Deanship of Graduate Studies

Al-Quds University

Hypergeometric Function Representation of Transport Coefficients for Drifting Maxwellian and Drifting bi-Maxwellian Plasmas

Walaa' Mohammad Najeeb Jubeh

M. Sc. Thesis

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This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science from the Department of Physics, Faculty of Science and Technology, Al-Quds University.

1439-2017

Deanship of Graduate Studies, Al-Quds University

Thesis Approval

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Master thesis submitted and accepted, **Date: 19 December 2017**
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3-External Examiner: Dr. Mohammed S. Abu-Jafar Signature Mohd S.

Jerusalem – Palestine
1439 AH/2017 AD

Dedication

I dedicate this thesis to my mother, husband, child, and family for their encouragement and trust

Declaration

I certify that this thesis submitted for the degree of Master, is the result of my own research, expect where otherwise acknowledged, and that this study (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed Walaa' S

Walaa' Mohammad Najeeb Jubeh

Date: 19/12/2017

Acknowledgments

First and foremost, I would like to express my sincere gratitude to Allah whom without his support this study wouldn't have come to life. I am also indebted to my mother for teaching me that patience is the key to success. I am grateful for my instructors in the Departments of Physics at Al-Quds University for their continuous guidance and support. Many thanks also go to Prof. Imad A. Barghouthi whose feedback and recommendations took my thesis to a whole new level. Also, I would also like to thank the examining committee members, Dr. Husain R. Alsmamra as internal examiner from Al-Quds University and Dr. Mohammed S. Abu Jafar as external examiner from An-Najah National University.

Abstract

Collisional momentum and energy transport in drifting Maxwellian and drifting bi-Maxwellian plasmas are calculated by using two approaches, Boltzmann collision integral and the Fokker-Planck approximations with special emphasis to the effect of Coulomb collision. The transport coefficients obtained from both approaches are identical to the leading order (proportional to the Coulomb logarithm) and are presented here in a closed form involving generalized hypergeometric functions.

In the derivation, we write the drift velocity **u** of the bi-Maxwellian plasma in terms of parallel and perpendicular components (i.e. $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$) with respect to the ambient magnetic field. The final results are presented in the form of triple hypergeometric function, and they are valid for arbitrary temperature anisotropies, arbitrary temperature differences between interacting gases, and arbitrary relative drift velocities both parallel and perpendicular to the magnetic field. Also, we calculate the transport coefficients for two special cases, firstly, when the drift velocity is parallel to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\parallel}$, and $\mathbf{u}_{\perp}=0$), and secondly, when the drift velocity is perpendicular to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\perp}$, $\mathbf{u}_{\parallel}=0$). For the first case, the transport coefficients are derived and presented in the form of double hypergeometric functions, these results are consistent with the findings of Hellinger and Trávníček (2009). For the second case, the transport coefficients are obtained and found to be in the form of double hypergeometric functions.

Similarly, we derive the collisional transport coefficients for Maxwellian plasmas with a general drift velocity with respect to the ambient magnetic field, these coefficients are presented in a closed form in terms of hypergeometric functions. We also extended the work of Schunk (1977) and calculated the transport coefficients by using Boltzmann collision integral for two special cases where the relative drift is either parallel or perpendicular to the magnetic field, which are the two most common cases in astronomy and space physics.

It is worthy to mention that, up to our knowledge, none of the derived transport coefficients for the above mentioned case are presented in closed form and in terms of hypergeometric function.

Contents

Chapter One

Introduction

Before discussing the transport Coefficients (momentum and energy) for drifting Maxwellian and drifting bi-Maxwellian plasmas, it is necessary to start with a description of the plasma as shown in section I. In section II, we present Boltzmann's equation and the basis of the standard form of the collision terms are presented in section III. This is followed by a derivation of the transport coefficients with special emphasis on the Coulomb collision in section IV. These transport coefficients are in the general formula and it is necessary to adopt an expression for species velocity distribution functions in order to evaluate these coefficients which are discussed in the next two chapters.

I. Definition of Plasma

Plasma is the fourth state of matter and it exists in many forms in nature. It has often been said that more than 99% of the visible matter in the universe is in the plasma state. However, plasma is a quasi-neutral gas consisting of positively and negatively charged particles (usually ions and electrons) with approximately equal charge densities, and its properties are dominated by electric, magnetic and other forces, and which exhibit collective behavior (Chen, 1984; Schwartz, Owen and Burgess, 2004).

Ions and electrons may interact via short range atomic forces (during collisions) and via long range electro-magnetic forces due to currents and charges in the plasma (Gravitational forces may also be important in some applications). The long range nature of the electromagnetic forces means that plasma can show collective behavior, for example, oscillations, instabilities, etc.

Plasmas can also contain some neutral particles (which interact with charged particles via collisions or ionization). Examples include the Earth's ionosphere, magnetosphere, upper atmosphere, interstellar medium, molecular clouds, etc. The simplest plasma is formed by ionization of atomic hydrogen, forming plasma of equal numbers of electrons and protons (Schwartz, Owen and Burgess, 2004; Fitzpatrick, 2014).

II. Boltzmann's Equation

In dealing with plasma or gas mixtures it is convenient to describe each species in the mixture by a separate velocity distribution function, f_s ($\mathbf{v_s}$, $\mathbf{r_s}$,t) . The velocity distribution function is defined such that f_s ($\mathbf{v_s}$, $\mathbf{r_s}$,t) d$\mathbf{v_s}$d$\mathbf{r_s}$ represents the number of particles of species s which at time t have velocities between $\mathbf{v_s}$ and $\mathbf{v_s}$+ d$\mathbf{v_s}$, and positions between $\mathbf{r_s}$ and $\mathbf{r_s}$+d$\mathbf{r_s}$. Alternatively, f_s can be viewed as a probability density in the r, $\mathbf{v_s}$, phase space (Barakat and Lemaire, 1990). The evolution in time of the species distribution function is determined by the net effect of collisions and the flow in phase space of particles under the influence of external forces. The mathematical description of this evolution is given by the well-known Boltzmann equation (Demars and Schunk, 1979):

$$\frac{\partial f_s}{\partial t} + \mathbf{v_s} \cdot \nabla f_s + \left[\mathbf{G} + \frac{q_s}{m_s} \left(\mathbf{E} + \frac{1}{c} \mathbf{v_s} \times \mathbf{B} \right) \right] \cdot \nabla_{\mathbf{v_s}} f_s = \frac{\delta f_s}{\delta t} \tag{1}$$

where q_s, and m_s, are the charge and mass of species s, **G** is the acceleration due to gravity, **E** is the electric field, **B** is the magnetic field, c is the speed of light, ∂/∂t is the time derivative, $\boldsymbol{\nabla}$ is the coordinate space gradient, and $\boldsymbol{\nabla}_{vs}$ is the velocity space gradient. The quantity δ f_s /δt in Boltzmann's equation represents the rate of change f_s in a given region of phase space as a result of collisions and called the collision term.

III. Collision Term

Collisions play a fundamental role in the dynamics and energetics of plasma. They are responsible for the production of ionization, the diffusion of plasma from high to low density regions, the conduction of heat from hot to cold regions, the exchange of energy between different species, and other processes. The collisional processes can be either elastic or inelastic. In an elastic collision, the mass, momentum and kinetic energy of the colliding particles are conserved, while this is not the case in an inelastic collision. The exact nature of the collision process depends both on the relative kinetic energy of the colliding particles and on the type of particles. In general, for low energies, elastic collisions dominate, but as the relative kinetic energy increases, inelastic collisions become progressively more important. The order of importance is from elastic to rotational, vibrational, and electronic excitation, and then to ionization as the relative kinetic energy increases. However, in our study we interested with binary elastic Coulomb collisions (Schunk and Nagy, 2009; Khazanov, 2011).

This section presents a short description of two approaches for collision terms that have been extensively used to describe the relevant Coulomb collision processes (Schunk, 1977).

III.1 Boltzmann Collision Integral

For binary elastic collisions between s and t species, the appropriate collision term $\delta f_s / \delta t$ is the Boltzmann collision integral, which can be presented as

$$\frac{\delta f_s}{\delta t} = \sum_t \int d\mathbf{v_t} d\Omega g_{st} \sigma_{st}(g_{st}, \theta) \left[f_s' f_t' - f_s f_t \right] \tag{2}$$

where $d\mathbf{v_t}$ is the velocity-space volume element of species t, g_{st} is the relative velocity of the colliding particles s and t, dΩ is an element of solid angle in the s particle reference frame, θ is the scattering angle, the primes denote quantities evaluated after a collision, and $\sigma(g_{st}, \theta)$ is the differential scattering cross-section (Goldstein,1980; Schunk and Nagy, 2009):

$$\sigma = \frac{q_s^2 q_t^2}{64\pi^2 \varepsilon_\circ^2 \mu_{st}^2} \frac{1}{g^4 \sin^4 \theta} \tag{3}$$

where q_s and q_t are the charge of species s and t species, respectively, $\mu_{st} = m_s m_t / (m_s + m_t)$ is the reduced mass, m_t is the mass of t particle, and ε_o is the permittivity of free space.

III.2 Fokker Planck Equation

Although the Boltzmann collision integral can be applied to charged particles, the complexity of this expression resulted in a search for simpler collision models. The motivation for simplifying the Boltzmann collision integral in the case of Coulomb collisions is that these are long – range interactions and therefore the change in velocity of species due to collision is small for most collisions (Schunk, 1977). Therefore, the Fokker- Planck equation can be derived directly from the Boltzmann collision integral, which is valid for binary collisions, under the assumption that a series of consecutive weak (small-angle deflection) binary collisions is a valid representation for the multiple Coulomb interaction (Goldston and Rutherford, 1995; Bittencourt,2004). In this case the distribution functions evaluated after the collision can be expressed in terms of those evaluated before the collision by expanding the Boltzmann collision integral and taking first terms in the Taylor series one gets the Fokker-Planck equation, which may be given in the Landau conservative form

$$\frac{\delta f_s}{\delta t} = -\sum_t \nabla_v \cdot \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 m_s} \int \frac{\mathbf{1}g^2 - \mathbf{gg}}{g^3} \cdot \left(\frac{f_s}{m_t} \frac{\partial f_t}{\partial \mathbf{v_t}} - \frac{f_t}{m_s} \frac{\partial f_s}{\partial \mathbf{v_s}} \right) d\mathbf{v_t} \tag{4}$$

where **1** is the unity tensor, and $\ln\Lambda$ is the Coulomb logarithm, which is typically between 10 to 25 for space plasmas.

The form of Fokker Plank equation derived by landau (Landau, 1936; Hochstim, 1969) and Rosenbluth (Rosenbluth et al., 1957). However, the Fokker-Planck approximation fails far from the thermal equilibrium and the Boltzmann integral has to be used (Hellinger and Trávníček, 2009).

IV. Transport Coefficients

Transport Coefficients represent the rate of change in a transport property such as the mass, momentum, energy, etc, as a result of collisions, Coulomb collisions in our case.

In the ideal situation one would like to solve the Boltzmann equation for each of the species in the gas mixture and thereby obtain the individual velocity distribution functions, but this can only be done for relatively simple situations. As a consequence, one is generally restricted to obtaining information on a limited number of low-order velocity moments of the species distribution function (Barakat and Schunk, 1981).

The procedure of multiplying the species distribution function by powers of velocity and then integrating over all velocities is called taking velocity moments. However, the definition of all higher-order velocity moments is not unique. For example, the temperature is a measure of the spread about some average velocity, and this average velocity must be selected before the temperature can be defined.

As an alternative to defining the transport properties with respect to the average gas velocity, Grad (1949, 1958) proposed that the transport properties of a given species be defined with respect to the average drift velocity of that species. In terms of the species average drift velocity $\mathbf{u_s}$, the random or thermal velocity is defined as

$$\mathbf{c_s} = \mathbf{v}_s - \mathbf{u_s} \tag{5}$$

and the physically significant moments of the species distribution function are given by Species drift velocity

$$\mathbf{u_s} = \langle \mathbf{v}_s \rangle \tag{6}$$

Absolute temperature

$$T_{\mathrm{s}} = \frac{3m_{\mathrm{s}}}{k} \langle c_s^2 \rangle \tag{7}$$

Parallel temperature

$$T_{\mathrm{s}\parallel} = \frac{m_{\mathrm{s}}}{k} \langle c_s^2 \rangle \tag{8}$$

Perpendicular temperature

$$T_{\mathrm{s}\perp} = \frac{m_{\mathrm{s}}}{2k} \langle c_s^2 \rangle \tag{9}$$

and the angle brackets denote the average

$$\langle A \rangle = \frac{1}{n_s} \int d\mathbf{v_s} \frac{\delta f_s}{\delta t} A \tag{10}$$

The symbols ∥ and ⊥ are used to identify quantities that are parallel and perpendicular to the magnetic field, respectively, as well as to identify quantities related to parallel and perpendicular thermal energy.

The starting point for the derivation of transport coefficients for gas mixtures is Boltzmann's equation Eq.(1). The transport coefficients are obtained by multiplying collision term in the right hand side of Boltzmann's equation with an appropriate function of velocity and then integrating over velocity space. The resulting transport coefficients describe the spatial and temporal behavior of the physically significant moments of the distribution function such as species concentration, drift velocity, temperature, stress tensor, and parallel and perpendicular heat flow.

If we multiply Eq.(2) or Eq.(4) by m_s, $m_s\mathbf{c_s}$, $m_s c^2/2$, $m_s c_{s\|}^2$ and $m_s c_{s\perp}^2/2$, and integrate over velocity space, we obtain rate of change of the mass, momentum, energy, parallel energy and perpendicular energy, respectively, for species s. are symbolically written as, $\delta n_s/\delta t$, $\delta M_s/\delta t$, $\delta E_s/\delta t$ $\delta E_{s\|}/\delta t$, and $\delta E_{s\perp}/\delta t$ respectively, for species s.

The transport coefficients by using the Boltzmann collision integral:

Density

$$\frac{\partial n_s}{\partial t} = 0 \tag{11}$$

Momentum

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \ln\Lambda \iint d\mathbf{c_s} d\mathbf{c_t}\, g_{st} \mathbf{g_{st}} f_s f_t \tag{12}$$

Energy

$$\frac{\delta E_s}{\delta t} = \sum_t -\mu_{st} \iint d\mathbf{c_s} d\mathbf{c_t}\, g_{st} f_s f_t (\mathbf{V_c} \cdot \mathbf{g_{st}}) Q_{st}^{(1)} \tag{13}$$

Parallel Energy

$$\frac{\delta E_{s\parallel}}{\delta t}=\sum_t 8\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2\left[\iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\mathbf{g_{st\parallel}}.(\mathbf{V_c}-\mathbf{u_s})+\iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\left(g_{st}^2-3g_{st\parallel}^2\right)f_s f_t\right] \quad (14)$$

Perpendicular Energy

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2\left[\iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\mathbf{g_{st\perp}}.(\mathbf{V_c}-\mathbf{u_s})_\perp f_s f_t+\frac{\mu_{st}}{2m_s}\iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\left(g_{st}^2-3g_{st\perp}^2\right)f_s f_t\right] \quad (15)$$

And the transport coefficients by using the Fokker Planck equation

Density

$$\frac{\partial n_s}{\partial t}=0 \quad (16)$$

Momentum

$$\frac{\delta \mathbf{M_s}}{\delta t}=-\sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \ln\Lambda\iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\mathbf{g_{st}} f_s f_t \quad (17)$$

Energy

$$\frac{\delta E_s}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s}\int \mathbf{c_s}\,\frac{1g^2-\mathbf{gg}}{g^3}\boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t} \quad (18)$$

Parallel Energy

$$\frac{\delta E_{s\parallel}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s}\int \mathbf{c_{s\parallel}}\,\frac{1g^2-\mathbf{gg}}{g^3}\boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t} \quad (19)$$

Perpendicular Energy

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s}\int \mathbf{c_{s\perp}}\,\frac{1g^2-\mathbf{gg}}{g^3}\boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)d\mathbf{c_s}d\mathbf{c_t} \quad (20)$$

In the next two chapters we will assume a specific form of the distribution function f_s and f_t and insert these functions in the above equations to obtain closed expressions for collisional transport coefficients.

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Chapter Two (Paper 1)

Hypergeometric function representation of transport coefficients for drifting bi-Maxwellian plasmas

Abstract

We derive the momentum, parallel energy, and perpendicular energy collisional transport coefficients for drifting bi–Maxwellian plasmas by using Boltzmann collision integral approach, and present them in the form of triple hypergeometric functions. In the derivation, we write the drift velocity $\mathbf{u}$ of the bi-Maxwellian plasma in terms of Parallel and perpendicular components (i.e. $\mathbf{u}= \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$), parallel and perpendicular with respect to the ambient magnetic field, and we consider the Coulomb collision interactions. We consider two special cases, firstly, when the drift velocity is parallel to the ambient magnetic field (i.e. $\mathbf{u}= \mathbf{u}_{\parallel}$), and secondly, when the drift velocity is perpendicular to the ambient magnetic field (i.e. $\mathbf{u}= \mathbf{u}_{\perp}$). For the first case, the transport equations and consequently, the transport coefficients are derived and presented in the form of double hypergeometric functions, these results are consistent with the findings of Hellinger and Trávníček (2009). For the second case, the transport coefficients are obtained and found to be in the form of double hypergeometric functions. When we combine these two special cases, i.e. for general $\mathbf{u}$, the transport coefficients are shown to be in the form of triple hypergeometric function. Also, we investigate the above problem by using another approach, i.e. Fokker Planck approximation. We obtain similar results for both approaches.

I. Introduction

Transport equations based on a bi-Maxwellian distribution function were first derived by Chew et al (1965) for collisionless anisotropic plasma, their study was extended by several authors (Kennal and Green,1966; Macmahon,1965; Frieman et al., 1966; Bowers and Haines, 1968; Oraevskii et al., 1968; Espedal, 1969) who derived the transport equations including transport phenomena such as collisionless plasma, viscosity, and heat flow.

All of these studies were dealing with collisionless anisotropic plasmas. Chodura and Pohl (1971) derived transport equation for an arbitrary anisotropic plasma taking care of collisionless as well as Coulomb collision effect. Since, then, Demars and Schunk (1979) have extended the work of Chodura and Pohl (1971) by deriving transport equations based on a bi-Maxwellian species distribution function for arbitrary anisotropic plasma (i.e. arbitrary temperature differences between the interacting gases and arbitrary temperature anisotropy). The relevant collision term have been calculated for resonant charge exchange interaction between an ion and it's neutral parent, inverse-power interaction potential that include non-resonant ion-neutral (Maxwell molecule) and Coulomb collision, and constant cross-section (hard sphere) interaction.

The last two studies valid just for small relative drift between the interacting gases. However, Barakat and Schunk (1981) removed this restriction and derived collision terms based on drift bi-Maxwellian gases that are valid for arbitrary drift velocities differences and for an arbitrary temperature differences between the interacting gases as well as arbitrary temperature anisotropy.

These transport equations were all derived based on velocity moments of Boltzmann's equation and the collision terms were all derived based on velocity moments of the Boltzmann collision integral.

Mitchener and Kruger (1973) and Hinton (1983) approximated Boltzmann collision integral by the Fokker Planck equation under the assumption that small angle deflections dominate. Hellinger and Trávníček (2009) calculated collision terms for bi-Maxwellian distribution function with drift along an ambient magnetic field by using Fokker Planck equation and obtained similar results by using Boltzmann collision integral method.

It is the purpose of this paper to extend the work of Hellinger and Trávníček (2009) by deriving transport coefficients based on drift (in general, i.e. $u= u_{\parallel} + u\perp$) bi-Maxwellian distribution function, and taking into consideration the Coulomb interactions.

We are interested in the derivation of collisional transport coefficients for drifting bi-Maxwellian velocity distribution function with respect to the background magnetic field because many applications in plasma physics are in special need to these coefficients. Usually, the differential velocity between different species is aligned with the ambient magnetic field. However, the drift velocity perpendicular to the ambient field is typically connected with non-gyrotropic velocity distribution function and could be also related to plasma inhomogeneity. For example, different studies investigated the behavior of O+ ions in the ionpsphere under the effect of ExB drift, ion-ion Coulomb collision and ion-neutral collisions (Barghouthi et al., 1994, 2003; Barghouthi, 2005), also many studies investigated the ion outflow along "open geomagnetic" field lines (Ganguli, 1996; Barghouthi, 2008; Nilsson et al., 2013). In order to go forward in above and similar studies we need well established formulas for these collisional coefficients.

This paper is organized as follows: Theoretical formulation (Boltzmann equation, Boltzmann collision integral, Fokker Planck equation, and transport coefficients) are presented in section II. In section III we presented transport coefficients for drifting bi-Maxwellian velocity distribution function. Special cases (drift velocities perpendicular and parallel to the ambient magnetic field) are presented in section IV. Our results and discussion are summarized in section V.

II. Theoretical Formulation

In dealing with plasma or gas mixture it is convenient to investigate the distribution of these particles or species, each species in the plasma is described by a separate velocity distribution function $f_s(\mathbf{r}, \mathbf{v}_s, t)$ which define such that $f_s(\mathbf{r},\mathbf{v}_s,t)d\mathbf{r}d\mathbf{v}_s$ represents the number of particles of species s which at time t have positions between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ and velocities between $\mathbf{v}_s$ and $\mathbf{v}_s + d\,\mathbf{v}_s$. The species distribution function changed with respect to time as a result of collisions and particle motions under the influence of external forces, this velocity distribution function is obtained by solving the following Boltzmann's equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v_s} \cdot \nabla f_s + \left[\mathbf{G} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v_s} \times \mathbf{B} \right) \right] \cdot \nabla_{\mathbf{v_s}} f_s = \frac{\delta f_s}{\delta t} \tag{1}$$

where q_s, and m_s, are the charge and mass of species s, **G** is the acceleration due to gravity, **E** is the electric field, **B** is the magnetic field, c is the speed of light, $\partial/\partial t$ is the time derivative, $\boldsymbol{\nabla}$ is the coordinate space gradient, $\boldsymbol{\nabla}_{\mathbf{vs}}$ is the velocity space gradient, and the operator $\delta f_s/\delta t$ represents the rate of change of f_s due to the collisions, this term is given in two forms: Boltzmann collision integral and Fokker Planck approximation.

II. 1 Boltzmann Collision Integral

For Coulomb collision between s and t particles, the appropriate collision operator in the right hand side of Boltzmann's equation is the Boltzmann collision integral, which can be presented as

$$\frac{\delta f_s}{\delta t} = \sum_t \int d\mathbf{v_t} d\Omega g_{st} \sigma_{st}(g_{st}, \theta)\left[f_s' f_t' - f_s f_t \right] \tag{2}$$

where $d\mathbf{v_t}$ is the velocity-space volume element of species t, g_{st} is the relative velocity of the colliding particles s and t, $d\Omega$ is an element of solid angle in the s particle reference frame, θ is the scattering angle, the primes denote quantities evaluated after a collision, and $\sigma(g_{st}, \theta)$ is the differential scattering cross-section (Goldston and Rutherford, 1995; Schunk and Nagy, 2009):

$$\sigma = \frac{q_s^2 q_t^2}{64\pi^2 \varepsilon_\circ^2 \mu_{st}^2} \frac{1}{g^4 \sin^4 \theta}$$

where q_s and q_t are the charge of species and t species, respectively, $\mu_{st} = m_s m_t / (m_s + m_t)$ is the reduced mass, m_t is the mass of t particle, and ε_o is the permittivity of free space.

II. 2 Fokker- Planck Equation

The collision operator can be represented by another equation that called Fokker-Planck equation, this equation can be derived directly from the Boltzmann collision integral (i.e. Eq. (2)) by taking the first order of Taylor expansion of it, this expansion is valid for binary collisions, under the assumption that a series of consecutive weak (small-angle deflection) binary collisions is a valid representation for the Coulomb interactions.

$$\frac{\delta f_s}{\delta t} = -\sum_t \nabla_v \cdot \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 m_s}\int \frac{\mathbf{1}g^2 - \mathbf{gg}}{g^3} \cdot \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{v_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{v_s}}\right) d\mathbf{v_t} \tag{3}$$

where 1 is the unity tensor, and $\ln\Lambda$ is the Coulomb logarithm, which is typically between 10 to 25 for space plasmas.

There are two approximations were employed in the transformation of the Boltzmann collision integral, i.e. Eq. (2), to a Fokker–Planck equation. The first is to remove the scattering angle singularity by evaluating the total momentum transfer cross-section such that scattering angles not smaller than θmin are included. The angle θmin is defined in terms of the ratio of the Debye length, $\lambda_D = \sqrt{kT_b / 4\pi e^2 N}$ to a temperature-averaged impact parameter, $b_o = q_s q_t / 3kT_b$, that is, $\sin^2(\theta_{\min}/\ 2) = [1+\Lambda]^{-1}$, where $\Lambda = \lambda_D / b_o$. The impact parameter, bo, is recognized as the impact parameter that corresponds to a deflection of θ=π/2 (Shizgal, 2004; Rosenbluth et al., 1957).

The momentum transfer cross-section is obtained from the integration over the scattering solid angle in the Boltzmann collision integral, Eq. (2). The momentum transfer cross section, which occurs in the calculation of collisional energy transfer is given by (Schunk and Nagy, 2009):

$$Q^{(1)} = 2\pi \int_{\theta_{\min}}^{2\pi} \sigma_{st}(g_{st}, \theta)(1-\cos\theta)\sin\theta d\theta \tag{4}$$

$$= 4\pi \left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g^2}\right)^2 \ln\Lambda$$

The second approximation is to assume that collisions with large impact parameters that are small scattering angles dominate and the Boltzmann collision integral can be replaced with the differential Fokker–Planck equation. The details of these calculations are provided elsewhere and are important for the interpretation of the results of this paper (Mitchener et al., 1973).

II. 3 Transport Coefficients

Transport Coefficients represent the change in a transport property (momentum, energy, etc.) as a result of collisions, Coulomb collisions in our case.

Although it would be nice to know the individual velocity distribution functions of the different species, the mathematical difficulties associated with obtaining closed-form solutions to Boltzmann's equation preclude this approach for most flow situations. As a consequence, one is generally restricted to obtaining information on a limited number of low-order velocity moments of the species distribution function.

Burgers (1969) and Grad (1949, 1958) proposed that the transport properties of a given species defined with respect to the average drift velocity of that species, $\mathbf{u}_s$, alternative to defining them with respect to the average as velocity, $\mathbf{v}_s$. This definition is more appropriate for large relative drifts between interacting species can occur. In terms of the species average drift velocity, the random or thermal velocity is defined as

$$\mathbf{c}_s = \mathbf{v}_s - \mathbf{u}_s$$

For most applications, the physically significant moments of the species distribution function are given by

Species drift velocity $\quad \mathbf{u}_s = \langle \mathbf{v}_s \rangle = (1/n_s)\int d\mathbf{v}_s\, f_s \mathbf{v}_s$

Parallel temperature $\quad T_{s\parallel} = m_s \langle c_{s\parallel}^2 \rangle / k = (1/n_s)\int d\mathbf{c}_s\, f_s m_s c_{s\parallel}^2 / k$

Perpendicular temperature $\quad T_{s\perp} = m_s \langle c_{s\perp}^2 \rangle / 2k = (1/n_s)\int d\mathbf{c}_s\, f_s m_s c_{s\perp}^2 / 2k$

where n_s is the number density of species s, k is Boltzmann's constant and the symbols ∥ and ⊥ are used to identify quantities that are parallel and perpendicular to the magnetic field, respectively.

The starting point for the derivation of transport coefficients is the collision term in the right hand side of Boltzmann equation. Moments of Boltzmann collision integral are obtained by multiplying the right hand side of Boltzmann equation with an appropriate function of velocity $Q_s=Q_s(\mathbf{c_s})$ and integrating over all velocity space. The corresponding moment of the Boltzmann collision Integral

$$\frac{\partial Q_s}{\partial t}=\int d^3c_s Q_s(c_s)\frac{\delta f_s}{\delta t}=\iiint d\mathbf{c_s}d\mathbf{c_t}d\Omega g_{st}\sigma_{st}(g_{st},\theta)\left[f_s' f_t' - f_s f_t\right]Q_s(c_s) \tag{5}$$

For $Q_s(c_s) = m_s\mathbf{c_s}, m_s c_{s\|}^2$ and $m_s c_{s\perp}^2 \backslash 2$, the obtained moments of the Boltzmann collision integral are momentum, parallel energy and perpendicular energy, are symbolically written as, δMs/δt, and $\delta E_{s\|} \big/\delta t$, $\delta E_{s\perp}/\delta t$ respectively, for species s.

Due to the reversibility of elastic collisions, we can interchange primed and unprimed quantities in the expression on the right side of Eq. (5) without changing the result

$$\frac{\partial Q_s}{\partial t}=\sum_t \iint d^3c_s d^3c_t g_{st} f_s f_t \int d\Omega\sigma_{st}(g_{st},\theta)\left[Q_s{}' - Q_s\right] \tag{6}$$

Where $Q_s{}'$ is the moment evaluated with the velocity found after the Coulomb collision. Integrals in Eq. (6) are called transfer integrals because of transfer of momentum and kinetic energy from one particle to the other particle due to the change in Q_s in a collision. Eq.(6) is easier than that Eq.(5) because they do not require the distribution functions after the collision.

The evaluation of the integral over dΩ in Eq. (6) has to be done using two steps. First, express $\left(Q_s{}' - Q_s\right)$ in terms of the center-of-mass velocity, $\mathbf{V_c}$, and the relative velocity,

$\mathbf{g_{st}} = \mathbf{v_s}-\mathbf{v_t}$, while the second step in evaluating the collision integral is to integrate over solid angle $d\Omega=\sin\theta d\theta d\varphi$ by using the spherical coordinates system in the center of mass reference frame with relative velocity before the collision (Barakat and Schunk, 1981; Schunk and Nagy, 2009; Burgers, 1969; Chapman and Cowling, 1970.The resulting system of transport coefficients is given by:

Momentum

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \ln\Lambda \iint d\mathbf{c_s} d\mathbf{c_t} g_{st} \mathbf{g_{st}} f_s f_t \tag{7}$$

Parallel Energy

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t 8\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \left[\iint d\mathbf{c_s} d\mathbf{c_t} g_{st} \mathbf{g_{st\parallel}} .(\mathbf{V_c}-\mathbf{u_s})_\parallel + \iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\left(g_{st}^2 - 3g_{st\parallel}^2\right) f_s f_t\right] \tag{8}$$

Perpendicular Energy

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \left[\iint d\mathbf{c_s} d\mathbf{c_t} g_{st} \mathbf{g_{st\perp}} .(\mathbf{V_c}-\mathbf{u_s})_\perp f_s f_t + \frac{\mu_{st}}{2m_s}\iint d\mathbf{c_s} d\mathbf{c_t} g_{st}\left(g_{st}^2 - 3g_{st\perp}^2\right) f_s f_t\right] \tag{9}$$

where $\mathbf{V_c} = \dfrac{m_s \mathbf{v_s} + m_t \mathbf{v_t}}{m_s + m_t}$ is the center of velocity.

Also these moments can be obtained by using the other form of collision term which is the Fokker Planck approximation by multiplying also it with an appropriate function of velocity $Q_s=Q_s(\mathbf{c_s})$ and integrating over all velocity space as follow

$$\frac{\delta Q_s}{\delta t} = -\sum_t \nabla_v . \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 m_s}\int \frac{1g^2 - \mathbf{gg}}{g^3} \boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right) Q(c_s) d\mathbf{c_s} d\mathbf{c_t} \tag{10}$$

After integration by parts, the corresponding transport coefficients can be expressed as

Momentum

$$\frac{\delta \mu_s}{\delta t} = \sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 n_s}\int \frac{1g^2 - \mathbf{gg}}{g^3} \boldsymbol{.}\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right) d\mathbf{c_s} d\mathbf{c_t} \tag{11}$$

Parallel Energy

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s} \int \mathbf{c}_{\mathbf{s}\parallel} \ \frac{1g^2 - \mathbf{gg}}{g^3} . \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}} \right) d\mathbf{c_s} d\mathbf{c_t} \tag{12}$$

Perpendicular Energy

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s} \int \mathbf{c}_{\mathbf{s}\perp} \frac{1g^2 - \mathbf{gg}}{g^3} . \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}} \right) d\mathbf{c_s} d\mathbf{c_t} \tag{13}$$

In this study we assume the distribution function to be drifting bi-Maxwellian function. This assumption will be used to evaluate the integrals in the equations (7, 8, 9, 11, 12, 13).

III. Transport coefficients for drifting bi-Maxwellian velocity distribution function

We assume that all considered species in the plasma have bi-Maxwellian velocity distribution functions with drift velocity Parallel and perpendicular components with respect to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_\parallel + \mathbf{u}_\perp$).

$$f_s = \frac{n_s}{\pi^{3/2} a_{s\parallel}\, a_{s\perp}^2} e^{-\frac{c_{s\parallel}^2}{a_{s\parallel}^2} - \frac{c_{s\perp}^2}{a_{s\perp}^2}} \tag{14}$$

$$f_t = \frac{n_t}{\pi^{3/2} a_{t\parallel}\, a_{t\perp}^2} e^{-\frac{c_{t\parallel}^2}{a_{t\parallel}^2} - \frac{c_{t\perp}^2}{a_{t\perp}^2}} \tag{15}$$

where $a_\parallel$ and $a_\perp$ are the average parallel and perpendicular thermal speeds of species s, that are equal to $\left(2kT_\parallel / m_s\right)^{1/2}$ and $\left(2kT_\perp / m_s\right)^{1/2}$, respectively.

In this section we will derived the transport coefficients by using the two approaches: Boltzmann collision integral and Fokker Planck equation, and verify that they are equivalent.

III. 1 Boltzmann collision integral

The first step in calculating the momentum coefficient by using Boltzmann collision integral is the multiply f_s by f_t ($f_s f_t$) and write it in the form

$$f_s f_t = \frac{n_s n_t}{\pi^3 a_{s\|} a_{s\perp}^2 a_{t\|} a_{t\perp}^2} \exp\left(-\frac{c_{s\|}^2}{a_{s\|}^2} - \frac{c_{s\perp}^2}{a_{s\perp}^2} - \frac{c_{t\|}^2}{a_{t\|}^2} - \frac{c_{t\perp}^2}{a_{t\perp}^2}\right) \tag{16}$$

The momentum coefficient according to Eq. (6)becomes

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t 4\pi\mu_{st}\left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g_{st}^2}\right)^2 \ln\Lambda \frac{n_s n_t}{\pi^3 a_{s\|} a_{s\perp}^2 a_{t\|} a_{t\perp}^2} \iint d\mathbf{c_s} d\mathbf{c_t} g_{st} \mathbf{g_{st}} exp\left(-\frac{c_{s\|}^2}{a_{s\|}^2} - \frac{c_{s\perp}^2}{a_{s\perp}^2} - \frac{c_{t\|}^2}{a_{t\|}^2} - \frac{c_{t\perp}^2}{a_{t\perp}^2}\right) \tag{17}$$

The integrations over d$\mathbf{c_s}$ and d$\mathbf{c_t}$ can be performed by changing the variables of $(\mathbf{c_s}, \mathbf{c_t})$ integration from to ($\mathbf{h}$, $\mathbf{l}$) by using variables defined as follows

$$\mathbf{c_{s\|}} = \mathbf{h}_\| + \frac{a_{s\|}^2}{a_{s\|}^2 + a_{t\|}^2}\mathbf{l} \tag{18}$$

$$\mathbf{c_{t\|}} = \mathbf{h}_\| - \frac{a_{t\|}^2}{a_{s\|}^2 + a_{t\|}^2}\mathbf{l} \tag{19}$$

$$\mathbf{c_{s\perp}} = \mathbf{h}_\perp + \frac{a_{s\perp}^2}{a_{s\perp}^2 + a_{t\perp}^2}\mathbf{l} \tag{20}$$

$$\mathbf{c_{t\perp}} = \mathbf{h}_\perp - \frac{a_{t\perp}^2}{a_{s\perp}^2 + a_{t\perp}^2}\mathbf{l} \tag{21}$$

Substituting Equations (18) to (21) into the equation (17) and by using Jacobian transformation $d\mathbf{c_s} d\mathbf{c}_t = d\mathbf{h} d\mathbf{l}$ the expression for momentum transport coefficients therefore becomes

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{\mu_{st} n_t}{\pi^3 a_{s\|} a_{s\perp}^2 a_{t\|} a_{t\perp}^2} \int \exp\left(-\frac{a_{s\|}^2 h_\|^2}{a_{s\|}^2 a_{t\|}^2} - \frac{a_{t\perp}^2 h_\perp^2}{a_{s\perp}^2 a_{t\perp}^2}\right) d\mathbf{h} \int g\mathbf{g} Q^{(1)} \exp\left(-l^2\left(\frac{1}{a_\|^2} + \frac{1}{a_\perp^2}\right)\right) a_\| \, a_\perp^2 d\mathbf{l} \tag{22}$$

Because the first integral depends only on the variable **z**, it can be evaluated immediately by using a Gaussian integral technique, so Eq.(22) reduce to

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{\mu_{st} n_t}{\pi^{3/2} a_\parallel\, a_\perp^2} \int g\mathbf{g} Q^{(1)} \exp\left(-y^2 \left(\frac{1}{a_\parallel^2} + \frac{1}{a_\perp^2} \right) \right) d\mathbf{y} \tag{23}$$

The calculation is further simplified by changing of variables and integrate over d**x** instead of d**l** where old and new variables are related by the following equations:

$$(x-\varepsilon)^2 = l^2 \left(\frac{1}{a_\parallel^2} + \frac{1}{a_\perp^2} \right) \tag{24}$$

$$a_\parallel = \sqrt{a_{s\parallel}^2 + a_{t\parallel}^2} \tag{25}$$

$$a_\perp = \sqrt{a_{s\perp}^2 + a_{t\perp}^2} \tag{26}$$

$$\mathbf{x} = \frac{\mathbf{g}_\parallel}{a_\parallel} + \frac{\mathbf{g}_\perp}{a_\perp} \tag{27}$$

$$\boldsymbol{\varepsilon} = \frac{\Delta \mathbf{u}_\parallel}{a_\parallel} + \frac{\Delta \mathbf{u}_\perp}{a_\perp} \tag{28}$$

With these changes, the integral in Eq. (23) become

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 n_t}{4\pi^{3/2} \varepsilon_\circ^2 \mu_{st}} \ln\Lambda \int e^{-(x-\varepsilon)^2} \frac{\mathbf{g}}{g^3} dx \tag{30}$$

and the exponential in Eq. (30) can be simplified as follows:

$$(x-\varepsilon)^2 = \left(\frac{\mathbf{g}_\parallel}{a_\parallel} + \frac{\mathbf{g}_\perp}{a_\perp} + \frac{(\mathbf{u_t} - \mathbf{u_s})_\parallel}{a_\parallel} + \frac{(\mathbf{u_t} - \mathbf{u_s})_\perp}{a_\perp} \right)^2 \tag{31}$$

$$(x-\varepsilon)^2 = \left(\frac{\mathbf{g}_\parallel^2}{a_\parallel^2} + \frac{\mathbf{g}_\perp^2}{a_\perp^2}\right) + \left(\frac{(\mathbf{u_t} - \mathbf{u_s})_\parallel^2}{a_\parallel^2} + \frac{(\mathbf{u_t} - \mathbf{u_s})_\perp^2}{a_\perp^2}\right) + \left(\frac{2\mathbf{g}(\mathbf{u_t} - \mathbf{u_s})_\parallel \cos\theta}{a_\parallel^2} + \frac{2\mathbf{g}(\mathbf{u_t} - \mathbf{u_s})_\perp \cos\theta}{a_\perp^2}\right) \quad (32)$$

Also we introduce

$$a_\parallel = \sqrt{2}v_{st\parallel}\text{ , and } a_\perp = \sqrt{2}v_{st\perp} \quad (33)$$

where

$$v_{st\parallel} = \sqrt{\frac{v_{s\parallel}^2 + v_{t\parallel}^2}{2}}\text{ , and } v_{st\perp} = \sqrt{\frac{v_{s\perp}^2 + v_{t\perp}^2}{2}} \quad (34)$$

are combined effective parallel and perpendicular velocities, respectively,

$$A_{st} = \frac{v_{st\perp}^2}{v_{st\parallel}^2} = \frac{m_t T_{s\perp} + m_s T_{t\perp}}{m_t T_{s\parallel} + m_s T_{t\parallel}} \quad (35)$$

is an effective temperature anistropy

And

$$\mathbf{g}_\parallel^2 = g^2 \cos^2\theta \quad (36)$$

$$\mathbf{g}_\perp^2 = g^2 - g^2 \cos^2\theta \quad (37)$$

then

$$(x-\varepsilon)^2 = \left(\frac{g^2 \cos^2\theta}{4v_{st\perp}^2}(A_{st} - 1) + \frac{g^2}{4v_{st\perp}^2}\right) + \left(\frac{(\mathbf{u_t} - \mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t} - \mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right) + \left(\frac{\sqrt{A_{st}}\, g(\mathbf{u_t} - \mathbf{u_s})_\parallel \cos\theta}{v_{st\parallel}\, v_{st\perp}} + \frac{g(\mathbf{u_t} - \mathbf{u_s})_\perp \cos\theta}{2\sqrt{A_{st}}\, v_{st\parallel}\, v_{st\perp}}\right) \quad (38)$$

we need also the substitution

$$v = \frac{g}{2v_{st\perp}} \quad (39)$$

$$V=\frac{\sqrt{A_{st}}\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\|}}{v_{st\|}} \tag{40}$$

$$w=\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\perp}}{\sqrt{A_{st}}\,v_{st\perp}} \tag{41}$$

$$A=\left(A_{st}-1\right) \tag{42}$$

$$\frac{\mathbf{g}}{g^3}d\mathbf{x}=\frac{1}{2v_{st\|}}\frac{\mathbf{v}}{v^3}d\mathbf{v} \tag{43}$$

$$\frac{g_{\|}^2}{g^3}d\mathbf{x}=\sqrt{A_{st}}\cos^2\theta\frac{d\mathbf{v}}{v} \tag{44}$$

$$\frac{g_{\perp}^2}{g^3}d\mathbf{x}=\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v} \tag{45}$$

So Eq. (29) can be expressed as

$$\frac{\delta\mathbf{M_s}}{\delta t}=-\sum_t\frac{q_s^2q_t^2n_t}{4\pi^{3/2}\varepsilon_\circ^2\mu_{st}}\ln\Lambda\int\exp\left[\left(\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\|}^2}{4v_{st\|}^2}+\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\perp}^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\frac{1}{2v_{st\|}}\frac{\mathbf{v}}{v^3}d\mathbf{v} \tag{46}$$

Because of the term $e^{-\left(\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\|}^2}{4v_{st\|}^2}+\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\perp}^2}{4v_{st\perp}^2}\right)}$ is constant with respect to variable x , we can get it out of the integration. The integration over all variables v using a spherical coordinate system in velocity space, then becomes

$$\frac{\delta\mathbf{M_s}}{\delta t}=-\sum_t\frac{q_s^2q_t^2n_t}{4\pi^{3/2}\varepsilon_\circ^2\mu_{st}}\ln\Lambda e^{-\left(\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\|}^2}{4v_{st\|}^2}+\frac{\left(\mathbf{u_t}-\mathbf{u_s}\right)_{\perp}^2}{4v_{st\perp}^2}\right)}\frac{2\pi}{2v_{st\|}}\int_0^{\pi}\int_0^{\infty}e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}\cos\theta\sin\theta dv d\theta \tag{47}$$

In order to solve this integral, we used the technique Maclaurin series expansion for the exponential terms with cosθ, and finally, write it in the triple hypergeometric function (Hellinger and Trávníček, 2009; Lebedev , 1965; Koepf, 2014).

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{(\mathbf{u_t} - \mathbf{u_s})_\parallel}{2} \frac{(\mathbf{u_t} - \mathbf{u_s})_\perp}{2} e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} + \frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)} F^3 \left(\begin{matrix} 1,2,\frac{3}{2} \\ 3,\frac{3}{2},\frac{3}{2} \end{matrix}, (1-A_{st}), \frac{(\mathbf{u_t} - \mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2} A_{st}, \frac{(\mathbf{u_t} - \mathbf{u_s})_\perp^2}{4A_{st} v_{st\perp}^2} \right) \tag{48}$$

where

$$\upsilon_{st} = \frac{q_s^2 q_t^2 n_t}{32\pi\varepsilon_\circ^2 \mu_{st} v_{st\parallel}^2 \; v_{st\perp}} \ln\Lambda \tag{49}$$

is a collision frequency of species s on species t.

Also, an equations for the energy transport coefficients (7) and (8) ,$\delta E_s/\delta t$, can be derived in a manner similar to that described above for $\delta\mu_s /\delta t$. The first steps in evaluating these integrals are expressing them in term of relative velocity, and the variable (x),

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st}} \left[\frac{\mu_{st}}{m_t} 4k_B \left(\frac{T_{t\parallel} - T_{s\parallel}}{2v_{s\parallel}^2} \right) \int e^{-(x-\varepsilon)^2} \frac{\mathbf{g}}{g^3} d\mathbf{x} - 2\mu_{st} \int e^{-(x-\varepsilon)^2} \frac{g_\parallel^2}{g^3} d\mathbf{x} + \mu_{st} \int e^{-(x-\varepsilon)^2} \frac{g_\perp^2}{g^3} d\mathbf{x} \right] \tag{50}$$

$$\begin{aligned} \frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}} \Bigg[& \frac{\mu_{st}}{m_t} 2k_B \left(\frac{T_{t\perp} - T_{s\perp}}{2v_{s\perp}^2} \right) \int e^{-(x-\varepsilon)^2} \frac{g_\perp^2}{g^3} d\mathbf{x} \\ & + \frac{2k_B T_{s\perp} (\mathbf{u_t} - \mathbf{u_s})}{\sqrt{2} v_{s\perp}} \int e^{-(x-\varepsilon)^2} \frac{\mathbf{g}}{g^3} dx + 2\mu_{st} \int e^{-(x-\varepsilon)^2} \frac{g_\parallel^2}{g^3} d\mathbf{x} \\ & - \frac{\mu_{st}}{2} \int e^{-(x-\varepsilon)^2} \frac{g_\perp^2}{g^3} d\mathbf{x} \Bigg] \end{aligned} \tag{51}$$

The next step in evaluating the energy collision integrals is the substitution of Eq.(38) in equations (50) and (51),

$$\frac{\delta E_{s\parallel}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st}}\left[\frac{\mu_{st}}{m_t}4k_B\left(\frac{T_{t\parallel}-T_{s\parallel}}{2v_{s\parallel}^2}\right)\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\right.$$

$$\frac{1}{2v_{st\parallel}}\frac{\mathbf{v}}{v^3}d\mathbf{v}-\frac{2k_B T_{s\parallel}\,(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2v_{s\perp}^2}\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\frac{1}{2v_{st\parallel}}\frac{\mathbf{v}}{v^3}d\mathbf{v}$$

$$-2\mu_{st}\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\sqrt{A_{st}}\cos^2\theta\frac{d\mathbf{v}}{v}+$$

$$\left.+\mu_{st}\int\ \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v}\right] \quad (52)$$

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}}\left[\frac{\mu_{st}}{m_t}2k_B\left(\frac{T_{t\perp}-T_{s\perp}}{2v_{s\perp}^2}\right)\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\right.$$

$$\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v}+\frac{2k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})}{\sqrt{2}v_{s\perp}}\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\frac{1}{2v_{st\parallel}}\frac{\mathbf{v}}{v^3}d\mathbf{v}$$

$$+2\mu_{st}\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\sqrt{A_{st}}\cos^2\theta\frac{d\mathbf{v}}{v}$$

$$\left.-\frac{\mu_{st}}{2}\int \exp\left[\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)+\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)\right]\sqrt{A_{st}}\sin^2\theta\frac{d\mathbf{v}}{v}\right] \quad (53)$$

By taking the integration over φ, the last two equations become

$$\frac{\delta E_{s\parallel}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st}}\left[\frac{\mu_{st}}{m_t}4k_B\left(\frac{T_{t\parallel}-T_{s\parallel}}{2v_{s\parallel}^2}\right)e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}\cos\theta\sin\theta dv d\theta\right.$$

$$-\frac{2k_B T_{s\parallel}\,(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2v_{s\perp}^2}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}\cos\theta\sin\theta dv d\theta$$

$$-2\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}v\cos^2\theta\sin\theta dv d\theta$$

$$\left.+2\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}v\sin^3\theta dv d\theta\right] \quad (54)$$

$$
\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}}\left[\frac{\mu_{st}}{m_t}2k_B\left(\frac{T_{t\perp}-T_{s\perp}}{2v_{s\perp}^2}\right)e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}v\sin^3\theta dv d\theta\right.
$$

$$
+\frac{2k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})_\perp}{\sqrt{2}v_{s\perp}}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}\cos\theta\sin\theta dv d\theta
$$

$$
+2\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}v\cos^2\theta\sin\theta dv d\theta
$$

$$
\left.-\frac{\mu_{st}}{2}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vV\cos\theta+vw\cos\theta\right)}v\sin^3\theta dv d\theta\right] \quad (55)
$$

And finally, write them in the triple hypergeometric function.

$$
\frac{\delta \mathbf{M_s}}{\delta t}=\sum_t \upsilon_{st}\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2}\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2}F^{(st)}_{12\frac{3}{2}\frac{3}{2}\frac{3}{2}} \quad (56)
$$

$$
\frac{\delta E_{s\parallel}}{\delta t}=\sum_t \upsilon_{st}k_B T_{s\parallel}\left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\parallel}}{T_{s\parallel}}-1\right)F^{(st)}_{1\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}-\frac{3\sqrt{\pi}}{4}\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp}{2v_{st\perp}}F^{(st)}_{123\frac{3}{2}\frac{3}{2}\frac{5}{2}}-2\left(F^{(st)}_{1\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{3}{2}}-F^{(st)}_{1\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}\right)\right] \quad (57)
$$

$$
\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{\upsilon_{st}}{A_{st}}k_B T_{s\perp}\left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\perp}}{T_{s\perp}}-1\right)F^{(st)}_{1\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}-\frac{3\sqrt{\pi}}{4}\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2v_{st\parallel}}\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}F^{(st)}_{123\frac{3}{2}\frac{3}{2}\frac{3}{2}}+\left(F^{(st)}_{1\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}-F^{(st)}_{1\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}\right)\right] \quad (58)
$$

Here $F^{(st)}_{abcd}$ are defined through triple hypergeometric functions

$$
F^{(st)}_{abcd}=e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}+\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}F^3\left(\begin{matrix}a,b\\c,c,d\end{matrix},(1-A_{st}),\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel^2}{4v_{st\parallel}^2}A_{st},\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4A_{st}v_{st\perp}^2}\right) \quad (59)
$$

III. 2 Fokker Planck Equation

The first step in evaluating the transport coefficients by using the Fokker Planck equation is the derivation of f_s and f_t with respect to $\mathbf{c_s}$, $\mathbf{c_t}$ respectively as follows

$$\frac{\partial f_s}{\partial \mathbf{c_s}} = -2 f_s \left(\frac{\mathbf{c}_{\mathbf{s}\parallel}}{a_{s\parallel}^2} + \frac{\mathbf{c}_{\mathbf{s}\perp}}{a_{s\perp}^2} \right) \tag{60}$$

$$\frac{\partial f_t}{\partial \mathbf{c_t}} = -2 f_t \left(\frac{\mathbf{c}_{t\parallel}}{a_{t\parallel}^2} + \frac{\mathbf{c}_{t\perp}}{a_{t\perp}^2} \right) \tag{61}$$

Using equations (60) and (61), the integration in equations (9,10,11) may be simplified by using matrix technique as follow

$$\frac{1g^2 - \mathbf{gg}}{g^3} \cdot \left(\frac{f_s}{m_t} \frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s} \frac{\partial f_s}{\partial \mathbf{c_s}} \right) = \frac{1}{g^3} \begin{bmatrix} \mathbf{g}_\perp^2 & -\mathbf{g}_\parallel \mathbf{g}_\perp \\ -\mathbf{g}_\perp \mathbf{g}_\parallel & \mathbf{g}_\parallel^2 \end{bmatrix} \cdot \frac{-2 f_s f_t}{m_s m_t} \begin{bmatrix} \frac{m_s \mathbf{c}_{t\parallel}}{a_{t\parallel}^2} - \frac{m_t \mathbf{c}_{\mathbf{s}\parallel}}{a_{s\parallel}^2} \\ \frac{m_s \mathbf{c}_{t\perp}}{a_{t\perp}^2} - \frac{m_t \mathbf{c}_{\mathbf{s}\perp}}{a_{s\perp}^2} \end{bmatrix}$$

$$= \frac{-2 f_s f_t}{m_s m_t g^3} \begin{bmatrix} (m_s + m_t)\mathbf{g}_\parallel \\ \frac{(m_s + m_t)\mathbf{g}_\perp}{2} \end{bmatrix} = \frac{-2 f_s f_t}{g^3} \frac{(m_s + m_t)}{m_s m_t} \begin{bmatrix} \mathbf{g}_\parallel \\ \frac{\mathbf{g}_\perp}{2} \end{bmatrix} = \frac{-2 f_s f_t}{\mu_{st} g^3} \left(\mathbf{g} - \frac{\mathbf{g}_\perp}{2} \right) \tag{62}$$

$$\mathbf{c}_{\mathbf{s}\parallel}\;\frac{1g^2-\mathbf{gg}}{g^3}.\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)=\frac{\mathbf{c}_{\mathbf{s}\parallel}}{g^3}\begin{bmatrix}\mathbf{g}_\perp^2 & -\mathbf{g}_\parallel\,\mathbf{g}_\perp\\ -\mathbf{g}_\perp\mathbf{g}_\parallel & \mathbf{g}_\parallel^2\end{bmatrix}.\frac{-2f_sf_t}{m_sm_t}\begin{bmatrix}\frac{m_s\mathbf{c}_{t\parallel}}{a_{t\parallel}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\parallel}}{a_{s\parallel}^2}\\ \frac{m_s\mathbf{c}_{t\perp}}{a_{t\perp}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\perp}}{a_{s\perp}^2}\end{bmatrix}$$

$$=\frac{-2f_sf_t}{g^3}\begin{bmatrix}\mathbf{g}.\left(\frac{\mu_{st}}{m_s}\frac{\mathbf{g}_\perp}{2}\right) & \frac{\mathbf{g}}{m_s+m_t}.\left(m_s\mathbf{c}_{\mathbf{s}\parallel}+m_t\mathbf{c}_{\mathbf{t}\parallel}\right)\\ \frac{\mathbf{g}}{m_s+m_t}.\left(-\frac{m_t\mathbf{g}_\parallel}{2}+\frac{(m_s+m_t)\mathbf{c}_{\mathbf{s}\parallel}}{2}-\frac{m_t\mathbf{c}_{\mathbf{t}\parallel}}{2}-\frac{m_sa_{s\parallel}^2\,\mathbf{c}_{t\parallel}}{2a_{t\parallel}^2}+\frac{m_t\left(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel}\right)}{2}\right) & 0\end{bmatrix}$$

$$=\frac{-2f_sf_t}{g^3}\mathbf{g}.\left(\frac{2k\left(T_{\mathbf{s}\parallel}-T_{\mathbf{t}\parallel}\right)}{(m_s+m_t)a_\parallel^2}\left(\mathbf{c}_{\mathbf{s}\parallel}-\mathbf{c}_{\mathbf{t}\parallel}\right)+\frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel-\frac{\mathbf{g}_\perp}{2}\right)-\frac{\mu_{st}\left(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel}\right)}{m_s}\frac{m_s\mathbf{c}_{s\parallel}}{2(m_s+m_t)}\right.$$

$$\left.-\frac{m_sa_{s\parallel}^2\,\mathbf{c}_{t\parallel}}{(m_s+m_t)a_{t\parallel}^2}+\frac{m_t\mathbf{c}_{s\parallel}}{(m_s+m_t)}+\frac{m_s\mathbf{c}_{\mathbf{t}\parallel}}{(m_s+m_t)}-\frac{\mu_{st}\left(m_sa_{t\parallel}^2-m_ta_{s\parallel}^2\right)}{(m_s+m_t)a_\parallel^2}\left(\mathbf{c}_{\mathbf{s}\parallel}-\mathbf{c}_{\mathbf{t}\parallel}\right)\right) \qquad (63)$$

$$\mathbf{c}_{\mathbf{s}\perp}\;\frac{1g^2-\mathbf{gg}}{g^3}.\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}}-\frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right)=\frac{\mathbf{c}_{\mathbf{s}\perp}}{g^3}\begin{bmatrix}\mathbf{g}_\perp^2 & -\mathbf{g}_\parallel\,\mathbf{g}_\perp\\ -\mathbf{g}_\perp\mathbf{g}_\parallel & \mathbf{g}_\parallel^2\end{bmatrix}.\frac{-2f_sf_t}{m_sm_t}\begin{bmatrix}\frac{m_s\mathbf{c}_{t\parallel}}{a_{t\parallel}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\parallel}}{a_{s\parallel}^2}\\ \frac{m_s\mathbf{c}_{t\perp}}{a_{t\perp}^2}-\frac{m_t\mathbf{c}_{\mathbf{s}\perp}}{a_{s\perp}^2}\end{bmatrix}$$

$$=\frac{-2f_sf_t}{g^3}\begin{bmatrix}\mathbf{g}.\left(\frac{\mu_{st}}{m_s}\frac{\mathbf{g}_\perp}{2}\right) & \frac{\mathbf{g}}{m_s+m_t}.\left(m_s\mathbf{c}_{\mathbf{s}\parallel}+m_t\mathbf{c}_{\mathbf{t}\parallel}\right)\\ \frac{\mathbf{g}}{m_s+m_t}.\left(-\frac{m_t\mathbf{g}_\parallel}{2}+\frac{(m_s+m_t)\mathbf{c}_{\mathbf{s}\parallel}}{2}-\frac{m_t\mathbf{c}_{\mathbf{t}\parallel}}{2}-\frac{m_sa_{s\parallel}^2\,\mathbf{c}_{t\parallel}}{2a_{t\parallel}^2}+\frac{m_t\left(\mathbf{u}_{\mathbf{s}\parallel}-\mathbf{u}_{\mathbf{t}\parallel}\right)}{2}\right) & 0\end{bmatrix}$$

$$=\frac{-f_sf_t\mathbf{g}}{g^3}.\left(\frac{2k\left(T_{\mathbf{s}\perp}-T_{\mathbf{t}\perp}\right)}{(m_s+m_t)a_\perp^2}\left(\mathbf{c}_{\mathbf{s}\perp}-\mathbf{c}_{\mathbf{t}\perp}\right)-\frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel-\frac{\mathbf{g}_\perp}{2}\right)-\frac{\mu_{st}\left(\mathbf{u}_{\mathbf{s}\perp}-\mathbf{u}_{\mathbf{t}\perp}\right)}{m_s}\frac{m_s\mathbf{c}_{s\perp}}{2}+m_t\mathbf{c}_{s\perp}+m_s\mathbf{c}_\perp\right.$$

$$\left.-\frac{\mu_{st}\left(m_sa_{t\perp}^2-m_ta_{s\perp}^2\right)}{a_\parallel^2}\left(\mathbf{c}_{\mathbf{s}\perp}-\mathbf{c}_{\mathbf{t}\perp}\right)\right) \qquad (64)$$

The transport Coefficients reduced to

$$\frac{\delta\mu_s}{\delta t}=-\sum_t\frac{q_s^2q_t^2\ln\Lambda}{4\pi\varepsilon_\circ^2\mu_{st}n_s}\int\frac{\mathbf{g}}{g^3}\,f_sf_td\mathbf{c_s}d\mathbf{c_t}+\sum_t\frac{q_s^2q_t^2\ln\Lambda}{8\pi\varepsilon_\circ^2\mu_{st}n_s}\int\frac{\mathbf{g}_\perp}{g^3}\,f_sf_td\mathbf{c_s}d\mathbf{c_t} \qquad (65)$$

$$\frac{\delta E_{s\parallel}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{2\pi\varepsilon_\circ^2 \mu_{st} n_s}\int f_s f_t \frac{\mathbf{g}}{g^3} d\mathbf{c}_\mathbf{s} d\mathbf{c}_\mathbf{t} \cdot \left(\frac{2k\left(T_{\mathbf{s}\parallel} - T_{\mathbf{t}\parallel}\right)}{(m_s + m_t)a_\parallel^2}\left(\mathbf{c}_{\mathbf{s}\parallel} - \mathbf{c}_{\mathbf{t}\parallel}\right) + \frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel - \frac{\mathbf{g}_\perp}{2}\right) - \frac{\mu_{st}\left(\mathbf{u}_{\mathbf{s}\parallel} - \mathbf{u}_{\mathbf{t}\parallel}\right)}{m_s} \right)$$
$$-\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{2\pi\varepsilon_\circ^2 \mu_{st} n_s}\int \frac{f_s f_t}{(m_s + m_t)}\frac{\mathbf{g}}{g^3} d\mathbf{c}_\mathbf{s} d\mathbf{c}_\mathbf{t} \cdot \left(\frac{m_s \mathbf{c}_{s\parallel}}{2} - \frac{m_s a_{s\parallel}^2\, \mathbf{c}_{\mathbf{t}\parallel}}{a_{t\parallel}^2} + m_t \mathbf{c}_{s\parallel} + m_s \mathbf{c}_{\mathbf{t}\parallel}\ \frac{\mu_{st}\left(m_s a_{t\parallel}^2 - m_t a_{s\parallel}^2\right)}{a_\parallel^2}\left(\mathbf{c}_{\mathbf{s}\parallel} - \mathbf{c}_{\mathbf{t}\parallel}\right) \right) \tag{66}$$

$$\frac{\delta E_{s\perp}}{\delta t} = -\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st} n_s}\int f_s f_t \frac{\mathbf{g}}{g^3} d\mathbf{c}_\mathbf{s} d\mathbf{c}_\mathbf{t} \cdot \left(\frac{2k\left(T_{\mathbf{s}\perp} - T_{\mathbf{t}\perp}\right)}{(m_s + m_t)a_\perp^2}\left(\mathbf{c}_{\mathbf{s}\perp} - \mathbf{c}_{\mathbf{t}\perp}\right) - \frac{\mu_{st}}{m_s}\left(\mathbf{g}_\parallel - \frac{\mathbf{g}_\perp}{2}\right) - \frac{\mu_{st}\left(\mathbf{u}_{\mathbf{s}\perp} - \mathbf{u}_{\mathbf{t}\perp}\right)}{m_s} \right)$$
$$-\sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st} n_s}\int \frac{f_s f_t}{(m_s + m_t)}\frac{\mathbf{g}}{g^3} d\mathbf{c}_\mathbf{s} d\mathbf{c}_\mathbf{t} \cdot \left(\frac{m_s \mathbf{c}_{s\perp}}{2} + m_t \mathbf{c}_{s\perp} + m_s \mathbf{c}_\perp - \frac{\mu_{st}\left(m_s a_{t\perp}^2 - m_t a_{s\perp}^2\right)}{a_\parallel^2}\left(\mathbf{c}_{\mathbf{s}\perp} - \mathbf{c}_{\mathbf{t}\perp}\right) \right) \tag{67}$$

For momentum, the first term is the same as we get from Boltzmann collision integral, and the second integral vanishes when integrating over the solid angle Ω. For parallel and perpendicular energy, the first integral the same as obtained from Boltzmann collision integral, and the second integral reduce to $\left(n_s n_t / g\right)$ and $\left[\left(\left(m_s + 2m_t\right) n_t a_{s\perp}^2\right) / \left(2g^3 (m_s + m_t)\right)\right.$

$\left. + m_s n_s a_{t\perp}^2 / \left(g^3 (m_s + m_t)\right) - n_s n_t \left(m_s a_{t\perp}^2 - m_t a_{s\perp}^2\right) / \left(a_\parallel^2 (m_s + m_t)\right)\right]$, respectively.

The transport coefficients are summarized as follow

$$\frac{\delta \mathbf{M}_\mathbf{s}}{\delta t} = \sum_t \upsilon_{st} \frac{\left(\mathbf{u}_\mathbf{t} - \mathbf{u}_\mathbf{s}\right)_\parallel}{2} \frac{\left(\mathbf{u}_\mathbf{t} - \mathbf{u}_\mathbf{s}\right)_\perp}{2} F^{st}_{12\frac{3}{2}\frac{3}{2}3} \tag{68}$$

$$\frac{\delta E_{s\parallel}}{\delta t} = \sum_t \upsilon_{st} k_B T_{s\parallel}\left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\parallel}}{T_{s\parallel}} - 1\right) F^{(st)}_{1\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}} - \frac{3\sqrt{\pi}}{4} \frac{\left(\mathbf{u}_\mathbf{t} - \mathbf{u}_\mathbf{s}\right)_\parallel^2}{4 v_{st\parallel}^2} \frac{\left(\mathbf{u}_\mathbf{t} - \mathbf{u}_\mathbf{s}\right)_\perp}{2 v_{st\perp}} F^{(st)}_{123\frac{3}{2}\frac{3}{2}\frac{5}{2}} - 2\left(F^{(st)}_{1\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{3}{2}} - F^{(st)}_{1\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}} \right) \right]$$
$$+ \sum_t \upsilon_{st} k_B T_{s\parallel} \frac{n_s n_t}{g} \tag{69}$$

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{\upsilon_{st}}{A_{st}}k_B T_{s\perp}\left[\frac{\mu_{st}}{m_t}\left(\frac{T_{t\perp}}{T_{s\perp}}-1\right)F^{(st)}_{1\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}-\frac{3\sqrt{\pi}}{4}\frac{(\mathbf{u_t}-\mathbf{u_s})_\parallel}{2v_{st\parallel}}\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}F^{(st)}_{123\frac{3}{2}\frac{3}{2}\frac{3}{2}}+\left(F^{(st)}_{1\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}-F^{(st)}_{1\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{5}{2}}\right)\right]$$

$$+\sum_t \frac{\upsilon_{st}}{A_{st}}k_B T_{s\perp}\left[\frac{m_s+2m_t}{2g^3}n_t a_{s\perp}^2+\frac{m_s n_s a_{t\perp}^2}{g^3}-\frac{n_s n_t\left(m_s a_{t\perp}^2-m_t a_{s\perp}^2\right)}{a_\parallel^2}\right] \tag{70}$$

Because of the Fokker Planck derivation from expanding the Boltzmann collision integral and taking first terms in the Taylor series, the transport coefficients by using the Fokker Planck equation for drifting bi-Maxwellian distribution functions with velocities parallel and perpendicular to the ambient magnetic field give approximately similar results when compared to the result of Boltzmann collision integral.

IV. Special cases

IV. 1 ($\mathbf{u}_\parallel = 0$, i.e. $\mathbf{u} = \mathbf{u}_\perp$), no drift velocity component parallel to the ambient magnetic field, and the drift velocity is perpendicular with respect to the ambient magnetic field, the integrals in equations (47,54, 55) reduce to:

$$\frac{\delta \mathbf{M_s}}{\delta t}=-\sum_t \frac{q_s^2 q_t^2 n_t}{4\pi^{3/2}\varepsilon_\circ^2\mu_{st}}\ln\Lambda e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\frac{2\pi}{2v_{st\parallel}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}\cos\theta\sin\theta dv d\theta \tag{71}$$

$$\frac{\delta E_{s\parallel}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s\mu_{st}v_{st\perp}^2}\left[\frac{\mu_{st}}{m_t}4k_B\left(\frac{T_{t\parallel}-T_{s\parallel}}{2v_{st\parallel}^2}\right)2\pi\sqrt{A_{st}}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}v\cos^2\theta\sin\theta dv d\theta\right.$$

$$-4\pi\sqrt{A_{st}}\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}v\cos^2\theta\sin\theta dv d\theta$$

$$\left.+4\pi\mu_{st}\sqrt{A_{st}}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}\sin^3\theta dv d\theta\right] \tag{72}$$

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \frac{q_s^2 q_t^2 n_t \ln\Lambda}{4\pi^{5/2}\varepsilon_\circ^2 m_s \mu_{st} v_{st\perp}}\left[\frac{\mu_{st}}{m_t}2k_B\left(\frac{T_{t\perp}-T_{s\perp}}{2v_{s\perp}^2}\right)e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\,e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}\sin^3\theta dv d\theta\right.$$

$$+\frac{2k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})_\perp}{\sqrt{2}v_{s\perp}}\frac{2\pi}{2v_{st\|}}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}\cos\theta\sin\theta dv d\theta$$

$$+2\pi\mu_{st}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}v\cos^2\theta\sin\theta dv d\theta$$

$$\left.-\frac{\mu_{st}}{2}e^{-\left(\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}\right)}2\pi\sqrt{A_{st}}\int_0^\pi\int_0^\infty e^{-\left(v^2+Av^2\cos^2\theta+vw\cos\theta\right)}\sin^3\theta dv d\theta\right] \tag{73}$$

The coefficients may be evaluated by expanding exponential terms with cosθ into infinite sums and integrating the resulting terms, and then writes the results in the form of double hypergeometric functions. The transport coefficients are summarized as follows

$$\frac{\delta\mathbf{M_s}}{\delta t}=\sum_t \upsilon_{st}\frac{u_{t\perp}-u_{s\perp}}{2}G^{(st)}_{1\frac{3}{2}\frac{5}{2}} \tag{74}$$

$$\frac{\delta E_{s\|}}{\delta t}=\sum_t \upsilon_{st}A_{st}\left[\frac{\mu_{st}}{m_t}k_B\left(T_{t\|}-T_{s\|}\right)G^{(st)}_{2\frac{1}{2}\frac{5}{2}}+2\mu_{st}v_{st}^2\left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}}-G^{(st)}_{2\frac{1}{2}\frac{5}{2}}\right)\right] \tag{75}$$

$$\frac{\delta E_{s\perp}}{\delta t}=\sum_t \upsilon_{st}\left[\frac{\mu_{st}}{m_t}k_B\left(T_{t\perp}-T_{s\perp}\right)G^{(st)}_{1\frac{1}{2}\frac{5}{2}}+\frac{k_B T_{s\perp}(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{2v_{st\perp}^2}G^{(st)}_{1\frac{3}{2}\frac{5}{2}}-\mu_{st}v_{st\perp}^2\left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}}-G^{(st)}_{2\frac{1}{2}\frac{5}{2}}\right)\right] \tag{76}$$

Where

$$\upsilon_{st}=\frac{q_s^2 q_t^2 n_t \ln\Lambda}{12\pi^{3/2}\varepsilon_\circ^2 \mu_{st} v_{st\|} v_{st\perp}^2} \tag{77}$$

is a collision frequency of species s on species t, and

$$G^{(st)}_{abc}=e^{-\frac{(\mathbf{u_t}-\mathbf{u_s})_\perp^2}{4v_{st\perp}^2}}F^{2..}_{1.1}\left(\begin{matrix}a & b\\ c & b\end{matrix},1-A_{st},\frac{(u_t-u_s)_\perp^2}{4v_{st\perp}^2}\right) \tag{78}$$

is generalized double hypergeometric or Kampe de Fériet functions.

The transport coefficients can be also calculated from the Fokker Planck equation. This calculation also leads to transport coefficients in the form of double hypergeometric function which nearly the same transport coefficients (74–76) as obtained from the Boltzmann collision integral.

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{u_{t\perp} - u_{s\perp}}{2} G^{(st)}_{1\frac{3}{2}\frac{5}{2}} \tag{79}$$

$$\frac{\delta E_{s\|}}{\delta t} = \sum_t \upsilon_{st} A_{st} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\|} - T_{s\|}\right) G^{(st)}_{2\frac{1}{2}\frac{5}{2}} + 2\mu_{st} v_{st}^2 \left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}} - G^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right] + \sum_t \upsilon_{st} k_B T_{s\|} \frac{n_s n_t}{g} \tag{80}$$

$$\begin{aligned} \frac{\delta E_{s\perp}}{\delta t} = \sum_t \upsilon_{st} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\perp} - T_{s\perp}\right) G^{(st)}_{1\frac{1}{2}\frac{5}{2}} + \frac{k_B T_{s\perp} (\mathbf{u_t} - \mathbf{u_s})_\perp^2}{2 v_{st\perp}^2} G^{(st)}_{1\frac{3}{2}\frac{5}{2}} - \mu_{st} v_{st\perp}^2 \left(G^{(st)}_{1\frac{1}{2}\frac{5}{2}} - G^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right] \\ + \sum_t \frac{\upsilon_{st}}{A_{st}} k_B T_{s\perp} \left[\frac{m_s + 2m_t}{2g^3} n_t a_{s\perp}^2 + \frac{m_s n_s a_{t\perp}^2}{g^3} - \frac{n_s n_t \left(m_s a_{t\perp}^2 - m_t a_{s\perp}^2\right)}{a_\|^2} \right] \end{aligned} \tag{81}$$

IV. 2 ($\mathbf{u}_\perp = 0$, i.e. $\mathbf{u} = \mathbf{u}_\|$), no drift velocity component perpendicular to the ambient magnetic field, and the drift velocity is parallel with respect to the ambient magnetic field, the transport coefficients take the form:

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t \upsilon_{st} \frac{u_{t\|} - u_{s\|}}{2} H^{(st)}_{1\frac{3}{2}\frac{5}{2}} \tag{82}$$

$$\frac{\delta E_{s\|}}{\delta t} = \sum_t \upsilon_{st} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\|} - T_{s\|}\right) H^{(st)}_{2\frac{1}{2}\frac{5}{2}} + k_B T_{s\|} \frac{\left(u_t - u_s\right)_\|^2}{2 v_{st\|}^2} H^{(st)}_{2\frac{3}{2}\frac{5}{2}} \right] + \sum_t \mu_{st} \upsilon_{st} v_{st\|}^2 \left(H^{(st)}_{1\frac{1}{2}\frac{5}{2}} - H^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \tag{83}$$

$$\frac{\delta E_{s\perp}}{\delta t} = \sum_t \frac{\upsilon_{st}}{A_{st}} \left[\frac{\mu_{st}}{m_t} k_B \left(T_{t\perp} - T_{s\perp}\right) H^{(st)}_{1\frac{1}{2}\frac{5}{2}} - \mu_{st} v_{st\perp}^2 \left(H^{(st)}_{1\frac{1}{2}\frac{5}{2}} - H^{(st)}_{2\frac{1}{2}\frac{5}{2}} \right) \right] \tag{84}$$

where $\upsilon_{st} = \dfrac{q_s^2 q_t^2 n_t \ln\Lambda}{12\pi^{3/2}\varepsilon_\circ^2 \mu_{st} v_{st\|}^3}$ is a collision frequency of species s on species t, and

$H_{abc}^{(st)} = e^{-\frac{(u_t-u_s)_\|^2}{4v_{st\|}^2}} F_{1.1}^{2..}\left(\begin{matrix} a & b \\ c & b \end{matrix}, 1-A_{st}, A_{st}\dfrac{(u_t-u_s)_\|^2}{4v_{st\|}^2}\right)$ is generalized double hypergeometric or Kamṕe de Fériet functions.

These results agree with the results of Hellinger and Trávníček (2009).

V. Results and Discussions

Coulomb collisions play a very important role in the kinetics of the inner magnetosphere, plasmasphere, ionosphere coupling processes. They are responsible for the plasma production in these regions as well as for the energy and momentum transfer between the different plasma species as a result of collisions. The mathematical description of the change in a transport property (momentum, energy, etc.) as a result of collisions called the transport coefficients which depend on the form of velocity distribution function of colliding species.

For temperature anisotropic plasmas (i.e., unequal species temperatures parallel and perpendicular to the ambient magnetic field, with the degree of the anisotropy given by the parallel to perpendicular temperature ratio)we obtained the transport coefficients (momentum, parallel energy, and perpendicular energy) based on a bi-Maxwellian velocity distribution functions with drift velocity **u**(parallel and perpendicular) with respect to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_\| + \mathbf{u}_\perp$) by using Boltzmann collision integral, and Fokker Planck approximation. The final results are presented in the form of triple hypergeometric function. The two approaches give nearly the same results, and valid for arbitrary temperature anisotropies, arbitrary temperature differences between interacting gases, and arbitrary relative drift velocities both parallel and perpendicular to the magnetic field.

We also calculated the transport coefficients by using Boltzmann collision integral for two special cases where the relative drift is either parallel or perpendicular to the magnetic field, which are the two most common cases in astronomy and space physics. Then we investigated the previously problem by using another approach, Fokker Planck approximation, we obtained nearly similar results. The transport coefficients are in the form of double hypergeometric functions. These results can be further generalized to an inverse power force interaction.

It should be noted that significant temperature anisotropies occur in plasma at all levels of ionization. The temperature anisotropy in the solar wind measured typically varies between a factor of 2 to 4 at the orbit of the Earth (cf. Brandt ,1970; Hundhausen ,1972) and developed in a region of the flow where only Coulomb collisions are important (i.e., the flow is effectively fully ionized), while in the terrestrial polar wind proton initial theoretical calculations indicate that the temperature anisotropy is about a factor of 20 at a distance of eight Earth radii (Holzer et al., 1971) and developed in a region of flow where Coulomb collisions and non-resonant ion- neutral interaction occur (i.e., the flow is partially ionized).

To sum up, we extended the work of Hellinger and Trávníček (2009) and calculated the transport coefficients for drifting bi-Maxwellian plasmas. Hellinger and Trávníček (2009) consider the plasma drift is along the ambient magnetic field, but in our study we have consider general drift ($\mathbf{u}=\mathbf{u}_\parallel+\mathbf{u}_\perp$) and investigated two special cases ($\mathbf{u}=\mathbf{u}_\parallel, \mathbf{u}_\perp=0$, and $\mathbf{u}=\mathbf{u}_\perp, \mathbf{u}_\parallel=0$).We have reproduced the results of Hellinger and Trávníček (2009) for case($\mathbf{u}=\mathbf{u}_\parallel, \mathbf{u}_\perp=0$). In our study we showed in detailed derivation for transport coefficients by using two approaches Boltzmann collision integral and Fokker Planck equation.

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Chapter Three (Paper 2)

Transport coefficients for drifting Maxwellian plasmas: The effect of Coulomb collisions

Abstract

We derive the collisional momentum and energy transport coefficients in Maxwellian plasmas with a general drift velocity with respect to the ambient magnetic field by using two approaches, the Fokker-Planck approximation and Boltzmann collision integral. We find the transport coefficients obtained from Fokker-Planck representation are similar to those obtained by using Boltzmann collision integral approach, and both results are presented in a closed form in terms of hypergeometric functions. This has been done for drifting Maxwellian plasmas with special emphasis on Coulomb collision, i.e. inverse-square force.

Also, we calculate the transport coefficients for two special cases, firstly, when the drift velocity is parallel to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\|}$, and zero perpendicular drift velocity), and secondly, when the drift velocity is perpendicular to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\perp}$, and zero parallel drift velocity). It is worthy to mention that, up to our knowledge, none of the derived transport coefficients for the above mentioned case are presented in closed form and in terms of hypergeometric function.

I. Introduction

Transport equations based on an isotropic Maxwellian distribution function were first derived by Tanenbaum (1967), Burgers (1969), and reviewed by Schunk (1977). They obtained these transport equations by using Boltzmann collision integral approach and presented them in terms of the Chapman–Cowling collision integrals (Chapman and

Cowling, 1970). These coefficients are valid for arbitrary temperature differences between the interacting gases, and are restricted to small relative drift velocity between the interacting gases. In this study, we removed the latter restriction and calculated transport coefficients for general drifting Maxwellian plasmas that are valid for arbitrary drift velocity differences as well as for temperature differences between the interacting plasma species. We also derived these transport coefficients for two special cases, the first one, when the drift velocity is parallel to the ambient magnetic field and the second one when the drift velocity is perpendicular to the ambient magnetic field. These coefficients are obtained by using two different approaches; Fokker-Planck approximation and Boltzmann collision integral.

This paper starts with a discussion of the theoretical formulation of Boltzmann's equation and the relevant collision terms i.e. Boltzmann collision integral and Fokker-Planck approximation. This is followed by showing the general forms of Boltzmann collision integral and Fokker-Planck approximation. Then, we derived the closed set of transport coefficients for drifting Maxwellian distribution function with emphasis on the effect of Coulomb collisions, and finally we investigated two special cases (i.e. drift velocities perpendicular and parallel to the ambient magnetic field) by using two forms of the collision terms. The last section discusses our results and future studies.

I.1 Theoretical Formulation

In dealing with plasma it is convenient to investigate the distribution function of these species, in general each species in the plasma is described by a separate velocity distribution function $f_s(\mathbf{r}, \boldsymbol{v}_s, t)$ which defined such that $f_s(\mathbf{r}, \mathbf{v}_s, t)\, d\boldsymbol{r} d\boldsymbol{v}_s$ represents the number density of particles of species s which at time t have positions between $\mathbf{r}$ and $\mathbf{r} + d\,\mathbf{r}$ and velocities between $\boldsymbol{v}_s$ and $\boldsymbol{v}_s + d\,\boldsymbol{v}_s$. The species distribution function changed with respect to time as a result of collisions and particle motions under the influence of external forces, the mathematical description of this effect is giving by Boltzmann's equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v_s} \cdot \nabla f_s + \left[\mathbf{G} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{1}{c}\mathbf{v_s} \times \mathbf{B}\right)\right] \cdot \nabla_{\mathbf{v_s}} f_s = \frac{\delta f_s}{\delta t} \tag{1}$$

where q_s, and m_s, are the charge and mass of species s, **G** is the acceleration due to gravity,**E** is the electric field, **B** is the magnetic field, c is the speed of light, $\partial/\partial t$ is the time derivative,$\boldsymbol{\nabla}$is the coordinate space gradient, $\boldsymbol{\nabla}_{vs}$is the velocity space gradient, and the quantity$\delta f_s/\delta t$ represents the rate of change of f_s due to the collisions, this term is given in different forms, in this study we are interested in Boltzmann collision integral and Fokker-Planck approximation forms.

I. 2 Boltzmann Collision Integral

For binary elastic Coulomb collision between s and t charged particles, the appropriate collision term is the Boltzmann collision integral, which can be presented as

$$\frac{\delta f_s}{\delta t} = \sum_t \int d\mathbf{v}_t d\Omega g_{st} \sigma_{st}(g_{st}, \theta)\left[f_s' f_t' - f_s f_t\right] \tag{2}$$

where $d\mathbf{v}_t$ is the velocity-space volume element of species t, g_{st} is the relative velocity of the colliding particles s and t, $d\Omega$ is an element of solid angle in the s particle reference frame, θ is the scattering angle, the primes denote quantities evaluated after a collision, and $\sigma(g_{st}, \theta)$ is the differential scattering cross-section (Goldstein, 1980; Schunk and Nagy, 2009):

$$\sigma = \frac{q_s^2 q_t^2}{64\pi^2 \varepsilon_\circ^2 \mu_{st}^2} \frac{1}{g^4 \sin^4\theta} \tag{3}$$

where q_s and q_t are the charges of species s and t species, respectively, $\mu_{st} = m_s m_t/(m_s + m_t)$ is the reduced mass, m_t and m_s are the masses of particles t and s, and ε_o is the permittivity of free space.

I. 3 Fokker- Planck Approximation

Sometimes Boltzmann collision integral appears to be difficult to evaluate, so that the Boltzmann collision integral (i.e. Eq. (2)) reduces to another simpler form by taking the first order of Taylor expansion of it under the assumption that a series of consecutive weak (small-angle deflection) binary collisions is a valid representation for the Coulomb interactions, the result is the Fokker-Planck approximation.

$$\frac{\delta f_s}{\delta t} = -\sum_t \nabla_v \cdot \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 m_s}\int \frac{1g^2 - \mathbf{gg}}{g^3} \cdot \left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{v_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{v_s}}\right) d\mathbf{v_t} \tag{4}$$

where 1 is the unity tensor, and $\ln\Lambda$ is the Coulomb logarithm, which is typically between 10 to 25 for space plasmas.

The moments of f_s are most conveniently defined in terms of the random or thermal velocity of the species s, $\mathbf{c_s}$, with respect to their own mean flow velocity, $\mathbf{u_s}$, as follow

$$\mathbf{c_c} = \mathbf{v_s} - \mathbf{u_s} \tag{5}$$

so that the integration over the velocity space $\mathbf{dc}_s = \mathbf{dv_s}$ and the only difference being a displacement of the origin of the velocity space (Grad ,1949,1958; Burgers, 1969). The advantage of it that if there are large drift velocity difference or temperature difference between interacting species, the velocity distribution function of a given species more likely to be Maxwellian about its own drift velocity than to be Maxwellian about the average velocity. Consequently, a series expansion of the species distribution function about Maxwellian will converge more rapidly if the species average drift velocity is used to define the transport properties (Schunk, 1977).

II. Transport Coefficients

The starting point for the derivation of transport coefficients for gas mixtures is Boltzmann's equation i.e. Eq. (1). The transport equations are obtained by multiplying the right hand side of the Boltzmann's equation by an appropriate function of velocity $Q_s = Q_s(\mathbf{c_s})$ and then integrating over all velocity space. The resulting transport coefficients describe the effect of collisions between different species.

If we multiply the right hand side of Eq. (1) by $Q_s = 1$, $m_s\mathbf{c_s}$, and $m_s c_s^2/2$ and integrate over velocity space, we obtain the rate of change of density, momentum and energy, and are symbolically written as $\delta n_s/\delta t$, $\delta M_s/\delta t$, and $\delta E_s/\delta t$, respectively, for species s.

For Boltzmann collision integral, the corresponding transport coefficients are given as

$$\frac{\delta Q_s}{\delta t} = \iiint d\mathbf{c_s} d\mathbf{c_t} d\Omega g_{st} \sigma_{st}(g_{st}, \theta)\left[f_s' f_t' - f_s f_t\right] Q_s \tag{6}$$

Due to reversibility of elastic collision, we can interchange primed and unprimed quantities in the Eq. (6) without changing the result (Schunk and Nagy, 2009).

$$\frac{\delta Q_s}{\delta t} = \iiint d\mathbf{c_s} d\mathbf{c_t} d\Omega g_{st} \sigma_{st}(g_{st}, \theta) f_s f_t \left[Q_s' - Q_s)\right] \tag{7}$$

where Q´ is a function of velocity after the collision. The evaluation of Eq. (7) is easier than that of Eq.(6), as it does not require the calculation of $f'_s f'_t$.

However, this integral can be evaluated by transform it from ($\mathbf{c_s}$,$\mathbf{c_t}$) to ($\mathbf{V_c}$, $\mathbf{g_{st}}$), where $\mathbf{V_c}$ is the center-of-mass velocity, and $\mathbf{g_{st}}$ is the relative velocity, which are

$$\mathbf{g_{st}} = \mathbf{v_s} - \mathbf{v_t} \tag{8}$$

$$\begin{aligned}\mathbf{V_c} &= \frac{m_s \mathbf{v_s} + m_t \mathbf{v_t}}{m_s + m_t} \\ &= \frac{\left(m_s \mathbf{c_s} + m_t \mathbf{c_t} + m_s \mathbf{u_s} + m_t \mathbf{u_t}\right)}{\left(m_s + m_t\right)}\end{aligned} \tag{9}$$

And the next step in evaluating the collision integral is to integrate over the solid angle dΩ = sinθ dθ dφ by using spherical polar coordinate system in the center of mass reference frame, so we obtain

Density

$$\frac{\partial n_s}{\partial t} = 0 \tag{10}$$

Momentum

$$\frac{\delta \mathbf{M_s}}{\delta t} = \sum_t -\mu_{st} \iint d\mathbf{c_s} d\mathbf{c_t} g_{st} \mathbf{g_{st}} f_s f_t Q_{st}^{(1)} \tag{11}$$

Energy

$$\frac{\delta E_s}{\delta t} = \sum_t -\mu_{st} \iint d\mathbf{c_s} d\mathbf{c_t} g_{st} f_s f_t (\mathbf{V_c}.\mathbf{g_{st}}) Q_{st}^{(1)} \tag{12}$$

where

$$\begin{aligned} Q^{(1)} &= 2\pi \int_{\theta_{\min}}^{2\pi} \sigma_{st}(g_{st}, \theta)(1-\cos\theta)\sin\theta d\theta \\ &= 4\pi \left(\frac{q_s q_t}{4\pi\varepsilon_\circ \mu_{st} g^2} \right)^2 \ln\Lambda \end{aligned} \tag{13}$$

Also, these moments can be obtained by using the Fokker-Planck approximation by multiplying it with an appropriate function of velocity $Q_s=Q_s(\mathbf{c_s})$ and integrating over all velocity space as follows:

$$\frac{\delta Q_s}{\delta t} = -\sum_t \nabla_v . \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 m_s} \int \frac{1g^2 - \mathbf{gg}}{g^3} . \left(\frac{f_s}{m_t} \frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s} \frac{\partial f_s}{\partial \mathbf{c_s}} \right) Q(c_s) d\mathbf{c_s} d\mathbf{c_t} \tag{14}$$

After integration by parts, the corresponding transport coefficients can be expressed as

Density

$$\frac{\partial n_s}{\partial t} = 0 \tag{15}$$

Momentum

$$\frac{\delta\mu_s}{\delta t} = \sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{8\pi\varepsilon_\circ^2 n_s} \int \frac{1g^2 - \mathbf{gg}}{g^3} . \left(\frac{f_s}{m_t} \frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s} \frac{\partial f_s}{\partial \mathbf{c_s}} \right) d\mathbf{c_s} d\mathbf{c_t} \tag{16}$$

Energy

$$\frac{\delta E_s}{\delta t} = \sum_t \frac{q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 n_s} \int \mathbf{c_s} \frac{1g^2 - \mathbf{gg}}{g^3} . \left(\frac{f_s}{m_t} \frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s} \frac{\partial f_s}{\partial \mathbf{c_s}} \right) d\mathbf{c_s} d\mathbf{c_t} \tag{17}$$

Note that for all elastic collisions the rate of change of density is zero because the particle's mass does not change.

The remaining integrals in equations (11, 12, 16, 17) can be evaluated after adopting approximate expressions for f_s and f_t. However, in this study we assume the distribution function to be drifting Maxwellian function. This assumption will be used to evaluate these integrals.

III. Transport Coefficients for Drifting Maxwellian Velocity Distribution Function

As noted in the last section, it is necessary to adopt approximate expression for the species velocity distribution functions, in order to evaluate the transport coefficients as presented in equations (11,12,16,17). So we assume all colliding species in the gas have drifting Maxwellian velocity distributions function. This case is known as the 5-moment approximation because each species in the gas mixture is characterized by five parameters (i.e. species density, three components of drift velocity, and temperature).

$$f_s = \left(\frac{m_s}{2\pi k T_s} \right)^{3/2} e^{-\frac{m_s c_s^2}{2kT_s}} \tag{18}$$

$$f_t = \left(\frac{m_t}{2\pi k T_t} \right)^{3/2} e^{-\frac{m_t c_t^2}{2kT_t}} \tag{19}$$

In the following sub-sections, we will derive the transport coefficients by using firstly Boltzmann collision integral and then Fokker-Planck approximation, and finally verify that they are equivalent.

III.1 Boltzmann Collision Integral

The rate of change of the momentum and energy are obtained from equations (11) and (12) respectively, the term $f_s f_t$ can be expressed as

$$f_s f_t = n_s n_t \left(\frac{m_s}{2\pi k T_s}\right)^{3/2} \left(\frac{m_t}{2\pi k T_t}\right)^{3/2} e^{-\frac{m_s c_s^2}{2kT_s} - \frac{m_t c_t^2}{2kT_t}} \tag{20}$$

The integrations over the velocity space can be performed by introducing the following variables as follows:

$$\mathbf{c_s} = \mathbf{c}_* - \frac{m_t T_s}{m_s T_t + m_t T_s} \mathbf{g}_* \tag{21}$$

$$\mathbf{c}_t = \mathbf{c}_* + \frac{m_s T_t}{m_s T_t + m_t T_s} \mathbf{g}_* \tag{22}$$

where

$$\mathbf{c}_* = \mathbf{V_c} - \mathbf{u_c} + \frac{T_t - T_s}{m_s T_t + m_t T_s} \mu_{st}^2 (\mathbf{\Delta u} + \mathbf{g}) \tag{23}$$

$$\mathbf{g}_* = -\mathbf{g} - \Delta\mathbf{u} \tag{24}$$

$$\Delta\mathbf{u} = \mathbf{u}_t - \mathbf{u}_s \tag{25}$$

$$\mathbf{u_c} = \frac{m_s \mathbf{u_s} + m_t \mathbf{u_t}}{m_s + m_t} \tag{26}$$

We also introduce

$$a^2 = \frac{2kT_sT_t}{m_sT_t + m_tT_s}; \quad \alpha^2 = \frac{2k(m_sT_t + m_tT_s)}{m_sm_t} \tag{27}$$

And

$$d\mathbf{c}_\mathbf{s} d\mathbf{c}_\mathbf{t} = d\mathbf{c}_* d\mathbf{g}_* \tag{28}$$

Substituting Equations from 21 to 28 into Eq.(20) and then into the expression for $\delta\mathbf{M}_s/\delta t$ (11) and $\delta E_s/\delta t$ (12) yields

$$\frac{\delta\mathbf{M}_\mathbf{s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^3 a^3\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int e^{-c_*^2/a^2} d\mathbf{c}_* \int \frac{\mathbf{g}}{g^3} e^{-g_*^2/\alpha^2} d\mathbf{g}_* \tag{29}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^3 a^3\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int e^{-c_*^2/a^2} d\mathbf{c}_* \int \frac{(\mathbf{V}_\mathbf{c}.\mathbf{g})}{g^3} e^{-g_*^2/\alpha^2} d\mathbf{g}_* \tag{30}$$

Integration with respect to d$\mathbf{c}_*$ can be evaluated immediately, using a spherical coordinate system (Gaussian integral)

$$\int e^{-c_*^2/a^2} d\mathbf{c}_* = \pi^{3/2}a^3 \tag{31}$$

It remains to carry out the integrations with respect to d$\mathbf{g}_*$. We expressed $\mathbf{g}_*$ in terms of $\mathbf{g}$ and $\mathbf{u}$

$$g_*^2 = g^2 + 2g\Delta g\Delta uco + (\Delta u)^2 \tag{32}$$

where θ is the angle between **g** and **v**.

With these changes, Eq. (29) and Eq. (30) become

$$\frac{\delta\mathbf{M}_\mathbf{s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^{3/2}\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int \frac{\mathbf{g}}{g^3} e^{-\left(g^2+2g\Delta g\Delta uco+(\Delta u)^2\right)/\alpha^2} d\mathbf{g} \tag{33}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{\mu_{st}n_t}{\pi^{3/2}\alpha^3}\left(\frac{q_sq_t}{4\pi\varepsilon_\circ\mu_{st}}\right)^2 \ln\Lambda \int \frac{(\mathbf{V}_\mathbf{c}.\mathbf{g})}{g^3} e^{-\left(g^2+2g\Delta g\Delta uco+(\Delta u)^2\right)/\alpha^2} d\mathbf{g} \tag{34}$$

Schunk (1977) calculated these remaining integral over g by expanded the exponential terms with cosθ, and he assumed a small relative drifts between the interacting gases (i.e., when the drift velocity differences are much smaller than thermal speeds), so that Schunk (1977) neglected the exponential term of $(\Delta u/\alpha)^2$, and finally the transport coefficients expressed in terms of the so-called Chapman–Cowling collision integrals (Chapman and Cowling ,1970).

In this study, we removed the latter restriction and calculated transport coefficients by using other strategy in which we introduced new variables as follow:

$$\mathbf{v} = \frac{\mathbf{g}}{\alpha};\ \mathbf{V} = \frac{2\Delta\mathbf{u}}{\alpha} \qquad (35)$$

The momentum and energy exchange collision terms reduce to

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{2\pi^{3/2}\varepsilon_\circ^2 \mu_{st}\alpha^3} e^{-\Delta u^2/\alpha^2} \int_0^\pi \int_0^\infty e^{-\left(v^2 + vV\cos\theta\right)} \alpha \sin\theta \cos\theta dv d\theta \qquad (36)$$

$$\frac{\delta E_{\mathbf{s}}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{2\pi^{3/2}\varepsilon_\circ^2 \mu_{st}\alpha^3} e^{-\Delta u^2/\alpha^2} \int_0^\pi \int_0^\infty \frac{1}{m_s + m_t} (m_s c_s + m_t c_t - m_t(u_s - u_t)) e^{-\left(v^2 + vV\cos\theta\right)} \alpha \sin\theta \cos\theta dv d\theta \quad (37)$$

This integral can be evaluated by using the technique Maclaurin series expansion for the exponential terms with cosθ, and then express them in terms of the hypergeometric function, so with these changes the final expressions for the coefficients are

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st}\alpha^2} e^{-\Delta u^2/\alpha^2} F\left(\frac{1}{2}; \frac{}{2}\left(\frac{\Delta\mathbf{u}}{2}\right)^2\right) \qquad (38)$$

$$\frac{\delta E_{\mathbf{s}}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 m_s m_t \alpha^2} e^{-\Delta u^2/\alpha^2} \left(3k(T_t - T_s) - m_t(\mathbf{u_s} - \mathbf{u_t})F\left(\frac{1}{2}; \frac{}{2}\left(\frac{\Delta\mathbf{u}}{2}\right)^2\right)\right) \qquad (39)$$

where $F\left(\frac{1}{2};\left(\frac{\Delta\mathbf{u}}{2}\right)^2 \atop 2\right)$ is hypergeometric function(Lebedev,1965; Koepf, 2014).

III. 2 Fokker-Planck Approximation

The first step in evaluating the transport coefficient for momentum by using the Fokker Planck approximation is the derivation of f_s *and* f_t with respect to $\mathbf{c_s}$, $\mathbf{c_t}$ respectively as follows

$$\frac{\partial f_s}{\partial \mathbf{c_s}} = -\frac{2\mathbf{c_s}}{a_s^2} f_s \tag{40}$$

$$\frac{\partial f_t}{\partial \mathbf{c_t}} = -\frac{2\mathbf{c}_t}{a_t^2} f_t \tag{41}$$

where

$$a_s^2 = \frac{2kT_s}{m_s} \quad ; \quad a_t^2 = \frac{2kT_t}{m_t} \tag{42}$$

Then

$$\left(\frac{f_s}{m_t}\frac{\partial f_t}{\partial \mathbf{c_t}} - \frac{f_t}{m_s}\frac{\partial f_s}{\partial \mathbf{c_s}}\right) = \frac{-2 f_s f_t}{m_s m_t}\left(\frac{m_s \mathbf{c}_t}{a_t^2} - \frac{m_t \mathbf{c_s}}{a_s^2}\right) \tag{43}$$

When this term is substituted into Equation (16) and use is made of the relations

$$\frac{1g^2 - \mathbf{gg}}{g^3} \cdot \left(\frac{m_s \mathbf{c}_t}{a_t^2} - \frac{m_t \mathbf{c_s}}{a_s^2}\right) = \frac{1}{(m_s + m_t)}\frac{\mathbf{g}}{g^3} \tag{44}$$

the result is

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi\varepsilon_\circ^2 \mu_{st}} \iint d\mathbf{c_s} d\mathbf{c_t} g_{st} \mathbf{g_{st}} f_s f_t \tag{45}$$

This is the same as we obtained from Boltzmann collision integral (i.e. Eq. (11)), so the final expression is

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 \mu_{st} \alpha^2} e^{-\Delta u^2/\alpha^2} F\left(\frac{1}{2}; \frac{2}{2}; \left(\frac{\Delta \mathbf{u}}{2}\right)^2\right) \tag{46}$$

Similarly, the energy coefficient $\delta E_s/\delta t$ can be calculated as we did for the momentum coefficient $\delta \mathbf{M}_s/\delta t$. We obtained approximately similar results as those obtained from Boltzmann collision integral.

$$\frac{\delta E_{\mathbf{s}}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 m_s m_t \alpha^2} \left[e^{-\Delta u^2/\alpha^2} \left(3k(T_t - T_s) - m_t(\mathbf{u_s} - \mathbf{u_t}) F\left(\frac{1}{2}; \frac{2}{2}; \left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right) + \left(\frac{m_s \mathbf{c}_s}{2} + m_t \mathbf{c}_s + m_s \mathbf{c}_t\right)\right] \tag{47}$$

The comparison between the results of Eq. (47) and Eq. (39) produce similar results and the little difference due to the Fokker-Planck approximation which obtained from expanding the Boltzmann collision integral and taking first terms in the Taylor series and neglect the other terms.

III. 3 Special Cases:

1) ($\mathbf{u}_\parallel = 0$, i.e. $\mathbf{u} = \mathbf{u}_\perp$), zero drift velocity parallel to the ambient magnetic field, and the drift velocity is perpendicular to the ambient magnetic field, the transport coefficients equations reduce to:

$$\frac{\delta \mathbf{M_s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 \mu_{st} \alpha^2} e^{-\Delta u_\perp^2/\alpha^2} F\left(\frac{1}{2}; \frac{2}{2}; \left(\frac{\Delta \mathbf{u}_\perp}{2}\right)^2\right) \tag{48}$$

$$\frac{\delta E_{\mathbf{s}}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln \Lambda}{4\pi \varepsilon_\circ^2 m_s m_t \alpha^2} \left[e^{-\Delta u_\perp^2/\alpha^2} \left(3k(T_t - T_s) - m_t(\mathbf{u_{s\perp}} - \mathbf{u_{t\perp}}) F\left(\frac{1}{2}; \frac{2}{2}; \left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right); + \left(\frac{m_s \mathbf{c}_s}{2} + m_t \mathbf{c}_s + m_s \mathbf{c}_t\right)\right] \tag{49}$$

2) ($\mathbf{u}_\perp$=0, i.e. $\mathbf{u}=\mathbf{u}_\parallel$), zero drift velocity component perpendicular to the ambient magnetic field, and the drift velocity is parallel to the ambient magnetic field, the transport coefficients take the form:

$$\frac{\delta \mathbf{M}_\mathbf{s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 \mu_{st}\alpha^2} e^{-\Delta u_\perp^2/\alpha^2} F\left(\frac{1}{2}; \frac{2}{2}; \left(\frac{\Delta \mathbf{u}_\perp}{2}\right)^2\right) \tag{50}$$

$$\frac{\delta E_\mathbf{s}}{\delta t} = -\sum_t \frac{n_t q_s^2 q_t^2 \ln\Lambda}{4\pi\varepsilon_\circ^2 m_s m_t \alpha^2}\left[e^{-\Delta u_\perp^2/\alpha^2}\left(3k\left(T_t - T_s\right) - m_t(\mathbf{u}_{\mathbf{s}\parallel} - \mathbf{u}_{\mathbf{t}\parallel})F\left(\frac{1}{2}; \frac{2}{2}; \left(\frac{\Delta \mathbf{u}}{2}\right)^2\right)\right) + \left(\frac{m_s \mathbf{c}_s}{2} + m_t \mathbf{c}_s + m_s \mathbf{c}_t\right)\right] \tag{51}$$

These coefficients derived by using Fokker-Planck approximation are, nearly, similar to the results obtained by using Boltzmann collision integral approach.

IV. Results and Discussions

For temperature isotropic plasmas, we obtained the transport coefficients (density, momentum, and energy) based on a drifting Maxwellian velocity distribution functions with drift velocity **u** with respect to the ambient magnetic field (i.e. $\mathbf{u}=\mathbf{u}_\parallel+\mathbf{u}_\perp$) by using Boltzmann collision integral, and Fokker Planck approximation. The final results are presented in the closed form in terms of hypergeometric functions. The two approaches produce approximately similar results.

We extended the work of Schunk (1977) and calculated the transport coefficients by using Boltzmann collision integral for two special cases where the relative drift is either parallel or perpendicular to the magnetic field, which are the two most common cases in astronomy and space physics. Then we investigated the previously problem by using another approach, Fokker Planck approximation, we obtained nearly similar results. The transport coefficients are presented in the form of hypergeometric functions. These results can be further generalized to an inverse power force interaction.

Finally, it should be noted that we derived the closed set of the collisional momentum and energy transport coefficients, however Chapman and Cowling (1970) calculated these coefficients approximately and for special case i.e. when the drift velocity differences between the various species are much smaller than typical thermal speeds, and they performed approximation for some specific collision processes.

Similarly, Jubeh and Barghouthi (2017) derived the above transport coefficients for bi-Maxwellian drifting plasma with special emphasis on the effect of Coulomb collisions. In an on-going study we are interested to derive, in closed form, the velocity diffusion coefficients for both cases, Maxwellian and bi-Maxwellian drifting plasma, and provide them in terms of Hypergeometric functions. These diffusion coefficients are going to be very useful to the solar and polar wind communities, especially in modeling the plasma behavior in these regions.

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Chapter Four

Results and Discussions

For application to problems dealing with isotropic and anisotropic plasmas with special emphasis on the effect of Coulomb collision (i.e. inverse-square force) we have derived a closed system of transport coefficients based on a Maxwellian and bi-Maxwellian drifting plasmas by using Boltzmann collision integral and Fokker-Planck approximation. The system of coefficients includes the mass, momentum and energy exchange collision terms, and the final results are presented in the closed form in terms of hypergeometric functions, as summarized:

1) We derive the momentum, parallel energy, and perpendicular energy collisional transport coefficients for bi-Maxwellian velocity distribution functions with drift velocity **u** with respect to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$) and present them in the form of triple hypergeometric functions. We also calculated the transport coefficients for two special cases where the relative drift is either parallel or perpendicular to the magnetic field, which are the two most common cases in astronomy and space physics (Jubeh and Barghouthi, 2017), as follow:

a) When the drift velocity is parallel to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\parallel}$), the transport coefficients are derived and presented in the form of double hypergeometric functions, these results are consistent with the findings of Hellinger and Trávníček (2009).

b) When the drift velocity is perpendicular to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\perp}$), the transport coefficients are obtained and found to be in the form of double hypergeometric functions.

2) We derive the collisional density, momentum and energy transport coefficients in Maxwellian plasmas with a general drift velocity with respect to the ambient magnetic field and presented in a closed form in terms of hypergeometric functions. However,

Chapman and Cowling (1970) calculated these coefficients approximately and for special case i.e. when the drift velocity differences between the various species are much smaller than typical thermal speeds, in this thesis, we removed the latter restriction and calculated transport coefficients for general drifting Maxwellian plasmas that are valid for arbitrary drift velocity differences as well as for temperature differences between the interacting plasma species. Also, we extended the work of Schunk (1977) and calculated the collisional transport coefficients for two special cases, as follow:

a) When the drift velocity is parallel to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\parallel}$, and $\mathbf{u}_{\perp}$=0).

b) When the drift velocity is perpendicular to the ambient magnetic field (i.e. $\mathbf{u} = \mathbf{u}_{\perp}$, and $\mathbf{u}_{\parallel}$=0).

Then we investigated the previously problems by using another approach, Fokker-Planck approximation, we obtained nearly similar results. These results can be further generalized to an inverse power force interaction.

In an on-going study we are interested to extend the work of Hinton (1983) and Hellinger and Trávníček (2012) and derive, in closed form, the velocity diffusion coefficients for both cases, Maxwellian and bi-Maxwellian drifting plasma, and provide them in terms of Hypergeometric functions. These diffusion coefficients are going to be very useful to the solar and polar wind communities, especially in modeling the plasma behavior in these regions.

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Appendix

Hypergeometric function

A generalized hypergeometric function ${}_pF_q$ ($a_1, \ldots a_p$; $b_1, \ldots, b_q$;x) is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}x \tag{1}$$

(The factor of (k+1) in the denominator is present for historical reasons of notation).

The function ${}_2F_1$ (a, b; c; x) corresponding to p=2, q=1, in general, arises the most frequently in physical problems, and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss, 1812; Barnes, 1908) that define by the power series

$${}_2\mathrm{F}_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{2}$$

It is undefined (or infinite) if c equals a non-positive integer. Here $(q)_n$ is the (rising) Pochhammer symbol, which is defined by:

$$(q)_n = \begin{cases} 1 & n = 0 \\ q(q+1)...(q+n+-1) & n > 0 \end{cases} \tag{3}$$

Example: The cosine function has the power series representation (Koepf, 2014)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{4}$$

To find its hypergeometric counterpart, we start with

$$a_k = \frac{(-1)^n}{(2n)!} x^{2n} \tag{5}$$

Then we get the term ratio

$$\frac{a_{k+1}}{a_k} = \frac{(-1)^{n+1}}{(2n+2)!} x^{2n+2} . \frac{(2n)!}{(-1)^n} x^{-2n} = \frac{1}{(n+1/2)(n+1)} \left(\frac{-x^2}{4} \right) \tag{6}$$

Since $a_0 = 1$, this leads finally to the hypergeometric representation

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = {}_0F_1\left(\left. \begin{matrix} - \\ \frac{1}{2} \end{matrix} \right| - \frac{x^2}{4} \right) \tag{7}$$

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 (Qureshi et al., 2015) , A special class of double hypergeometric functions or Kamp´e de F´eriet functions is considered here (Hellinger and Trávníček, 2009). These functions can be represented as double series

$$\mathrm{F}_{1.1}^{2..}\left(\begin{matrix} a,b \\ c;d \end{matrix}, x, y \right) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_{n+k}}{(c)_{n+k}(d)_k} \frac{x^n}{n!} \frac{y^k}{k!} \tag{8}$$

In 1967, the hypergeometric function was extended to three variables, resulting in the formula called the triple hypergeometric function F_A^3 (x,y,z) and it defined by

$$\mathrm{F}_{\mathrm{A}}^{3.}(x,y,z) = \sum_{n,k,m=0}^{\infty} \frac{(a)_{n+k+m}(b_1)_n(b_2)_k(b_3)_m}{(c_1)_n(c_2)_k(c_3)_p} \frac{x^n}{n!} \frac{y^k}{k!} \frac{z^m}{m!} \tag{9}$$

where F_A^3 (x,y,z) is called the Lauricella's triple hypergeometric function (Choi and Rathie, 2013).

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معاملات النقل للجسيمات البلازمية ذات التوزيع ماكسويل و التوزيع ثنائي ماكسويل

تتكون هذه الرسالة من اربعة فصول،حيث الفصل الاول مقدمة عامة في فيزياء البلازما و معادلة بولتزمان، و الفصل الثاني بحثا منشورا، بينما الفصل الثالث بحث مرسل للنشر، و الفصل الرابع و الاخير يتضمن النتائج و الدراسات المستقبلية.

في الفصل الثاني من هذه الرسالة تم اشتقاق معاملات النقل في فيزياء البلازما جيث اعتبرنا الجسيمات البلازمية تتوزع حسب توزيع ثنائي ماكسويل و تقع تحت تأثير تصادمات كولوم، و تم تعبير عن هذه المعاملات (الكثافة، كمية التحرك، الطاقة) بدلالة اقتران هيبيرجيوميترك (Hypergeometric function) ، و قمنا باشتقاق هذه المعاملات باستخدام طريقتين: تكاملات تصادم بولتزمان (Boltzmann collision integral) و معادلة فوكر- بلانك (Fokker– Planck equation) ، اضافة الى أنه تم حساب معاملات النقل لحالتين الخاصيتيين التاليتيين: اولا، عندما تكون سرعة الانزياح موازية للمجال المغناطيسي، و ثانيا، عندما تكون سرعة الانزياح باتجاه العمودي على المجال المغناطيسي. في الحالة الاولى، تم اشتقاق معاملات النقل بدلالة اقتران ثنائي هيبيرجيوميترك Double) (Hypergeometric function، و هذه النتائج تتفق مع نتائج هلنجر و ترافنيسك (٢٠٠٩). و بالنسبة للحالة الثانية، تم كتابة معاملات النقل ايضا بدلالة اقتران ثنائي هيبيرجيوميترك.

و في الفصل الثالث قمنا بنفس الدراسة و لكن افترضنا أن الجسيمات موزعة حسب توزيع ماكسويل بانزياح بشكل عام (التوزيع الطبيعي للجسيمات) و تم التعبير عن معاملات النقل بشكل كامل بدلالة اقتران هيبيرجيوميترك، علما بان الدراسات السابقة قامت باشتقاق هذه المعاملات بشكل تقريبي، كما تم اشتقاق معاملات النقل لحالتين خاصيتين عندما تكون سرعة الانزياح اما موازية او عمودية على المجال المغناطيسي، و هما الحالتان الأكثر شيوعا في علم الفلك و الفضاء.

نستطيع القول قدمنا معاملات نقل جديدة يمكن توظيفها في دراسة الرياح الشمسية و الرياح القطبية و العديد من الظواهر الفلكية الاخرى.