\, s_1(n) \\ s_2(n+1) &= \delta^2\, \gamma\,(1-\alpha)\, p(n-1) \end{aligned} \tag{3.4.6}$$

(by using equation (3.4.5))

Substituting for $s_1(n+1)$, $s_2(n+1)$ in expression (3.4.5) and (3.4.6) into formula (3.4.1) gives

$$p(n+1) = \alpha\gamma\delta\, p(n) + \beta\gamma\delta^2 (1-\alpha)\, p(n-1)$$

or

$$p(n+2) = \alpha\gamma\delta\, p(n+1) + \beta\gamma\delta^2 (1-\alpha)\, p(n) \qquad (3.4.7)$$

These equations represent a model that describes the number of plants in generation n .

The auxiliary equation of (3.4.7) is given by

$$m^2 - \alpha\gamma\delta\, m + \beta\,\gamma\,\delta^2 (1-\alpha) = 0$$

which has the roots:

$$m_1 = \frac{\alpha\,\gamma\,\delta}{2}\left[1+\sqrt{1+\frac{4\beta(1-\alpha)}{\gamma\alpha^2}}\right]$$

$$m_2 = \frac{\alpha\,\gamma\,\delta}{2}\left[1-\sqrt{1+\frac{4\beta(1-\alpha)}{\gamma\alpha^2}}\right]$$

since $1-\alpha > 0$, we observe that m_1 and m_2 are real roots. Furthermore $m_1 > 0$ and $m_2 < 0$.

Thus, the solution is given by:

$$p_1(n) = \left(\frac{\alpha\,\gamma\,\delta}{2}\left[1+\sqrt{1+\frac{4\beta(1-\alpha)}{\gamma\alpha^2}}\right]\right)^n$$

and $p_2(n) = \left(\frac{\alpha\,\gamma\,\delta}{2}\left[1-\sqrt{1+\frac{4\beta(1-\alpha)}{\gamma\alpha^2}}\right]\right)^n$.

To insure propagation (i.e. $p(n)$ increases indefinitely as $n \to \infty$) we need to have $m_1 > 1$, we are not going to do the same with m_2 since it is negative and leads to undesired oscillation in the size of the plants population .

Hence , we have

$$\frac{\alpha\,\gamma\,\delta}{2}\left[1+\sqrt{1+\frac{4\beta(1-\alpha)}{\gamma\alpha^2}}\right] > 1$$

or $$\frac{\alpha\,\gamma\,\delta}{2}\left[\sqrt{1+\frac{4\beta(1-\alpha)}{\gamma\alpha^2}}\right] > 1-\frac{\alpha\,\gamma\,\delta}{2} \qquad (3.4.8)$$

squaring both sides and simplifying yields

$$\gamma \quad > \quad \frac{1}{\alpha\,\delta+\beta\,\delta^2(1-\alpha)}$$

Now if $\beta = 0$,that is if no two -year - old seeds germinate in May then (3.4.8) becomes

$$\gamma \quad > \quad \frac{1}{\alpha\,\delta} \qquad (3.4.9)$$

The relation (3.4.9) says that plant propagation occurs if the product of the number of seeds produced per plant in August, the fraction of one - year - old seeds that germinate in May , and the fraction of seeds that survive a given winter , exceeds one $\alpha\,\delta\,\gamma > 1$.

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