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On Generalized Almost Pythagorean Triples

Ameera Jamil Mohammed Salahat

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Prepared by:

Ameera Jamil Mohammed Salahat

B. Sc: Bethlehem University/ Palestine

Supervisor: Dr: Ibrahim Mahmoud Alghrouz

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Prepared by: Ameera Jamil Mohammed Salahat
Registration No: 21512757

Supervisor: Dr Ibrahim Mahmoud Alghrouz

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The names and the signatures of the examining committee members are as follows:

1-Head of Committee	Dr. Ibrahim Alghrouz	Signature: ………………
2-Internal Examiner	Dr. Jamil Jamal	Signature: ………………
3-External Examiner	Dr. Taha Abu Kaff	Signature: ………………

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Dedication

To my parents, my husband, my brothers, my sisters, my sons, my colleagues “teachers” and each person gave me support and assistance.

Ameera Salahat

Declaration

I certify that this submitted for the degree of master is the result of my own research, except where otherwise acknowledge. And that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed: ……………….

Ameera Jamil Mohammed Salahat

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Abstract

In this thesis, we studied the Pythagorean Triples, primitive Pythagorean Triples, Almost Pythagorean Triples, Nearly Pythagorean Triples and Almost – isosceles Pythagorean Triple. Also we do a program in Java Language to generate infinitely many Pythagorean Triples and Almost Pythagorean Triples by depending on the procedures and theorems that generate these triples, which will be explained in detail in this research.

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Introduction

Around 600 BC Pythagoras and his disciples made rather through studies of the integers in various ways:

Even numbers, Odd numbers, Prime numbers, and composite numbers, where the prime number is a number greater than 1 whose only divisions are 1 and the number itself.

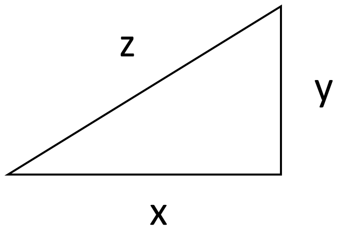
Numbers that are not prime are called composite except that the number 1 is considered neither prime nor composite.

The Pythagoreans also linked numbers with geometry, they introduced the idea of polygonal numbers, triangular number, square numbers, pentagonal numbers, etc.

The reason for this geometrical nomenclature is clear when the numbers are represented by dots arranged in the form of triangles, squares, pentagonal, etc.

Another link with geometry came from the famous theorem of Pythagoras which states that in any right triangle the square of the length of the hypotenuse is the sum of the squares of the lengths of the two legs.

Such triangles are now called Pythagorean triangles. The corresponding triple of numbers (x, y, z) representing the lengths of the sides is called a Pythagorean triple



$$x^2 + y^2 = z^2$$

A Babylonian tablet has been found, dating from about 1700 BC, which contain an extensive list of a Pythagorean triples, some of the numbers being quite large. Pythagoras was the first to give a method for determining infinity many triples, see [4].

Now, we will give a brief summary about chapter one, two and three.

In chapter one we introduce the definition of Pythagorean triple, Primitive Pythagorean triple and study some lemmas and theorems about Pythagorean triples and primitive Pythagorean triples. Also identify on the formula that produce all primitive Pythagorean triples, and study some applications about Pythagorean triple and primitive Pythagorean triple. In [9], they stated and proved the fundamental theorem of arithmetic to prove Lemma 1.1.10. which states that every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.

Addition to that in [2], they stated and proved the This lemma to prove Theorem 1.2.8 which states that if p is prime and a is an integer not divisible by p, then there exist integers x and y such that

$$ax \equiv y \pmod{p} \text{ with } 0 < |x| < \sqrt{p} \text{ and } 0 < |y| < \sqrt{p}$$

And also in [2] they used some theorems and lemmas that are using also in the proof of Theorem 1.2.8 as the following:

Let m and a be integers such that $m \neq 0$ and $\gcd(a, m) = 1$, we say that a is quadratic residue modulo m if the congruence

$$x^2 \equiv a \pmod{m}$$

is solvable.

If $p > 2$ is a prime and $\gcd(a, p) = 1$, we introduce the Legendre symbol $\left(\frac{a}{p}\right)$ by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & , a \text{ is a quadratic residue} \\ -1 & , \text{other wise.} \end{cases}$$

In [2] they proved that if p is an odd prime, then

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}},$$

also, in chapter one we studied the Pell equation

$$x^2 - 2y^2 = -1,$$

and the relation between Pell equation and primitive Pythagorean triples (PPTs), moreover we studied the relation between Pythagorean triples and reducible quadratic polynomials.

In chapter two we studied the definition of almost Pythagorean triple (APT), nearly Pythagorean triples (NPT) and almost isosceles Pythagorean triples(AI-PT) . Also, studied some lemmas and theorems to identify the properties of them and knowing the relation between APT and NPT. At the end of this chapter we generated infinitely many APTs and NPTs and applied the results in order to develop algorithm for constructing infinitely many AI-PT.

Finally, **in chapter three** we studied an algorithm for generating Pythagorean triples (PTs) by deal with two equation which are $y = x + a$ and $z = y + b$ and getting on some results that are using for generating Pythagorean triples. Also, we generate almost Pythagorean triples (APT) by depending on two theorems we proved its. However, in the end of this chapter gave a program by using Java language to generate infinity many (PTs) and (APTs).

Chapter One

Pythagorean Triples

The Greek mathematician Pythagoras theorem says the area of the square upon the hypotenuse of the right triangle is equal to the sum of the area of the squares upon the other sides. Conversely, any triangle for which the sum of the squares of the length of two shortest sides equals to the square of the third side is a right triangle.

Consequently, to find all right triangle with integral side lengths, we need to find all triples of positive integers $x, y,$ and z satisfying the equation. For more details, see [9].

$$x^2 + y^2 = z^2 \tag{1.1}$$

1.1 Primitive Pythagorean Triples.

In this section we want to define the primitive Pythagorean triples and identify on the formula that produces all the primitive Pythagorean triples under certain conditions.

Definition 1.1.1:

Triples of positive integers satisfying the equation (1.1) are called Pythagorean triples.

Example 1.1.2:

The triples (3,4,5), (6,8,10) and (5,12,13) are Pythagorean triples because

$$3^2 + 4^2 = 5^2 \text{ , } 6^2 + 8^2 = 10^2 \text{ and } 5^2 + 12^2 = 13^2$$

Some Pythagorean triples are scalar multiple of other triples such that (6,8,10) is twice (3,4,5).

In the next definition we define the primitive Pythagorean triple which has no common factor.

Definition 1.1.3:

A Pythagorean triple (x, y, z) is called primitive if $\gcd(x, y, z) = 1$, we will use PPT to stand for a Primitive Pythagorean Triple.

Example 1.1.4:

A Pythagorean triples (3,4,5) and (5,12,13) are primitive whereas the Pythagorean triple (6,8,10) is not.

Theorem 1.1.5:

Any integral multiple of primitive Pythagorean triple is a Pythagorean triple.

Proof:

Let (x, y, z) be a primitive Pythagorean triple and d be any positive integer, then

$$(dx)^2 + (dy)^2 = d^2x^2 + d^2y^2$$

$$=d^2(x^2 + y^2)$$

but $x^2 + y^2 = z^2$, then

$$(dx)^2 + (dy)^2 = d^2z^2 = (dz)^2,$$

therefore ,(dx, dy, dz) is a Pythagorean triple.

Hence, all Pythagorean triples can be found by forming an integral multiple of a Primitive Pythagorean triple (PPT).■

Now, to find all primitive Pythagorean triples we need some lemmas and theorems. The first one tells us that any two integers of a primitive Pythagorean triple are relatively prime.

Lemma 1.1.6:

If (x, y, z) is primitive Pythagorean triple, then $\gcd(x, y) = \gcd(x, z) = \gcd(y, z) = 1$.

Proof:

Suppose that (x, y, z) is a primitive Pythagorean triple and $\gcd(x, y) = d > 1$, then $\exists\ a$ prime number p such that $p|\gcd(x, y)$, so that $p|x$ and $p|y$ which implies that $p|(x^2 + y^2) = z^2$, but p is prime and $p|z^2$, then we conclude that $p|z$. Therefore, $p|\gcd(x, y, z)$, which is a contradiction, since $\gcd(x, y, z) = 1$, therefore $\gcd(x, y) = 1$.

To prove that $\gcd(x, z) = 1$, by contrary, suppose $\gcd(x, z) = d > 1$>. By the same steps as in above there exists a prime number p such that $p|\gcd(x, z)$, so that $p|x$ and $p|z$, thus $p|(z^2 - x^2) = y^2$, so $p|y^2$ but p is prime number, we conclude $p|y$. Which is a contradiction since $\gcd(x, y, z) = 1$, therefore $\gcd(x, z) = 1$.

Similarly, we can show that $\gcd(y, z) = 1$.■

The second lemma, helps us to determine the formula for PPT as the follows: see [6],

Lemma 1.1.7:

For a Pythagorean triple (x, y, z), the following properties are equivalent:

(1) $x, y,$ and z have no common factor, i.e. the triple is primitive,

(2) $x, y,$ and z are pair wise relatively prime,

(3) two of $x, y,$ and z are relatively prime.

Proof:

In Lemma 1.1.6 we proved that (1) implies (2). Also, it is obvious that (2) implies (3). To prove that (3) implies (1), assume that (3) is true with $\gcd(x, y) = 1$ and suppose that $\gcd(x, y, z) \neq 1$, then $\exists$ a prime number p such that $p|x$, $p|y$ and $p|z$, but x and y are relatively prime, a contradiction.

Before we introduce the third lemma we present the following theorem, in order to prove the next lemma. see [1].■

Theorem 1.1.8:

If m is an integer, then $m^2 \equiv 0\ (mod\ 4)$ if m is even and $m^2 \equiv 1 (mod 4)$ if m is odd.

Proof:

Let $m \in Z$. Then either m is even or m is odd.

If m is even, then $\exists k \in Z$, such that $m = 2k$. Then $m^2 = 4k^2$, so $4|m^2$ and hence $m^2 \equiv 0(mod4)$.

If m is odd, then $\exists k \in Z$, such that $m = 2k + 1$. Then $m^2 = 4k^2 + 4k + 1$, so $m^2 - 1 = 4(k^2 + k)$ hence$4|(m^2 - 1)$, and hence $m^2 \equiv 1(mod4)$.Therefore, if m is an integer, then $m^2 \equiv 0(mod4)$ or $m^2 \equiv 1\ (mod\ 4)$. ■

Now, we present the third lemma which tells us that one of the PT is odd and the other is even or vice versa. For more details, see [9].

Lemma 1.1.9:

If (x, y, z) is PPT, then one of *x* and *y* is odd and the other is even.

Proof:

To show that one of *x* and *y* is odd and the other is even. Suppose that *x* and *y* are both odd, then by Theorem 1.1.8, $x^2 \equiv 1(mod4)$ and $y^2 \equiv 1(mod4)$.
So, $z^2 = x^2 + y^2 \equiv 2(mod\ 4)$.
This is impossible because the square of any integer must be congruent either to 0 or 1 modulo 4, as in Theorem 1.1.8. Thus, *x* and *y* are not both odd. Also *x* and *y* are not both even, since if both *x* and *y* are even, then $2|x$ and $2|y$, and so $\gcd(x, y) \neq 1$, which contradicts Lemma 1.1.6. Hence one of *x* and *y* must be odd and the other is even.■

The fourth Lemma that we need is a consequence of the fundamental theorem of arithmetic it tells that if the multiple of two relatively prime integers is a square, then each of them must both be square.

Lemma 1.1.10:

Let r, s and t are positive integers such that $\gcd(r, s) = 1$ and $r.s = t^2$ then $\exists\, k, l$ such that $r = k^2$ and $s = l^2$.

Proof:

If $r\,.\,s = 1$, then clearly $r = 1$ and $s = 1$. Also, if $r = 1$ or $s = 1$, then $r = t^2$ or $s = t^2$, and clearly Lemma is true. So we may suppose that $r > 1$ and $s > 1$. Then the prime – power factorizations of r, s and t are

$$r = {P_1}^{a_1}{P_2}^{a_2} \dots \dots \dots {P_u}^{a_u}$$

$$s = p_{u+1}^{a_{u+1}} p_{u+2}^{a_{u+2}} \dots \dots \dots {P_v}^{a_v}$$

and

$$t = {q_1}^{b_1}{q_2}^{b_2} \dots \dots \dots {q_k}^{b_k}$$

since, $\gcd(r, s) = 1$, then the primes occurring in the factorizations of r and s are distinct. Since $rs = t^2$, we have

$${P_1}^{a_1}{P_2}^{a_2} \dots \dots {P_u}^{a_u} p_{u+1}^{a_{u+1}} p_{u+2}^{a_{u+2}} \dots \dots {P_v}^{a_v} = {q_1}^{2b_1}{q_2}^{2b_2} \dots \dots \dots {q_k}^{2b_k}.$$

From the fundamental theorem of arithmetic, the prime-powers occurring on the two sides of the above equation are the same. Hence each p_i must be equal to q_j for some j with matching exponents, so that $a_i = 2b_j$.

Consequently, every exponent a_i is even and therefore $a_i/2$ is an integer. Now, let

$$k = {P_1}^{a_1/2}{P_2}^{a_2/2} \dots \dots \dots {P_u}^{a_u/2}$$

and

$$l = p_{u+1}^{a_{u+1}/2} p_{u+2}^{a_{u+2}/2} \dots \dots \dots {P_v}^{a_v/2}$$

Clearly

$$r = k^2 \text{ and } s = l^2 . \;\blacksquare$$

Now, we introduce a desired theorem which describes all primitive Pythagorean triples.

Theorem 1.1.11:

The positive integers x, y and z form a primitive Pythagorean triples, with y even if and only if there are relatively prime positive integers k and $l, k > l$, and $k \not\equiv l(mod 2)$, with k odd and l even or vice versa, such that

$$x = k^2 - l^2 \quad , \quad y = 2kl \quad \text{and} \quad z = k^2 + l^2 \qquad (1.2)$$

Proof:

($\Rightarrow$)Let (x, y, z) be PPT with y is even , then by Lemma 1.1.9, x is odd and so x^2 is odd. But y^2 is even then $x^2 + y^2$ is odd so z^2 is odd which implies z is odd.

Hence $x + z$, $x - z$ are both even.

Define r and s by

$$r = \frac{z+x}{2}, \ s = \frac{z-x}{2}$$

Since

$$x^2 + y^2 = z^2$$

we have

$$y^2 = z^2 - x^2$$

$$y^2 = (z - x)(z + x) \qquad (1.3)$$

dividing equation (1.3) by 4, we have

$$(\frac{y}{2})^2 = \left(\frac{z-x}{2}\right)\left(\frac{z+x}{2}\right) = rs.$$

Now, we want to show that $\gcd(r, s) = 1$, so let $\gcd(r, s) = d$, since $d|r$ and $d|s$, then $d|(r + s) = z$ and $d|(r - s) = x$, thus $d|\gcd(x, z)$. But $\gcd(x, z) = 1$, then $d = 1$. Since $(\frac{y}{2})^2 = rs$ and $\gcd(r, s) = 1$, then by Lemma 1.1.10 there exist, $k, l \in Z$ such that $r = k^2$ and $s = l^2$. Writing x, y and z in terms of k and l we have

$$x \quad = r - s \quad = k^2 - l^2,$$

$$y = \sqrt{4rs} \quad = \sqrt{4k^2l^2} = 2kl,$$

And hence

$$z = r + s \quad = k^2 + l^2.$$

Also we must show that $\gcd(k, l) = 1$. So If $\gcd(k, l) = d$, then $d|k^2 - l^2$, $d|2kl$ and $d|k^2 + l^2$, that is $d|x, d|y$ and $d|z$. hence $d|\gcd(x, y, z)$, but we know that $\gcd(x, y, z) = 1$, then $d = 1$. Since $r = \frac{z+x}{2} > \frac{z-x}{2} = s$, then $k > l$.

Finally, we must show that $k \not\equiv l(mod2)$. To show that suppose $k \equiv l(mod2)$, then $k - l$ is divisible by 2. We have the following cases:

(1) If both k and l are even, then $2|k$ and $2|l$ and so $\gcd(k,l) \neq 1$, a contradiction.

(2) If both k and l are odd, then x, y and z are all even, this contradicts $\gcd(x,y,z) = 1$.

(3) If one of them odd and the other is even, say, k is odd and l is even. Then $k - l$ is odd, but $2|k - l$, a contradiction.So, $k \not\equiv l(mod2)$, From (1) and (2) above one of k and l is odd and the other is even.

($\Leftarrow$) Suppose the triple (x, y, z) satisfy

$$x = k^2 - l^2 \text{ , } y = 2kl \text{ and } z = k^2 + l^2,$$

where k and $l \in Z,\ k > l,\ \gcd(k,l) = 1$ and $k \not\equiv l(mod2)$. We must show that x, y and z form a primitive Pythagorean triples, with y even.

We first note that (x, y, z) is PT

$$\begin{aligned} x^2 + y^2 &= (k^2 - l^2)^2 + (2kl)^2 \\ &= k^4 - 2k^2l^2 + l^4 + 4k^2l^2 \\ &= k^4 + 2k^2l^2 + l^4 \\ &= (k^2 + l^2)^2 \\ &= z^2. \end{aligned}$$

Now, we want to show $\gcd(x,y,z) = 1$. This follows by Lemma 1.1.7 by showing that x and z are relatively prime. Suppose $\gcd(x,z) = d > 1$. Then $\exists$ a prime number $p > 1$ such that $p|d$ and so $p|x$ and $p|z$. Hence $p|x + z = 2k^2$ and $p|z - x = 2l^2$. If $p|2$, then $p = 2$. But $k \not\equiv l(mod2)$, so k and l not both odd or not both even, then $k^2 + l^2$ and $k^2 - l^2$ are both odd, i.e x and z are both odd. Hence, $p \neq 2$, Suppose $p \neq 2$, then $p|k^2$ and $p|l^2$, but p is prime, then $p|k$ and $p|l$, and so $\gcd(k,l) \neq 1$, a contradiction. Thus, $\gcd(x,z) = 1$, and so by Lemma 1.1.7. $\gcd(x,y,z) = 1$).■

Example 1.1.12:

Let $k = 5$ and $l = 2$, so that $\gcd(k,l) = 1$, $k \not\equiv l(mod2)$ and $k > l$, hence by theorem 1.1.9 we have

$$x = k^2 - l^2 = 5^2 - 2^2 = 21$$

$$y = 2kl = 2.5.2 = 20$$
$$z = k^2 + l^2 = 5^2 + 2^2 = 29$$

Table 1.1: The list of the PPT generated by using theorem 1.1.11 with $k \leq 6$

k	***l***	***x***	***y***	***Z***
2	1	3	4	5
3	2	5	12	13
4	1	15	8	17
4	3	7	24	25
5	2	21	20	29
5	4	9	40	41
6	1	35	12	37
6	5	11	60	61

1.2 Applications

For the rest of this section, we are interested in the Diophantine equation $x^2 - dy^2 = n$, where d and n are integers and d is a positive integer which is not a perfect square. The special case of the Diophantine equation $x^2 - dy^2 = n$ with $n = 1$ is called pell's equation.

The problem of finding the solutions of this equation has a long history. Special cases of Pell's equation are discussed in ancient works by Archimedes and Diophantus. Moreover, the twelfth century Indian mathematicians Bhakra described a method for finding the solutions of Pell's equation. A letter written in 1657 Fermat posed to the mathematicians of Europe" the problem of showing that there are infinity many integral solutions of the equation $x^2 - dy^2 = 1$ where d is a positive integer greater than 1 which is not a perfect square. Soon afterwards, the English mathematicians Wails and Brounckes developed a method to find these solutions, but did not provide a proof that their methods work. Euler provided all the theory needed for a proof in a paper published in 1767 and Lagrange published such a proof in 1768. See [9]

An Important results in number theory developed from the study of values of $x^2 - dy^2$ where d is fixed non square integer, the special case of Pell equation when $d = 2$ shows up in the setting of Pythagorean triples, but other values of d are important for other problems. See [6].

If we have a solution of the equation

$$x^2 - 2y^2 = \pm 1$$

in positive integers x and y, then x is odd and $\gcd(x, y) = 1$. Let $k = x + y$ and $l = y$. then the triple $(k^2 - l^2, 2kl, k^2 + l^2)$ is a primitive Pythagorean triple with $k > l > 0$, $\gcd(k, l) = 1$ and $k \not\equiv l \ (mod\ 2)$.

To show this, Since $k = x + y$ and $l = y$, then $k = x + y > y = l > 0$, so $k > l > 0$. Now, if $d|k$ and $d|l$, then d divides $k^2 - l^2$, so $d|x$ and $d|y$,but $\gcd(x, y) = 1$, so $d = 1$, so that $\gcd(k, l) = 1$. Since $k - l = (x + y) - y = x$ and x is odd, then $x \not\equiv 0 \ (mod\ 2)$, and so $k - l \not\equiv 0 (mod 2)$, i.e $k \not\equiv l \ (mod\ 2)$.

Now, we claim that $(k^2 - l^2, 2kl, k^2 + l^2)$ is PPT. Clearly

$$\begin{aligned}
&(k^2 - l^2)^2 + (2kl)^2 \\
&\quad = k^4 - 2k^2l^2 + l^4 + 4k^2l^2 \\
&\quad = k^4 + 2k^2l^2 + l^4 \\
&\quad = (k^2 + l^2)^2.
\end{aligned}$$

Hence this triple is PT.

Using Lemma 1.1.7, it is sufficient to show that $gcd(k^2 - l^2, 2kl) = 1$.

Suppose $\exists$ a prime number $p|\gcd(k^2 + l^2, 2kl)$. Then $p|(k^2 + l^2) - 2kl$ and $p|(k^2 + l^2) + 2kl$. So, this implies $p|(k - l)^2$ and $p|(k + l)^2$. But $(k - l)^2 = x^2$ and x is odd, so $p \neq 2$. Also, since p is prime and $p|(k - l)^2 = x^2$, then $p|x$. But $p|(k + l)^2$ implies $p|k + l$, so $p|(k + l) - (k - l)$, i.e. $p|2l$, but $p \neq 2$, so $p|l$. Hence $p|y$ and so $p|\gcd(x, y)$, but $\gcd(x, y) = 1$ so $p = 1$.And so the triple $(k^2 - l^2, 2kl, k^2 + l^2)$ is a PPT.

The most well-known Pythagorean triples (3,4,5) and (5,12,13) have consecutive terms. Clearly 3 and 4 in the first triple, 12 and13 in the others, so by using the parametric formula (1.2) for primitive Pythagorean triples, we can address questions concerning relation among the sides of primitive right triangle, as the following:

What are all the Pythagorean triples (a, b, c) with a pair of consecutive terms either (*a* and *b* or *b* and *c*)? To answer the question, we will consider the following cases:

First Case:

Let us consider the case when a and b are consecutive.

For a and b to differ by 1 means that $a - b = \pm 1$, but from the discussion we have $a = k^2 - l^2$ and $b = 2kl$, so

$$(k^2 - l^2) - 2kl = \pm 1 \qquad (1.4)$$

By adding $2l^2$ for both sides we get,

$$k^2 - l^2 - 2kl + 2l^2 = \pm 1 + 2l^2$$

which implies to

$$k^2 - 2kl + l^2 = \pm 1 + 2l^2$$

By factorizing the left hand side and write it as a complete square we have,

$$(k - l)^2 = \pm 1 + 2l^2$$

If we subtract $2l^2$ form both side we get,

$$(k-l)^2 - 2l^2 = \pm 1$$

Therefore, the relation (1.4) can be written as

$$(k-l)^2 - 2l^2 = \pm 1 \qquad (1.5)$$

where $k-l$ is positive and odd and l is positive.

That is, finding Pythagorean triples whose legs differ by 1 is the same as finding positive integer solution to the Pell equation $x^2 - 2y^2 = \pm 1$.

Example 1.2.1:

Show that Pell equation is satisfied when $x = 1$ and $y = 1$, and find the Pythagorean triple (a, b, c) which satisfies it?

Solution:

Suppose $x = 1$ and $y = 1$, it's clear that the greatest common factor between $x \; and \; y$ equal one. Now substitute $x = 1$ and $y = 1$ in Pell equation we get

$$1^2 - 2(1)^2 = -1$$

Thus, $x = 1$ and $y = 1$ satisfies the Pell equation. So, let $k = x + y$ and $l = y$, that implies $l = 1$ and $k = 2$. Therefore, the primitive Pythagorean triple is (3,4,5). since

$$a = k^2 - l^2 = 2^2 - 1^2 = 3$$

$$b = 2kl = 2 * 2 * 1 = 4$$

$$c = k^2 + l^2 = 2^2 + 1^2 = 5.$$

Next table shows 5 examples of Pythagorean triples that we can find it in the same procedure in the previous example. For more details, See [4]

Table 1.2: consecutive legs.

x	***y***	***k***	***l***	***a***	***b***	***c***
1	1	2	1	3	4	5
3	2	5	2	21	20	29
7	5	12	5	119	120	169
17	12	29	12	697	696	925
41	29	70	29	4059	4060	5741

The above Procedure to calculate Pythagorean triples when the differ between two legs in a primitive triple equal 1, but even if two legs in a primitive triple don't differ by 1 the formula (1.5) is still $(k-l)^2 - 2l^2 = n$, for any integer n .

Second case:

For *x* and *y* differ by 2, by the same steps when *x* and *y* differ by 1, we conclude that

$$(k-l)^2 - 2l^2 = \pm 2$$

The equation

$$x^2 - 2y^2 = \pm 2$$

is not solvable, since $x^2 = 2y^2 \pm 2$, then $2|x^2$ and so $2|x$, but x is odd. A contradiction. then the triple $(k^2 - l^2, 2kl, k^2 + l^2)$ is not a primitive, where $k = x + y$ and $l = y$.
In generalized if $x^2 - 2y^2 = \pm 2k$ where k is an integer then the triple is not PPT.

So the possible differences between legs in a primitive triple are precisely the odd values of $x^2 - 2y^2$ for positive integers x and y.

Third case:

When

$$x^2 - 2y^2 = p \qquad (1.6)$$

where p is a prime.

The last case tell us if p is a prime congruent to $\pm 1 (mod 8)$, then the general pell equation $x^2 - 2y^2 = p$ is solvable. Thus to show that we need to prove the next theorem. For more details, see [2]

Theorem 1.2.2:

Let p be an odd prime number then $p = a^2 - 2b^2$ for some integers a and b if and only if $p \equiv \pm 1 (mod 8)$.

Proof

Suppose p is an odd prime number such that $p = a^2 - 2b^2$ for some integers a and b, then $a^2 \equiv 2b^2\ (modp)$. Also let b' be an integer such that $bb' \equiv 1(modp)$, thus $(ab')^2 = a^2(b')^2 \equiv 2b^2(b')^2(modp)$ and so $(ab')^2 \equiv 2(bb')^2(modp)p \equiv 2(modp)$.

Now from [2], we used corollary that state if $p > 2$ is a prime and $\gcd(a, p) = 1$, then the Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & , a \text{ is a quadratic residue,} \\ -1 & , \text{other wise,} \end{cases}$$

and so 2 is quadratic residue modulo p and the congruence $(ab')^2 \equiv 2(modp)$ is solvable, thus $(\frac{2}{p}) = 1$

Also, from [2] we used theorem that states, $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$, Hence $\frac{p^2-1}{8} = 2k$, where k is an integer. Thus

$$p^2 - 1 \equiv 0 \quad (mod8)$$

$$p^2 \equiv 1 \quad (mod8)$$

$$\Rightarrow p \equiv 1 \quad (mod8) \text{ or } p \equiv -1 \quad (mod8).$$

Suppose p is an odd prime number such that $p \equiv \pm 1(mod8)$ and let a be an integer such that $a^2 \equiv 2(modp)$, so there is no positive integers $x\ and\ y$ with $x < y < \sqrt{p}$ such that $p|a^2x^2 - y^2$, therefore $p|(a^2 - 2)x^2 + 2x^2 - y^2$, so $p|2x^2 - y^2$. hence $0 < 2x^2 - y^2 < 2p$, implies $p = 2x^2 - y^2$.

Now, we take some examples on this case, and what are the solutions that satisfies the Pell equation (1.3), and what are the PPT that satisfied from this equation.

Example 1.2.3:

The equation

$$x^2 - 2y^2 = \pm 3$$

has no integral solution because the prime 3 cannot be written as the form $8k \pm 1$ (by theorem 1.2.2) so no PPT has it legs differing by 3.

Example 1.2.4:

The equation

$$x^2 - 2y^2 = \pm 5$$

has no integral solution because the prime 5 cannot be written as the form $8k \pm 1$ so no PPT has it legs differing by 5.

Example 1.2.5:

The equation

$$x^2 - 2y^2 = 7$$

has integral solution because, the prime 7 can be written as $8(1) - 1 = 7$, Thus, the equation

$$x^2 - 2y^2 = 7$$

has the integral solution $x = 3$ and $y = 1$.

To find the PPT, let

$$k = x + y = 4 \text{ and } l = y = 1$$

So the PPT is

$$k^2 - l^2 = 4^2 - 1^2 = 15$$

$$2kl = 2.4.1 = 8$$

And

$$k^2 + l^2 = 4^2 + 1^2 = 17\ ,$$

therefore, the PPT is (15,8,17).

Hence, from discussion of the cases above we conclude the following for the equation

$$x^2 - 2y^2 = \pm n,$$

1) If n is even, then there no PPT with consecutive legs.
2) If $n = 1$, then there are many PPT as in Table(1.2)
3) If n is a prime number, then we have PPT triples with consecutive legs if $n = 8k \pm 1$, for some $k \in Z$.
4) Not every odd n can derive PPT with consecutive legs.

Fourth Case:

Let us now, turning to a leg and hypotenuse which differ by 1. The story is much simpler if the hypotenuse use is odd, so it can only differ by 1 from the even leg, since

$$y = 2kl \text{ and } z = k^2 + l^2$$

as we discussed before, then $k^2 + l^2 - 2kl = 1$.

But $k^2 + l^2 - 2kl = (k - l)^2$. i.e $(k - l)^2 = 1$, which implies that $k = l + 1$. So the PPT $(k^2 - l^2, 2kl, k^2 + l^2)$ becomes,

$$(2l + 1, 2l^2 + 2l,\ 2l^2 + 2l + 1).$$

Since,

$$k^2 - l^2 = (l + 1)^2 - l^2 = l^2 + 2l + 1 - l^2 = 2l + 1,$$

$$2kl = 2(l + 1)l = 2l^2 + 2l$$

and

$$k^2 + l^2 = (l + 1)^2 + l^2 = l^2 + 2l + 1 + l^2 = 2l^2 + 2l + 1.$$

The next table shows the first four examples can be found by the previous procedure.

Table1.3: consecutive leg and hypotenuse.

$\boldsymbol{l}$	$\mathbf{2l + 1}$	$\mathbf{2l^2 + 2l}$	$\mathbf{2l^2 + 2l + 1}$
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41

From above discussion we studied the PTs when the difference between x and y equal 1 and the difference between y and z equal 1,but now we introduce a theorem that study the PTs when the absolute value of difference between x and y equal $2k^2 - 1$, where $k > 0$. For more details, see [5].

Theorem 1.2.6:

For any positive integer k , there are infinitely many PTs (x, y, z) ,satisfying $|y - x| = 2k^2 - 1$.

Proof:

We assume $a_1 = 1$ and $b_1 = k$. The Fibonacci type numbers are

$$\{ a_1, b_1, a_1 + b_1, a_1 + 2b_1\} = \{ 1, k, 1 + k, 1 + 2k\}$$

Notice that we depend on the middle two terms of $\{f_n\}$ to find the PTs, therefore,

$$x = (1 + k)^2 - k^2 = 2k + 1$$

$$y = 2k\,(1 + k) = 2k + 2\,k^2,$$

and

$$z = (1 + k)^2 + k^2 = 2k^2 + 2k + 1$$

thus the PT $T_1^{(k)} = (2k + 1\,, 2k + 2k^2, 2k^2 + 2k + 1)$

so the difference $S_1^{(k)} = |y_1 - x_1|$

$$= 2\,k^2 + 2k - 2k - 1$$

$$= 2k^2 - 1.$$

Secondly if $a_2 = a_1 + 2b_1, b_2 = a_1 + b_1$ then the Fibonacci type numbers are

$$\{a_2, b_2, a_2 + b_2, a_2 + 2b_2\}$$

$$= \{a_1 + 2b_1, a_1 + b_1, 2a_1 + 3b_1, 3a_1 + 4b_1\}$$

$$= \{1 + 2k\,, 1 + k, 2 + 3k, 3 + 4k\}.$$

By depending on the middle two terms of Fibonacci type numbers we have

$$x = (2 + 3k)^2 - (1 + k)^2$$

$$= 4 + 12k + 9k^2 - 1 - k^2 - 2k$$

$$x = 8k^2 + 10k + 3$$

$$y = 2\,(1 + k)\,(2 + 3k)$$

$$= 2\,(2 + 3k + 2k + 3k^2)$$

$$y = 6k^2 + 10k + 4,$$

and

$$z = (2 + 3k)^2 + (1 + k)^2$$
$$= 4 + 12k + 9k^2 + 1 + k^2 + 2k$$
$$z = 10k^2 + 14k + 5,$$

thus the PT $T_2^{(k)}$ is

$$(8k^2 + 10k + 3, 6k^2 + 10k + 4, 10k^2 + 14k + 5)$$

With $S_2^{(k)} = |y_2 - x_2| = |6k^2 + 10k + 4 - 8k^2 - 10k - 3|$
$= 2k^2 - 1.$

Now, for any $n > 1$, let $a_n = a_{n-1} + 2b_{n-1}$ and $b_n = a_{n-1} + b_{n-1}$ assume that the PT $T_n^{(k)} = (x_n, y_n, z_n)$

$$= \left(a_n (a_n + 2b_n), 2b_n (a_n + b_n), {b_n}^2 + (a_n + b_n)^2\right),$$

generated by Fibonacci type numbers $\{ a_n, b_n, a_n + b_n, a_n + 2b_n\}$ satisfies $S_n^{(k)} = |2k^2 - 1|$, then the next PT $T_{n+1}^{(k)}$ generated by

$$\{ a_{n+1}, b_{n+1}, a_{n+1} + b_{n+1}, a_{n+1} + 2b_{n+1}\},$$

forms

$$T_{n+1}^{(k)} = \left(a_{n+1} (a_{n+1} + 2b_{n+1}), 2b_{n+1} (a_{n+1} + b_{n+1}), {b_{n+1}}^2 + (a_{n+1} + b_{n+1})^2\right).$$

And also we have,

$$S_{n+1}^{(k)} = |y_{n+1} - x_{n+1}|$$
$$= |2(a_n + b_n) (2a_n + 3b_n) - (a_n + 2b_n) (3a_n + 4b_n)|$$
$$= |a_n^2 - 2b_n^2|$$
$$= |2b_n (a_n + b_n) - a_n (a_n + 2b_n)|$$
$$= \left|S_n^{(k)}\right| = |2k^2 - 1|.$$

So, we have infinity many PTs (x_n, y_n, z_n), such that, $|y_n - x_n| = 2k^2 - 1$.

Example 1.2.7:

If $a_1 = 1, a_2 = k$, $(1 \leq K \leq 4)$ then

$T_n^{(k)}$ with $S_n^{(k)} = |2k^2 - 1|$ are $(3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41)$.

Table 1.4: The PT with Fibonacci numbers and the difference $S_n^{(k)} = |2k^2 - 1|$

n	$f'_n s$	$T_n^{(1)}$ with $S_n^{(1)} = 1$	$f'_n s$	$T_n^{(2)}$ with $S_n^{(2)} = 7$	$f'_n s$	$T_n^{(3)}$ with $S_n^{(3)} = 17$	$f'_n s$	$T_n^{(4)}$ with $S_n^{(4)} = 31$
1	1,1,2,3	3,4,5	1,2,3,5	5,12,13	1,3,4,7	7,24,25	1,4,5,9	9,40,41
2	3.2.5.7	21.20.29	5,3,8,11	55,48,73	7,4,11,15	105,88,137	9,5,14,19	171,140,221
3	7,5,12,17	119,120,169	11,8,19,27	297,304,425	15,11,26,37	555,572,797	19,14,33,47	893,924,1285

1.3 The relation between Pythagorean triples and reducible quadratic polynomial

Consider the equation

$$x^2 + mx \pm n = 0 \qquad (1.7)$$

where m and n are positive integers. If the equation (1.7) have integer roots , then $m^2 \pm 4n$ are perfect squares. Let $m^2 - 4n = d^2$ and $m^2 + 4n = e^2$, clearly d and e are integers.

Now,

$$e^2 - d^2 = 8n = (e-d)(e+d),$$

so 2 divides $(e-d)$ or 2 divides $(e+d)$, that is $d \equiv e \bmod 2$,Thus, $\frac{e-d}{2}$ and $\frac{e+d}{2}$ are integers. Also,

$$d^2 + e^2 = m^2 - 4n + m^2 + 4n = 2m^2$$

then $m^2 = \frac{d^2+e^2}{2} = \left(\frac{e+d}{2}\right)^2 + \left(\frac{e-d}{2}\right)^2$ Thus $\left(\frac{e-d}{2}, \frac{e+d}{2}, m\right)$ is PT.

In fact$\left(\frac{e-d}{2}, \frac{e+d}{2}, m\right)$ is PPT because if $\exists$ a prime p so that $p|\gcd\left(\frac{e-d}{2}, \frac{e+d}{2}, m\right)$ then $p|\frac{e+d}{2}$, $p|\frac{e-d}{2}$ and $p|m$, so $p|(\frac{e-d}{2})(\frac{e+d}{2})$ i.e. p$|\frac{e^2-d^2}{2}$, but $e^2 - d^2 = 8n$ So $p|2n$, and $p \not\equiv 2$ (from Theorem 1.1.11), so $p|n$. Hence, $p|\gcd(n,m) = 1$, so $p = 1$.

Example 1.2.8:

Show that the polynomial $x^2 + 5 \times \pm 6$ corresponds to the Pythagorean triple (3,4,5)?

Solution:

As above discussion, let $d^2 = 5^2 - 4 \times 6 = 1$ and $e^2 = 5^2 + 4 \times 6 = 49$

i.e. $d = 1$ and $e = 7$. So the PPT corresponding to this equation is $\left(\frac{7-1}{2}, \frac{7+1}{2}, 5\right) = (3,4,5)$.

Next table shows some corresponding Pythagorean triples and reducible $x^2 + m \times \pm n$.

Table 1.5: Pythagorean triples and reducible quadratic polynomial $x^2 + m \times \pm n$.

$\boldsymbol{m}$	$\boldsymbol{n}$	$\boldsymbol{x^2 + m \times + n}$	$\boldsymbol{x^2 + m \times - n}$	$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$
5	6	$(x+2)(x+3)$	$(x-1)(x+6)$	(3,4,5)
13	30	$(x+3)(x+10)$	$(x-2)(x+15)$	(5,12,13)
17	60	$(x+5)(x+12)$	$(x-3)(x+20)$	(8,15,17)
29	210	$(x+14)(x+15)$	$(x-6)(x+35)$	(20,21,29)

Chapter Two

Almost and Nearly isosceles Pythagorean triples

A Pythagorean triple (PT) is an integer solution (a, b, c) satisfying the polynomial

$$x^2 + y^2 = z^2$$

and it is said to be primitive (PPT) if $\gcd(a, b, c) = 1$.There have been many ways for finding solutions of $x^2 + y^2 = z^2$and one of the well-known methods is due to Euclid, BC 300. The investigation of integer solutions of $x^2 + y^2 = z^2$has been expanded to various aspects. One direction is to deal with polynomial $x^2 + y^2 = z^2 \pm 1$ which called Almost Pythagorean Triple (APT) or Nearly Pythagorean Triple (NPT) depending on the sign $\pm$.

Another side is to study the integer solution (a, b, c) of $x^2 + y^2 = z^2$ having some special conditions, such that when $a = b$ but in this case, there is no integer solution of $x^2 + y^2 = z^2$ so we can investigate the integer solution (a, b, c) with $|a - b| = 1$. We shall call this solution (a, b, c) is an Almost Isosceles Pythagorean Triple (AI-PT) and typical examples are (3,4,5) and (20,21,29).

In this work we generate infinitely many APTs and NPTs and then apply the results in order to develop algorithms for constructing infinitely many AI-PT. For more details, see [5]

2.1 Almost and Nearly Pythagorean Triples

At the beginning of this section we define an Almost and a Nearly Pythagorean Triples by the following definition.

Definition 2.1.1:

1. An integer solution (a, b, c) of $x^2 + y^2 = z^2 + 1$, is called Almost Pythagorean Triple (APT).
2. An integer solution (a, b, c) of $x^2 + y^2 = z^2 - 1$ is called Nearly Pythagorean Triple (NPT).

Example 2.1.2:

The triples (5,5,7), (4,7,8) and (8,9,12) are APT, because

$$5^2 + 5^2 = 7^2 + 1, \qquad 4^2 + 7^2 = 8^2 + 1$$

and

$$8^2 + 9^2 = 12^2 + 1$$

and the triples (10,50,51), (20,200,201) and (30,450,451) are NPT, because

$$10^2 + 50^2 = 51^2 - 1 \ , \ 20^2 + 202^2 = 201^2 - 1$$

and

$$30^2 + (450)^2 = (451)^2 - 1.$$

Now, we want to introduce some lemmas and theorems that studying APT and NPT and, giving infinitely many APTs and NPTs of many forms. For more details, see [5]

Lemma 2.1.3:

1. If (a, b, c) is an APT then $(2ac, 2bc, 2c^2 + 1)$ is a NPT.
2. If (a, b, c) is a NPT then $(2a^2 + 1, 2ab, 2ac)$ is an APT.

Proof:

Suppose (a, b, c) is an APT. To show that the triple $(2ac, 2bc, 2c^2 + 1)$ is a NPT we must show that

$$(2ac)^2 + (2bc)^2 = (2c^2 + 1)^2 - 1.$$

The left hand side is

$$(2ac)^2 + (2bc)^2 = 4a^2c^2 + 4b^2 c^2$$

$$= 4c^2 (a^2 + b^2),$$

but (a, b, c) is an APT implies

$$a^2 + b^2 = c^2 + 1$$

and so

$$4c^2 (a^2 + b^2) = 4c^2 (c^2 + 1)$$

$$= 4c^4 + 4c^2$$

$$= (4c^4 + 4c^2 + 1) - 1$$

$$= (2c^2 + 1)^2 - 1,$$

thus $(2ac, 2bc, 2c^2 + 1)$ is a NPT.

Now suppose that (a, b, c) is a NPT, by the same way, as above, we prove,

$$(2a^2 + 1)^2 + (2ab)^2 = (2ac)^2 + 1.$$

The left-hand side is

$$(2a^2 + 1)^2 + (2ab)^2 = 4a^4 + 4a^2 + 1 + 4a^2b^2$$

$$= (4a^4 + 4a^2 + 4a^2b^2) + 1$$

$$= 4a^2 (a^2 + 1 + b^2) + 1$$

$$= 4a^2 (a^2 + b^2 + 1) + 1.$$

But (a, b, c) is a NPT so $a^2 + b^2 = c^2 - 1$, thus

$$4a^2 (a^2 + b^2 + 1) + 1 = 4a^2 (c^2 - 1 + 1) + 1$$

$$= 4a^2c^2 + 1$$

hence $(2a^2 + 1, 2ab, 2ac)$ is an APT. ■

Theorem 2.1.4:

If a is an even integer, then we have the following:

(1) $(a, b, b+1)$ is an APT if $b = \frac{a^2}{2} - 1$ while it is a NPT if $b = a^2/2$.

(2) $(2a^2 + 1, a^3, a(a^2 + 2))$ is an APT and $\left(a^3, a^2\left(\frac{a^2}{2} - 1\right), \frac{a^4}{2} + 1\right)$ is a NPT.

Proof:

(1) a) Suppose a is an even integer and $b = \frac{a^2}{2} - 1$. To show that $(a, b, b+1)$ is an APT we must show that

$$a^2 + b^2 = (b+1)^2 + 1$$

but $b = \frac{a^2}{2} - 1$ implies $a^2 = 2b + 2$ and so

$$\begin{aligned} a^2 + b^2 &= 2b + 2 + b^2 \\ &= (b^2 + 2b + 1) + 1 \\ &= (b+1)^2 + 1. \end{aligned}$$

Therefore, $(a, b, b+1)$ is an APT.

b) Suppose $b = \frac{a^2}{2}$ then $a^2 = 2b$ and so

$$\begin{aligned} a^2 + b^2 &= 2b + b^2 \\ &= (b^2 + 2b + 1) - 1 \\ &= (b+1)^2 - 1. \end{aligned}$$

Hence $(a, b, b+1)$ is an NPT.

(2) We want to show that $(2a^2 + 1, a^3, a(a^2 + 2))$ is an APT and

$\left(\mathrm{a}^3, a^2\left(\frac{a^2}{2}-1\right), \frac{a^4}{2}+1\right)$ is an NPT.

To show that the triple $(2a^2 + 1, a^3, a(a^2+2))$ is an APT , we must show that

$$(2a^2+1)^2 + (a^3)^2 = \left(a(a^2+2)\right)^2 + 1.$$

But

$$\begin{aligned}(2a^2+1)^2 + (a^3)^2 &= 4a^4 + 4a^2 + 1 + a^6 \\ &= (4a^4 + 4a^2 + a^6) + 1 \\ &= a^2(a^4 + 4a^2 + 4) + 1 \\ &= a^2(a^2+2)^2 + 1 \\ &= (a(a^2+2))^2 + 1,\end{aligned}$$

So the triple $(2a^2 + 1, a^3, a(a^2+2))$ is an APT.

Now, we want to prove that the triple $(a^3, a^2\left(\frac{a^2}{2}-1\right), \frac{a^4}{2}+1)$ is an NPT. We must show that

$$(a^3)^2 + \left(a^2\left(\frac{a^2}{2}-1\right)\right)^2 = (\frac{a^4}{2}+1)^2 - 1.$$

But

$$\begin{aligned}(a^3)^2 + \left(a^2\left(\frac{a^2}{2}-1\right)\right)^2 &= a^6 + a^4\left(\frac{a^4}{4} - a^2 + 1\right) \\ &= \mathrm{a}^6 + \frac{a^8}{4} - a^6 + a^4 \\ &= \frac{a^8}{4} + a^4 \\ &= \left(\frac{a^8}{4} + a^4 + 1\right) - 1 \\ &= \left(\frac{a^4}{2} + 1\right)^2 - 1.\end{aligned}$$

Thus the triple $(\mathrm{a}^3, a^2\left(\frac{a^2}{2}-1\right), \frac{a^4}{2}+1)$ is a NPT.■

Table 2.1: Many APTs and NPTs of many forms when a is even.

$\boldsymbol{a}$	$\left(\boldsymbol{a},\frac{\boldsymbol{a}^2-2}{2},\frac{\boldsymbol{a}^2}{2}\right)$	$\left(\boldsymbol{a}^3,\frac{\boldsymbol{a}^2(\boldsymbol{a}^2-2)}{2},\frac{\boldsymbol{a}^4+2}{2}\right)$	$\left(\boldsymbol{a},\frac{\boldsymbol{a}^2}{2},\frac{\boldsymbol{a}^2+2}{2}\right)$	$\left(2\boldsymbol{a}^2+1,\boldsymbol{a}^3,\boldsymbol{a}(\boldsymbol{a}^2+2)\right)$
2	(2,1,2)	(8,4,9)	(2,2,3)	(9,8,12)
4	(4,7,8)	(64,112,129)	(4,8,9)	(33,64,72)
6	(6,17,18)	(216,612,649)	(6,18,19)	(73,216,228)
8	(8,31,32)	(320,1984,2049)	(8.32.33)	(129,320,528)
10	(10,49,50)	(1000,4900,5001)	(10,50,51)	(201,1000,1020)

We note that the previous theorem gives infinitely many APTs and NPTs (a, b, c) such that $c - b = 1$.

The following theorem generates APT and NPT $(a, b, c,)$ with $c - b = 5$.

Theorem 2.1.5:

(1) If $a \equiv \pm 2 \ (mod\ 10)$ and $b = \frac{a^2-24}{10}$,then $(a, b, b + 5)$ is a NPT

(2) If $a \equiv \pm 4 \ (mod\ 10)\ and\ b = \frac{a^2-26}{10}$,then $(a, b, b + 5)$ is an APT.

Proof:

(1) Suppose $b = \frac{a^2-24}{10}$, then $a^2 = 10b + 24$.

Now,

$$a^2 + b^2 = 10b + 24 + b^2$$
$$= (b^2 + 10b + 25) - 1$$
$$= (b + 5)^2 - 1$$

Hence $(a, b, b + 5)$ is an NPT.■

(2) Suppose $b = \frac{a^2-26}{10}$, then $a^2 = 10b + 26$,

Now,

$$a^2 + b^2 = 10b + 26 + b^2$$

$$= (b^2 + 10b + 25) + 1$$

$$= (b + 5)^2 + 1$$

Hence $(a, b, b + 5)$ is an APT.■

Table 2.2: Many APTs and NPTs of the form $(a, b, b + 5)$

a	**NPT** $\left(\mathbf{a}, \frac{\mathbf{a^2-24}}{\mathbf{10}}, \frac{\mathbf{a^2+26}}{\mathbf{10}}\right)$	**a**	**APT** $\left(\mathbf{a}, \frac{\mathbf{a^2-26}}{\mathbf{10}}, \frac{\mathbf{a^2+24}}{\mathbf{10}}\right)$
8	(8,4,9)	14	(14,17,22)
12	(12,12,17)	16	(16,23,28)
18	(18,30,35)	24	(24,55,60)

Theorem 2.1.5 together with lemma 2.1.3 yields infinitely many NPTs and APTs (see table 2) though there are APT and NPT $(a, b, b + k)$ with $k = 1, 5$.

Example:2.1.6

Show that there is no NPT $(a, b, b + k)$ exist if $k = 2$ or 3 .

Solution: suppose that $(a, b, b + 2)$ is a NPT, then

$$a^2 + b^2 = (b + 2)^2 - 1$$

$$a^2 + b^2 = b^2 + 4b + 4 - 1$$

$$a^2 = 4b + 3,$$

so $a^2 \equiv 3\ (mod 4)$. But by Theorem 1.1.8, $a^2 \equiv 0\ (mod\ 4)$ or $a^2 \equiv 1\ (mod\ 4)$, then no solution *a* exists and therefore if $k = 2$ then no NPT $(a, b, b + k)$ exists.

Now if $k = 3$, suppose that $(a, b, b + 3)$ is a NPT, then

$$a^2 + b^2 = (b + 3)^2 - 1$$

$$a^2 + b^2 = b^2 + 6b + 9 - 1$$

$$a^2 = 6b + 8,$$

thus $a^2 \equiv 2\ (mod\ 6)$. But, in general, if m is in integer, then m^2 have one of the following cases:

$$m^2 \equiv 0\ (mod\ 6)\ \ or\ m^2 \equiv 1\ (mod\ 6)$$

$$or\ m^2 \equiv 3\ (mod\ 6)\ or\ m^2 \equiv 4\ (mod\ 6)$$

so, $a^2 \equiv 2 \ (mod\ 6)$, has no integer solution. Thus, no NPT of the form $(a, b, b + 3)$ exists.

Example:2.1.7

Show that there are APTs $(a, b, b + k)$ if $k = 2 , 3$.

Solution: If $(a, b, b + 2)$ is an APT, then

$$a^2 + b^2 = (b + 2)^2 + 1$$
$$a^2 + b^2 = b^2 + 4b + 4 + 1$$
$$a^2 = 4b + 5,$$

and so $a^2 \equiv 1 \ (mod\ 4)$. But by Theorem 1.1.8, $a^2 \equiv 1 \ (mod\ 4)$ has a solution and so there are APTs of the form $(a, b, b + 2)$. For example, (5,5,7), (7, 11, 13), …

If $(a, b, b + 3)$ is an APT, then

$$a^2 + b^2 = (b + 3)^2 + 1$$
$$a^2 + b^2 = b^2 + 6b + 9 + 1$$
$$a^2 = 6b + 10.$$

Thus $a^2 \equiv 4 \ (mod 6)$, but $a^2 \equiv 4 \ (mod\ 6)$ has a solution as in the above discussion. For example, (8, 9, 12), (10, 15, 18) are APT. Therefore, we conclude that if $k = 2$ or 3 there are APT, but no NPT exists. To generalize this, we give the following theorem.

Theorem 2.1.8:

For any $k > 0$, the APTs of the form $(a, b, b + k)$ always exists. If $k - 1$ is even and square then there exist NPTs of the form $(a, b, b + k)$.

Proof:

A triple $(a, b, b + k)$ is an APT, if

$$a^2 + b^2 = (b + k)^2 + 1$$
$$a^2 + b^2 = b^2 + 2bk + k^2 + 1$$
$$a^2 = 2bk + k^2 + 1$$

That is,

$$b = \frac{a^2 - k^2 - 1}{2k}$$

Then

$$a^2 \equiv k^2 + 1 \equiv (k \pm 1)^2 \qquad (\,mod\ 2k)$$

and so,

$$a^2 \equiv (k \pm 1)^2 \qquad (mod\ 2k).$$

Take

$$a = 2mk \pm (k \pm 1)\text{ , for } m \in Z$$

then,

$$b = \frac{a^2 - k^2 - 1}{2k},$$

then

$$b = \frac{(2mk + (k \pm 1))^2 - k^2 - 1}{2k}$$

$$b = \frac{4m^2k^2 + 4mk\,(k \pm 1) + (k \pm 1)^2 - k^2 - 1}{2k}$$

$$= \frac{4m^2k^2 + 4mk^2 \pm 4mk + k^2 \pm 2k + 1 - k^2 - 1}{2k}$$

$$= 2m^2k + 2km \pm 2m \pm 1$$

$$= 2m\,(mk \pm (k \pm 1) + 1 \qquad for\ m \in Z.$$

Now we can prove that if

$$a = 2mk \pm (k \pm 1)\text{ , for } m \in Z$$

and

$$b = 2m\,(mk \pm (k \pm 1) + 1,\text{for } m \in Z,$$

then the triple $(a, b, b + k)$ is an APT. This can be done by showing the left hand side and the right hand side of

$$a^2 + b^2 = (b + k)^2 + 1$$

are the same.

The left hand side is

$$a^2 + b^2 = (2mk \pm (k \pm 1))^2 + (2m\,(mk \pm (k \pm 1) + 1\,)^2$$

$$= 4m^2k^2 \pm 4mk\,(k \pm 1) + (k \pm 1)^2$$

$$+ 4m^2(mk \pm (k \pm 1) + 1)^2$$
$$= 4m^2k^2 \pm 4mk\,(k \pm 1) + (k \pm 1)^2$$
$$+ 4m^2\big((mk + 1) \pm (k \pm 1)\big)^2$$
$$= 4m^2k^2 \pm 4mk^2 \pm 4mk + k^2 \pm 2k + 1$$
$$+ 4m^2((mk + 1)^2 \pm 2(mk + 1)(k \pm 1) + (k \pm 1)^2)$$
$$= 4m^2k^2 \pm 4mk^2 \pm 4mk + k^2 \pm 2k + 1$$
$$+4m^4k^2 \pm 8m^3k^2 \pm 8m^3k \pm 4m^2k^2 \pm 8m^2k$$
$$+4m^2 \pm 4m^2k \pm 4mk \pm 4m + 1$$
$$= 4m^4k^2 \pm 8m^3k^2 \pm 8m^3k \pm 4m^2k^2 \pm 12m^2k$$
$$\pm 4mk^2 \pm 8mk \pm 4m^2 \pm 4m + k^2 + 2k + 2.$$

Now, the right hand side is

$$(b + k)^2 + 1 = (2m(mk \pm k \pm 1) + 1 + k)^2 + 1$$
$$= (2m^2k \pm 2mk \pm 2m + 1 + k)^2 + 1$$
$$= 4m^4k^2 \pm 4m^3k^2 \pm 4m^3k + 2m^2k + 2m^2k^2$$
$$\pm 4m^3k^2 \pm 4m^2k^2 \pm 4m^2k \pm 2mk$$
$$\pm 2mk^2 \pm 4m^3k \pm 4m^2k \pm 4m^2$$
$$\pm 2m \pm 2mk \pm 4m^2k \pm 2mk$$
$$\pm 2m + 1 + k + 2\,m^2k^2 \pm 2mk^2$$
$$\pm 2mk + k + k^2 + 1$$
$$= 4m^4k^2 \pm 8m^3k^2 \pm 8m^3k \pm 4m^2k^2 \pm 12m^2k$$
$$\pm 4mk^2 \pm 8mk \pm 4m^2 \pm 4m + k^2 + 2k + 2,$$

hence, $a^2 + b^2 = (b + k)^2 + 1$ and so $(a, b, b + k)$ is an APT for all $k > 0$.
In particular $(k + 1, 1, k + 1)$ is an APT for all $k > 0$.

(2) If $k - 1$ is even and square then there exist NPT of the form $(a, b, b + k)$.
To show that, let $k - 1 = 2v = u^2$, where $u, v \in N$.

For $(a, b, b + k)$ to be a NPT, we must have $a^2 + b^2 = (b + k)^2 - 1$

$$a^2 + b^2 = b^2 + 2bk + k^2 - 1$$
$$a^2 = 2bk + k^2 - 1\,,$$

that is

$$a^2 \equiv k^2 - 1 \qquad (mod\ 2k)$$

$$b = \frac{a^2 - k^2 + 1}{2k}.$$

Since $k - 1 = 2v = u^2$, then

$$\begin{aligned} a^2 &\equiv (2v + 1)^2 - 1 && (mod\ 2\ (2v + 1)) \\ &\equiv 4v^2 + 4v + 1 - 1 && (mod\ 4v + 2) \\ &\equiv 4v^2 + 4v && (mod\ 4v + 2) \\ &\equiv 4v^2 + 2v + 2v && (mod\ 4v + 2) \\ &\equiv v\,(4v + 2) + 2v && (mod\ 4v + 2) \\ &\equiv 2v = u^2 && (mod\ 2u^2 + 2) \end{aligned}$$

then

$$a^2 \equiv u^2 \qquad (mod\ 2u^2 + 2)$$

therefore,

$$a^2 = (2u^2 + 2)m + u^2, \qquad \text{for } m \in Z,$$

and

$$\begin{aligned} b &= \frac{2m(u^2 + 1) + u^2 - (u^2 + 1)^2 + 1}{2\,(u^2 + 1)} \\ &= \frac{2mu^2 + 2m + u^2 - u^4 - 2u^2 - 1 + 1}{2\,(u^2 + 1)} \\ &= \frac{2mu^2 + 2m - u^2 - u^4}{2\,(u^2 + 1)} \\ &= \frac{2m(u^2 + 1) - u^2(u^2 + 1)}{2\,(u^2 + 1)} \\ &= \frac{2m\,u^2 + 1)}{2\,(u^2 + 1)} - \frac{u^2(\,u^2 + 1)}{2\,(u^2 + 1)} \\ &= m - \frac{u^2}{2}, \end{aligned}$$

also,

$$\begin{aligned} b + k &= m - \frac{u^2}{2} + u^2 + 1 \\ &= m + \frac{u^2}{2} + 1. \end{aligned}$$

Now we can prove that if

$$a^2 = 2km + k - 1, \qquad \text{for some } m \in Z$$

and

$$b = m - \frac{k-1}{2}, \qquad m \in Z,$$

then the triple $(a, b, b+k)$ is an NPT. This can be done by showing the left hand side and the right hand side of

$$a^2 + b^2 = (b+k)^2 - 1,$$

thus, using $k - 1 = u^2$ we get,

$$(b+k)^2 - b^2 - 1 = ((m+1) + \frac{u^2}{2})^2 - (m - \frac{u^2}{2})^2 - 1$$

$$= (m+1)^2 + (m+1)u^2 + \frac{u^4}{4} - \left[m^2 - mu^2 + \frac{u^4}{4}\right] - 1$$

$$= m^2 + 2m + 1 + mu^2 + u^2 + \frac{u^4}{4} - m^2 + mu^2 - \frac{u^4}{4} - 1$$

$$= 2m + 2mu^2 + u^2$$

$$= a^2.$$

So, $(b+k)^2 - b^2 - 1 = a^2$ and $a^2 + b^2 = (b+k)^2 - 1$, so the triple $(a, b, b+k)$ is an NPT.■

Examples 2.1.9:

(1) The triples (31,43,53), (51,125,135) are APTs with $k = 10$.

(2) Similarly (34, 47, 58), (56, 137, 148) are APTs with $k = 11$.

So, we have infinitely many APTs (a, b, c) such that k is any integer.

On the other hand, let $k = 1, 5, 17, 37$. Since $k - 1$ even and square then Theorem 2.2.6 yields NPT $(a, b, b+k)$ satisfying $a^2 = 2km + k^2 - 1$ and $b = (a^2 - k^2 + 1) / 2k$

The following table contains some examples on NPT $(a, b, b+k)$, when $k =$ 1,5,17 and 37:

Table 2.3:The NPTs $(a, b, b+k)$, when $k = 1$,5,17 and 37.

$\boldsymbol{k}$	$\boldsymbol{a^2 \equiv k^2 - 1 (mod 2k)}$	$\boldsymbol{a} > k$	$\boldsymbol{b}$	$(\boldsymbol{a, b, b+k})$ **NPT**
1	$a^2 \equiv 0 \pmod{2}$	2		(2,2,3)
		4		(4,8,9)
		6		(6,18,19)
5	$a^2 \equiv 4 \pmod{10}$	8	4	(8,4,9)

		12	12	(12,12,17)
		18	30	(18,30,35)
17	$a^2 \equiv 16 \pmod{34}$	30	18	(30,18,35)
		38	34	(38,34,51)
		64	112	(64,112,129)
37	$a^2 \equiv 36 \pmod{74}$	68	44	68,44,81)
		80	68	(80,68,105)
		142	254	(142,254,291)

Corollary 2.1.10:

Let $n \equiv 0 \ (mod\ 10)$. If $a = n + 10\,k$ and $b = \frac{n^2}{2} + 10\,k\,(n + 5\,k)$ for any $k \geq 0$ then $(a, b, b+1)$ is a NPT.

Proof:

Suppose $a = n + 10\,k$ and $b = \frac{n^2}{2} + 10\,k\,(n + 5k)$, then

$$a^2 = (n + 10\,k)^2$$

Thus,

$$a^2 = n^2 + 20\,nk + 100\,k^2$$

and,

$$b^2 = \frac{n^4}{4} + 100\,k^2\,(n + 5\,k)^2 + 10\,n^2\,k\,(n + 5\,k)$$

$$= \frac{n^4}{4} + 100\,k^2\,(n^2 + 10\,nk + 25\,k^2)$$

$$+ 10\,n^3\,k + 50\,n^2\,k^2$$

$$= \frac{n^4}{4} + 100\,k^2\,n^2 + 1000\,nk^3 + 2500\,k^4$$

$$+ 10\,n^3\,k + 50\,n^2\,k^2.$$

Now,

$$a^2 + b^2 = n^2 + 20\,nk + 100\,k^2 + \frac{n^4}{4}$$

$$+ 100\,k^2 n^2 + 1000\,nk^3 + 2500\,k^4$$

$$+ 10\,n^3\,k + 50\,n^2\,k^2\,,$$

also, $(b+1)^2 - 1$ is equal $(b + 1)^2 - 1 = (\frac{n^2}{2} + 10\,k\,(n + 5\,k) + 1)^2 - 1$

$$= \left(\frac{n^2}{2} + 1\right)^2 + 100\,k^2\,(n + 5k)^2 + 2\,(\frac{n^2}{2} + 1)\,10\,k\,(n + 5k) - 1$$

$$= \frac{n^4}{4} + n^2 + 1 + 100\,k^2\,(n + 5k)^2 + (n^2 + 2)\,(10\,kn + 50k^2) - 1$$

$$= \frac{n^4}{4} + n^2 + 100\,k^2(n + 5k)^2 + 10\,kn^3 + 50\,n^2\,k^2 + 20nk + 100\,k^2$$

$$= \frac{n^4}{4} + n^2 + 100\,k^2\,(n^2 + 10\,nk + 25k^2) + 10\,kn^3 + 50\,n^2k^2 + 20\,kn$$

$$+ 100\,k^2$$

$$= \frac{n^4}{4} + n^2 + 100\,k^2\,n^2 + 1000\,nk^3 + 2500\,k^4 + 10\,kn^3 + 50\,n^2k^2$$

$$+ 20\,kn + 100\,k^2$$

Therefore, $a^2 + b^2 = (b+1)^2 - 1$ .So $(a, b, b+1)$ is a NPT. ■

Examples 2.1.11:

If we let $n = 10$, then $a = 10 + 10(1) = 20$ $b = \frac{100}{2} + 10(1)(10 + 5) = 200$ and $c = 201$, so the NPT is $(20, 200, 201)$.

The following table contains some examples are applicable on previous theorem:

Table 2.4: When $a = n + 10\,k$ and $b = \frac{n^2}{2} + 10\,k\,(n + 5\,k)$,then $(a, b, b+1)$ is a NPT.

n	$a = n + 10$	$b = \frac{n^2}{2} + 10(n+5)$	$(a, b, b+1)$
10	20	200	$(20, 200, 201)$
20	30	450	(30,450,451)
30	40	800	(40,800,801)
40	50	1700	(50,1700,1701)
50	60	2350	(60,2350,2351)

Now we discuss another way to construct NPTs from PPT as in the following theorem.

Theorem 2.1.12:

Let (x, y, z) be a PPT. Then there are many NPT (a, b, c) with $c - b = z$.

Proof:

Let (x, y, z) be PPT, then by Theorem (1.1.11), we can write $x = k^2 - l^2$, $y = 2kl$, $z = k^2 + l^2$, where $k > l > 0$, $\gcd(k, l) = 1$ and $k \not\equiv l \ (mod\ 2)$. Since $k \not\equiv l\ (mod\ 2)$, then not both *k* and *l* even or not both odd, so suppose *k* is even and *l* is odd.

Let $k = 2r$ and $l = 2s + 1$ such that $r, s \in N$. Since z is a sum of squares then

$z = k^2 + l^2 \equiv 1\ (mod\ 4)$ and *z* is odd. The triple (a, b, c) is a NPT with $c - b = z$ if

$$a^2 + b^2 = (b + z)^2 - 1$$

Thus, $$a^2 + b^2 = b^2 + 2bz + z^2 - 1$$

$$a^2 = 2bz + z^2 - 1$$

And $$b = \frac{a^2 - z^2 + 1}{2z}$$

Since $a^2 = 2bz + z^2 - 1$, then we have

$$a^2 \equiv z^2 - 1 \pmod{2z}$$

$$a^2 \equiv z^2 - 1 \pmod{2}$$

$$a^2 \equiv -1 \quad (\text{mod} z)$$

$$a^2 \equiv 0 \quad (\text{mod}2).$$

Now, to prove that there exist a NPT of the form $(a, b, b + z)$ we must prove that $a^2 \equiv -1\ (mod\ z)$ has integer solution, where z is prime. Now, if z is a prime then $a^2 \equiv -1\ (mod\ z)$ has integers solutions since $z \equiv 1\ (mod\ 4)$. So, with $b = \frac{a^2-z^2+1}{2z}$, there exists a NPT of the form $(a, b, b + z)$.

On the other hand, if we write z as a product of prime as$z = p, \dots \dots . p_j$, where $(p_j$odd primes and $1 \le i \le j)$, then $z \equiv 1\ (mod\ 4)$ implies that either every $p_i \equiv 1(mod4)$ or, there are even number of p_i such that $p_i \equiv -1\ \ (mod4)$ for $1 \le i \le j$. Thus, in [2] they proved that if p is an odd prime, then the Legendre symbol $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$ So, the Legendre symbol $\left(\frac{-1}{z}\right) = \left(\frac{-1}{p_i}\right) \dots \dots \dots \dots \left(\frac{-1}{p_j}\right) = 1$, Thus -1 is a quadratic residue modulo p so $a^2 \equiv -1\ (mod\ z)$ has integer solution. Hence there is a NPT $(a, b, b + z)$. ■

Example 2.1.13:

Let (3, 4, 5) be a PPT, what are the NPTs that satisfies with $c - b = 5$.

Solution:

Let $z = c - b = 5$, by previous theorem $a^2 \equiv -1$ (mod5) has an integer solution. Since $a^2 > z^2 = 25$, take $a = 8$, then $b = \frac{64-24}{10} = 4$, and $c = b + 5 = 4 + 5 = 9$, then the triple (8,4,9) is a NPT.

Also, take $a = 12$, then $b = 12$ and $c = 17$. Thus the triple (12,12,17) is a NPT. The PPT (x, y, z) with $z \le 40$ are (3, 4, 5), (5, 12, 13), (8, 15, 17), (7, 24, 25), (20, 21, 29) and (12, 35, 37).

If $z = 5, 17, 37$ then table 2.3 contains the list of NPTs. When $z = 13, 25, 29$, NPTs are shown in table 2.5**:**

Table 2.5:The NPTs $(a, b, b+z)$ when $z = 13, 25, 29$.

z	$a^2 = z^2 - 1(mod2z)$	$a(mod2z)$	$a > z$	b	$(a, b, b+z)$ **NPT**
13	$a^2 \equiv 168 \equiv 64(mod26)$	± 8	18	6	(18,6,19)
			34	38	(34,38,51)
			44	68	(44,68,81)
25	$a^2 \equiv 624 \equiv 324(mod50)$	± 18	32	8	(32,8,33)
			68	80	(68,80,105)
			82	122	(82,122,147)
29	$a^2 \equiv 840 \equiv 144(mod58)$	± 12	46	22	(46,22,51)
			70	70	(70,70,99)
			128	268	(128,268,297)

Now we introduce the definition of isosceles APT (iso-APT) and isosceles NPT (iso-NPT) For more details see. [5]

Definition 2.1.14:

The almost Pythagorean triple (a, b, c) is called iso-APT if $a = b$ and also the nearly Pythagorean triple (a, b, c) is called iso-NPT if $a = b$.

Though there is no isosceles PT, there are many iso-APTs and iso-NPTs. Indeed iso-APT and iso-NPT as (a, a, c) which satisfy

$$a^2 + a^2 = c^2 \pm 1,$$

so the pair (a, c) is an integer solution of

$$2x^2 - y^2 = \pm 1.$$

Which is the pell polynomial.

Let (a_1, c_1) , (a_2, c_2) be two integers solutions of

$$2x^2 - y^2 = 1 \text{ or } 2x^2 - y^2 = -1,$$

then

$$-1 = (2a_1^2 - c_1^2)\,(2a_2^2 - c_2^2)$$

$$= 4a_1^2a_2^2 - 2a_1^2c_2^2 - 2a_2^2c_1^2 + c_1^2c_2^2.$$

Add and subtract $4a_1a_2c_1c_2$ we get

$$-1 = 4a_1^2a_2^2 - 2a_1^2c_2^2 - 2a_2^2c_1^2 + c_1^2c_2^2 - 4a_1a_2c_1c_2 + 4a_1a_2c_1c_2$$

$$= -2a_1^2c_2^2 - 2a_2^2c_1^2 - 4a_1a_2c_1c_2 + 4a_1^2a_2^2 + c_1^2c_2^2 + 4a_1a_2c_1c_2$$

$$= -2(a_1^2c_2^2 + a_2^2c_1^2 + 2a_1a_2c_1c_2) + 4a_1^2a_2^2 + c_1^2c_2^2 + 4a_1a_2c_1c_2$$

$$= -2(a_1c_2+a_2c_1)^2 + (2a_1a_2+c_1c_2)^2$$

thus, let $x = a_1c_2+a_2c_1$ and $y = 2a_1a_2+c_1c_2$ then (x, y) satisfies $2x^2 - y^2 = -1$.

Now if we define the product of (a_1, c_1) and (a_2, c_2) as the above discussion, then this product satisfied the equation $2x^2 - y^2 = -1$.

Definition 2.1.15:

Let (a_1, c_1) and (a_2, c_2) be two pairs of integers. then

$$(a_1, c_1)(a_2, c_2) = (a_1c_2+a_2c_1, 2a_1a_2+c_1c_2).$$

Example 2.1.16:

Consider the $2x^2 - y^2 = -1$, clearly, (2,3) is a root of this equation. By above discussion then (2,3) . (2,3) is also a root of the equation. But (2,3) . (2,3) = (12,17) and

(5,7) is a root of $2x^2 - y^2 = 1$, then (5,7). (5,7) = (70,99) satisfies $2x^2 - y^2 = -1$.

So the first few nonnegative solutions of $2x^2 - y^2 = \pm 1$ are

$$\{ (0,0)_- , (1,1)_+ , (2,3)_- , (5,7)_+ , (12,17)_- ,$$
$$(29,41)_+ , (70,99)_- , (169, 239)_+ , \ldots \ldots \quad \ldots \ldots \ldots . . \}$$

where the subscripts $+$, $-$ indicate solutions of $2x^2 - y^2 = +1$ and $2x^2 - y^2 = -1$ respectively.

Theorem 2.1.17:

Let $S_n = (a_n, c_n)$ for $S_{n+1} = 2S_n + S_{n-1}$ with $S_0 = (0,1)$, $S_1 = (1,1)$ then the following hold:

(1) $a_{n+1} = a_n + c_n$ and $c_{n+1} = a_{n+1} + a_n$ and $2a_n a_{n-1} - c_n c_{n-1} = (-1)^n$ so $S = \{S_n\}, n \geq 0$ is a sequence of solutions $2x^2 - y^2 = (-1)^{n+1}$

(2) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ then $S_n = S_{n-1} A = S_0 A^n$ by considering S_n as a matrix.

(3) Let S_+, S_- be subset of S consisting of S_{n+}, S_{n-}, respectively.

If $S_n \in S_\pm$ then $S_{n+1} \in S_\mp$ and $S_{n+2} \in S_\pm$.

Proof:

Since $S_{n+1} = 2S_n + S_{n-1}$,then

$$(a_{n+1}, c_{n+1}) = (2a_n + a_{n-1}, 2c_n + c_{n-1})$$

Claim $c_{n+1} = a_{n+1} + a_n$, by Using mathematical induction,

if $n = 1$, then

$$c_2 = a_2 + a_1 = 2 + 1 = 3 \text{ which is the same as } c_2 = 2c_1 + c_0.$$

Now suppose the claim is true for n , then

$$c_n = a_n + a_{n-1},$$

to prove the claim is true for $n + 1$, let

$$\begin{aligned} c_{n+1} &= 2c_n + c_{n-1} \\ &= 2(a_n + a_{n-1}) + (a_{n-1} + a_{n-2}) \\ &= 2a_n + 2a_{n-1} + a_{n-1} + a_{n-2} \\ &= 2a_n + a_{n-1} + 2a_{n-1} + a_{n-2} \\ &= a_{n+1} + a_n. \end{aligned}$$

By similar way we prove that $a_{n+1} = a_n + c_n$.

1) for $n = 1$, then

$$a_2 = a_1 + c_1 = 1 + 1 = 2 \quad \text{which is true}$$

2) suppose is true for n , i.e

$$a_n = a_{n-1} + c_{n-1}$$

3) To prove that for $n+1$, such that $a_{n+1} = a_n + a_n$, then

$$\begin{aligned} a_{n+1} &= 2a_n + a_{n-1} \\ &= 2(a_{n-1} + c_{n-1}) + (a_{n-2} + c_{n-2}) \\ &= 2a_{n-1} + 2c_{n-1} + a_{n-2} + c_{n-2} \\ &= 2a_{n-1} + a_{n-2} + 2c_{n-1} + c_{n-2} \\ &= a_n + c_n, \end{aligned}$$

clearly $S_n = (a_n, c_n)$ is a solutions of $2x^2 - y^2 = (-1)^{n+1}$. so $2a_n^2 - c_n^2 = (-1)^{n+1}$.

Want to prove

$$2a_n a_{n-1} - c_n c_{n-1} = (-1)^n \qquad (2.1)$$

By induction, (2.1) is true for $n = 1$ $\left(\text{since } 2a_1 a_0 - c_1 c_0 = -1 = (-1)^1\right)$, now suppose (2.1) is true for $n-1$, i.e $2a_{n-1}a_{n-2} - c_{n-1}c_{n-2} = (-1)^{n-1}$ is true.

To prove (2.1) for n, consider

$$\begin{aligned} 2a_{n+1}^2 - c_{n+1}^2 &= 2(2a_n + a_{n-1})^2 - (2c_n + c_{n-1})^2 \\ &= 2(4a_n^2 + 4a_n a_{n-1} + a_{n-1}^2) - (4c_n^2 + 4c_n c_{n-1} + c_{n-1}^2) \\ &= 8a_n^2 + 8a_n a_{n-1} + 2a_{n-1}^2 - 4c_n^2 - 4c_n c_{n-1} - c_{n-1}^2 \\ &= 4(2a_n^2 - c_n^2) + (2a_{n-1}^2 - c_{n-1}^2) + 4(2a_n a_{n-1} - c_n c_{n-1}) \\ &= 4(-1)^{n+1} + (-1)^n + 4(2a_n a_{n-1} - c_n c_{n-1}) \end{aligned}$$

Thus

$$(-1)^{n+2} = 2a_{n+1}^2 - c_{n+1}^2 = 4(-1)^{n+1} + (-1)^n + 4(2a_n a_{n-1} - c_n c_{n-1})$$

So

$$(-1)^{n+2} = 4(-1)^{n+1} + (-1)^n + 4(2a_n a_{n-1} - c_n c_{n-1})$$

Thus

$$4(2a_n a_{n-1} - c_n c_{n-1}) = ((-1)^{n+2} - (-1)^n - 4(-1)^{n+1})$$

$$= -4(-1)^{n+1}$$

$$2a_n a_{n-1} - c_n c_{n-1} = -(-1)^{n+1} = (-1)^{n+2} = (-1)^n.$$

Proof: (2) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $S_0 = [0,1]$, $S_1 = [1,1]$,

Now

$$S_0\, A = [0 \;\; 1] \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = [1 \;\; 1] = S_1$$

and

$$S_1\, A = [1 \;\; 1] \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = [2 \;\; 3] = S_2 = S_0\, A^2$$

So if assume $S_{n-1}\, A = S_n = S_0\, A^n$,then

$$S_0\, A^{n+1} = S_n\, A$$

$$= [a_n \;\; c_n] \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = [a_n + c_n, 2a_n + c_n]$$

$$= (a_{n+1} \;\; c_{n+1}) = S_{n+1}.$$

Hence, $S_0\, A^{n+1} = S_{n+1}$. Moreover for $S_n = [a_n \;\; c_n]$ we have

$$S_{n+1} = [a_n + c_n \;\; 2a_n + c_n]$$

satisfies

$$2(a_n + c_n)^2 - (2a_n + c_n)^2 = -(2a_n^2 - c_n^2).$$

Similarly from $S_{n+2} = S_{n+1}\, A = [a_n + c_n \;\; 2a_n + c_n] \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$= [a_n + c_n + 2a_n + c_n \quad 2(a_n + c_n) + 2\, a_n + c_n]$$

$$= [3a_n + 2c_n , 4a_n + 3c_n]$$

We have

$$2\,(3a_n + 2c_n)^2 - (4a_n + 3c_n)^2 = 2a_n^2 - c_n^2$$

thus if $S_n \in S_-$ then $S_{n+1} \in S_+$ and $S_{n+2} \in S_-$. This completes the proof. ■

Next, we introduce the definition of the multiplication of iso-NPT or iso-APT by

$$(a_1, a_1, c_1)(a_2, a_2, c_2) = (a_1c_2 + a_2c_1\,, a_1c_2 + a_2c_1, 2a_1a_2 + c_1c_2).$$

Theorem 2.2.18:

Let $X_1 = (a_1, a_1, c_1)$ and $X_2 = (a_2, a_2, c_2)$,

(1) If X_1 and X_2 are iso-NPT, then $\mathrm{X}_1.\mathrm{X}_2$ is also an iso-NPT.

(2) If X_1 and X_2 are iso-APT, then $\mathrm{X}_1.\mathrm{X}_2$ is an iso-NPT.

(3) If X_1 is an iso-APT (or iso-NPT) and X_2 is an iso-NPT (or iso-APT) then $\mathrm{X}_1.\mathrm{X}_2$ is an iso-APT.

Proof:

(1) Since $\mathrm{X}_1$and X_2 are iso-NPT, then

$$2\mathrm{a}_1^2 = \mathrm{c}_1^2 - 1 \quad \text{and} \quad 2\mathrm{a}_2^2 = \mathrm{c}_2^2 - 1$$

Since

$$\mathrm{X}_1.\mathrm{X}_2 = (a_1c_2 + a_2c_1\,, a_1c_2 + a_2c_1, 2a_1a_2 + c_1c_2).$$

Then to show that $\mathrm{X}_1.\mathrm{X}_2$ is an iso-NPT we must prove that

$$2\,(\mathrm{a}_1\mathrm{c}_2 + \mathrm{a}_2\mathrm{c}_1)^2 = (2\mathrm{a}_1\mathrm{a}_2 + \mathrm{c}_1\mathrm{c}_2)^2 - 1$$

The LHS is

$$2\,(a_1c_2 + a_2c_1)^2 = 2a_1^2c_2^2 + 2a_2^2c_1^2 + 4a_1a_2c_1c_2$$

But

$$2a_1^2 = c_1^2 - 1 \quad \text{and} \quad 2a_2^2 = c_2^2 - 1$$

Thus

$$\begin{aligned} 2a_1^2c_2^2 + 2a_2^2c_1^2 + 4a_1a_2c_1c_2 &= (c_1^2 - 1)\,c_2^2 + (c_2^2 - 1)\,c_1^2 + 4a_1a_2c_1c_2 \\ &= c_1^2c_2^2 - c_2^2 + c_1^2c_2^2 - c_1^2 + 4a_1a_2c_1c_2 \\ &= 2\,c_1^2c_2^2 - c_1^2 - c_2^2 + 4a_1a_2c_1c_2 \end{aligned}$$

Also, the RHS is

$$(2a_1a_2 + c_1c_2)^2 - 1 = 4a_1^2a_2^2 + c_1^2c_2^2 + 4a_1a_2c_1c_2 - 1$$
$$= 4\left(\frac{c_1^2-1}{2}\right)\left(\frac{c_2^2-1}{2}\right) + c_1^2c_2^2 + 4a_1a_2c_1c_2 - 1$$
$$= c_1^2c_2^2 - c_1^2 - c_2^2 + 1 + c_1^2c_2^2 + 4a_1a_2c_1c_2 - 1$$
$$= c_1^2c_2^2 - c_1^2 - c_2^2 + +4a_1a_2c_1c_2$$

So LHS is the same as RHS. Hence $X_1.X_2$is an iso-NPT. ■

(2) If X_1 and X_2 are iso-APT then

$$2a_1^2 = c_1^2 + 1 \quad \text{and} \quad 2a_2^2 = c_2^2 + 1$$

Since

$$X_1.X_2 = (a_1c_2 + a_2c_1, a_1c_2 + a_2c_1, 2a_1a_2 + c_1c_2).$$

To show that $X_1.X_2$is iso-NPT we must prove that

$$2(a_1c_2 + a_2c_1)^2 = (2a_1a_2 + c_1c_2)^2 - 1.$$

The LHS is

$$2(a_1c_2 + a_2c_1)^2 = 2a_1^2c_2^2 + 2a_2^2c_1^2 + 4a_1a_2c_1c_2,$$

but we have

$$2a_1^2 = c_1^2 + 1 \quad \text{and} \quad 2a_2^2 = c_2^2 + 1,$$

thus

$$2a_1^2c_2^2 + 2a_2^2c_1^2 + 4a_1a_2c_1c_2 = (c_1^2 + 1)\,c_2^2 + (c_2^2 + 1)\,c_1^2 + 4a_1a_2c_1c_2$$
$$= c_1^2\,c_2^2 + c_2^2 + c_1^2\,c_2^2 + c_1^2 + 4a_1a_2c_1c_2$$
$$= 2\,c_1^2\,c_2^2 + c_1^2 + c_2^2 + 4a_1a_2c_1c_2.$$

The RHS is

$$(2a_1a_2 + c_1c_2)^2 - 1 = 4a_1^2\,a_2^2 + c_1^2\,c_2^2 + 4a_1a_2c_1c_2 - 1$$
$$= 4\left(\frac{c_1^2+1}{2}\right)\left(\frac{c_2^2+1}{2}\right) + c_1^2\,c_2^2 + 4a_1a_2c_1c_2 - 1$$
$$= c_1^2\,c_2^2 + c_1^2 + c_2^2 + 1 + c_1^2\,c_2^2 + 4a_1a_2c_1c_2 - 1$$
$$= 2c_1^2\,c_2^2 + c_1^2 + c_2^2 + 4a_1a_2c_1c_2,$$

so LHS is the same as RHS. Therefore $X_1.X_2$ is an iso-NPT. ■

(3) Suppose X_1 is an iso-APT and X_2 is an iso-NPT then

$$2a_1^2 = c_1^2 + 1 \quad \text{and} \quad 2a_2^2 = c_2^2 - 1$$

Since

$$X_1.X_2 = (a_1c_2 + a_2c_1, a_1c_2 + a_2c_1, 2a_1a_2 + c_1c_2).$$

To show that $X_1.X_2$is iso-APT we must prove that

$$2(a_1c_2 + a_2c_1)^2 = (2a_1a_2 + c_1c_2)^2 + 1$$

The LHS is

$$2(a_1c_2 + a_2c_1)^2 = 2a_1^2c_2^2 + 2a_2^2c_1^2 + 4a_1a_2c_1c_2,$$

Since

$$2a_1^2 = c_1^2 + 1 \quad \text{and} \quad 2a_2^2 = c_2^2 - 1$$

then,

$$\begin{aligned} 2a_1^2c_2^2 + 2a_2^2c_1^2 + 4a_1a_2c_1c_2 &= (c_1^2 + 1)\, c_2^2 + (c_2^2 - 1)\, c_1^2 + 4a_1a_2c_1c_2 \\ &= c_1^2\, c_2^2 + c_2^2 + c_1^2\, c_2^2 - c_1^2 + 4a_1a_2c_1c_2 \\ &= 2c_1^2\, c_2^2 + c_2^2 - c_1^2 + 4a_1a_2c_1c_2. \end{aligned}$$

The RHS is

$$\begin{aligned} (2a_1a_2 + c_1c_2)^2 + 1 &= 4a_1^2\, a_2^2 + c_1^2\, c_2^2 + 4a_1a_2c_1c_2 + 1 \\ &= 4\left(\frac{c_1^2+1}{2}\right)\left(\frac{c_2^2-1}{2}\right) + c_1^2\, c_2^2 + 4a_1a_2c_1c_2 + 1 \\ &= c_1^2\, c_2^2 - c_1^2 + c_2^2 - 1 + c_1^2\, c_2^2 + 4a_1a_2c_1c_2 + 1 \\ &= 2c_1^2\, c_2^2 - c_1^2 + c_2^2 + 4a_1a_2c_1c_2 \end{aligned}$$

So LHS is the same as RHS. So $X_1.X_2$is an iso-APT.■

2.2 Almost Isosceles Pythagorean Triples

The nonexistence of isosceles integer solution of $x^2 + y^2 = z^2$ open the investigations for finding solutions that look more and more like isosceles. By an almost isosceles Pythagorean triple (AI-PT). For more details, see [5].

We mean an integer solution (a,b,c) of $x^2+y^2=z^2$ such that a and b differ by 1.

The triples (3,4,5), (20,21,29) and (119,120,169) are typical examples of AI-PT.

Now, let (a,b,c) be an AI-PT with $b = a+1$,

if $c=b+k$ for $k \in N$, then

$$a^2 + b^2 = c^2$$

$$a^2 + (a+1)^2 = (b+k)^2$$

$$a^2 + (a+1)^2 = (a+1+k)^2$$

$$a^2+a^2+2a+1 = a^2+a+ak+a+k+1+ak+k+k^2.$$

So

$$a^2 = 2ak + 2k + k^2$$

thus $$a^2 - 2ka - (k^2 + 2k) = 0$$

the solution a $= k \pm \sqrt{2k\,(k+1)}$ is an integer if $2k\,(k+1)$ is a perfect square.

In fact, if $k=1$ then $2k\,(k+1) = 4$, so $a=3$ and $b=4$ yields an AI-PT (3,4,5).

If we let $2k\,(k+1) = u^2$ for some $u \in N$, then $a = k \pm u$. Let $a=k+u$, then

$$a^2 - 2ka - (k^2 + 2k) = 0$$

$$(k+u)^2 - 2k\,(k+u) - k^2 - 2k = 0$$

$$k^2 + 2ku + u^2 - 2k^2 - 2ku - k^2 - 2k = 0$$

$$u^2 - 2k^2 - 2k = 0\,.$$

multiplying both sides by 2 and subtract 1 to both sides we get,

$$2\,u^2 - (2k+1)^2 = -1.$$

If $v = 2k+1$ then

$$2\,u^2 - v^2 = -1,$$

so the pairs (u,v) correspond to the pairs $(u_n,v_n) \in S_-$ in the Theorem 3.2.13 where $k_n = \frac{v_n-1}{2}$. Hence the set $S_- = \{(2,3), (12,17), (70,99), \ldots\}$ together with $k_n = \frac{v_n-1}{2}$, $a_n = u_n + k_n$, $b_n = a_n + 1$ and $c_n = b_n + k_n$ provides Table 2.6 of AI-PT (a_n,b_n,c_n).

Table 2.6: The AI-PT (a_n, b_n, c_n) where $(u_n, v_n) \in S_-$

n	(u_n, v_n)	k_n	(a_n, b_n, c_n)
2	(2,3)	1	(3,4,5)
4	(12,17)	8	(20,21,29)
6	(70,99)	49	(119,120,169)
8	(408,577)	288	(696,697,985)

Theorem 2.2.1:

(1) When $(u_n, v_n) \in S_-$, let $a_n = u_n + \frac{v_n - 1}{2}$, $b_n = u_n + \frac{v_n + 1}{2}$ and $c_n = u_n + v_n$

then (a_n, b_n, c_n) is an AI-PT with $c_n - b_n = \frac{v_n - 1}{2}$

(2) If $(u_n, v_n) \in S_+$, then $(a_n, b_n, c_n)$is an AI-PT if

$$a_n = \frac{v_n - 1}{2} \ , \ b_n = \frac{v_n + 1}{2} \ \text{ and } \ c_n = u_n$$

Proof:

(1) If $(u_n, v_n) \in S_-$, then v_n is odd since ($v_n = 2k_n + 1$) and $v_n = 2v_{n-1} + v_{n-2}$ as in Theorem 2.1.17. Claim $c_n - b_n = \frac{v_n - 1}{2}$, Then

$$\begin{aligned} c_n - b_n &= c_n - u_n - \frac{v_n + 1}{2} \\ &= u_n + v_n - u_n - \frac{v_n + 1}{2} \\ &= v_n - \frac{v_n + 1}{2} \\ &= \frac{v_n - 1}{2} = k_n. \end{aligned}$$

Claim: (a_n, b_n, c_n) is an AI-PT, Clearly

$$\begin{aligned} a_n + 1 &= u_n + \frac{v_n - 1}{2} + 1 \\ &= u_n + \frac{v_n - 1}{2} + 1 \\ &= u_n + \frac{v_n + 1}{2} \\ &= b_n . \end{aligned}$$

To show that (a_n, b_n, c_n) is PT, we must prove that $a_n^2 + b_n^2 = c_n^2$, To do this, consider

$$2\,(a_n^2 + b_n^2) = 2\,(a_n^2 + (a_n+1)^2)$$

$$= 2\,(a_n^2 + a_n^2 + 2a_n + 1)$$

$$= 2\,(2a_n^2 + 2a_n + 1)$$

$$= 2\,(2(u_n + \tfrac{v_n-1}{2})^2 + 2(u_n + \tfrac{v_n-1}{2}) + 1)$$

$$= 4\,((u_n + \tfrac{v_n-1}{2})^2 + 4\,(u_n + \tfrac{v_n-1}{2}) + 2$$

$$= 4u_n^2 + 4u_n\,(v_n - 1) + (v_n - 1)^2 + 4u_n + 2(v_n - 1) + 2$$

$$= 4u_n^2 + 4u_nv_n - 4u_n + {v_n}^2 - 2v_n + 1 + 4\,u_n + 2v_n - 2 + 2$$

$$= 4u_n^2 + 4u_nv_n + {v_n}^2 + 1$$

$$= 2u_n^2 + 2u_n^2 + 1 + 4u_nv_n + {v_n}^2$$

$$= 2u_n^2 + {v_n}^2 + 4u_nv_n + {v_n}^2$$

$$= 2u_n^2 + 4u_nv_n + 2{v_n}^2$$

$$= 2\,(u_n + v_n)^2$$

$$= 2c_n^2.$$

Since $(u_n, v_n) \in S_-$, satisfies $2u_n^2 - v_n^2 = -1$,so $(a_n, b_n, c_n)$is an AI-PT. ■

(2) Similarly Theorem 2.1.17 says if $(u_n, v_n) \in S_+$, then $(u_{n-1}, v_{n-1}) \in S_-$, where

$$(u_{n-1}, v_{n-1}) = (u_n, v_n)\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}^{-1}$$

$$= (u_n, v_n)\begin{pmatrix}-1 & 2\\ 1 & -1\end{pmatrix}$$

$$= (-u_n + v_n, 2u_n - v_n),$$

thus

$$u_{n-1} = -u_n + v_n$$

$$v_{n-1} = 2u_n - v_n.$$

Hence

$$a_n = u_{n-1} + \tfrac{1}{2}\,(v_{n-1} - 1)$$

$$= -u_n + v_n + \tfrac{1}{2}\,(2u_n - v_n - 1)$$

$$= -u_n + v_n + u_n - \tfrac{v_n}{2} - \tfrac{1}{2}$$

$$= \tfrac{v_n}{2} - \tfrac{1}{2} = \tfrac{v_n-1}{2}$$

$$b_n = u_{n-1} + \frac{v_{n-1} + 1}{2}$$

$$= -u_n + v_n + \tfrac{1}{2}\,(2u_n - v_n + 1)$$

$$= -u_n + v_n + u_n - \tfrac{v_n}{2} + \tfrac{1}{2}$$

$$= \frac{v_n}{2} + \frac{1}{2} = \frac{v_n - 1}{2}.$$

And

$$c_n = u_{n-1} + v_{n-1}$$

$$c_n = -u_n + v_n, 2u_n - v_n$$

$$= u_n$$

By part (1) then $(a_n, b_n, c_n)$is an AI-PT .■

Chapter Three

Generating Pythagorean Triples and Almost Pythagorean Triples

In this chapter we will investigate Pythagorean triples (x, y, z) with equation $y = x + 1$ and led to some interesting relationships which allow Pythagorean triples to be generated iteratively and obtain these relationships and other algorithms for generating Pythagorean triples. Also we determine all Almost Pythagorean Triples (APT) by solving the given Diophantine equation. However, the result does not explicitly and readily give a particular almost Pythagorean triples. In this note, using basic algebraic operations and Frink's result we give an explicit formula that readily gives a particular almost Pythagorean triples. We also give a certain integer sequence as a result of the generated formula.

3.1 A new algorithm for generating Pythagorean Triples

In this section we depend on two equations which are $y = x + a$ and $z = y + b$ and from these equations we get on some results that are using for generating Pythagorean triples and accordingly for this we deal with two cases which we will explain in this chapter.[8]

If we have a Pythagorean triple (x, y, z) that satisfies the Diophantine equation $x^2 + y^2 = z^2$ and if we let $z = y + b$ then,

$$x^2 + y^2 = (y + b)^2$$

$$x^2 + y^2 = y^2 + 2by + b^2$$

$$x^2 = 2by + b^2.$$

Hence

$$y = \frac{x^2-b^2}{2b} \text{ and } z = y + b = \frac{x^2+b^2}{2b}$$

therefore, the triple can therefore be expressed as

$$\left(x, \left(\frac{X^2-b^2}{2b}\right), \left(\frac{X^2+b^2}{2b}\right)\right). \qquad (3.1)$$

Also, let $y = x + a$ then

$$x + a = \frac{x^2 - b^2}{2b}$$

and thus

$$x^2 - 2bx - (b^2 + 2ab) = 0 \qquad (3.2)$$

by using equation (3.2) we may generate infinitely many Pythagorean triples.
Consider the following cases:

Case (1): If *a* is fixed

In this case if we let a be fixed and change the values of b, for example, if we let $a = 1$ and take $b_1 = 1$ then the equation (3.2) becomes

$$x^2 - 2x - 3 = (x - 3)(x + 1) = 0$$

Now take $x_1 = 3$ and substitute in the triple $\left(x, \left(\frac{X^2-b^2}{2b}\right), \left(\frac{X^2+b^2}{2b}\right)\right)$,we get the triple (3,4,5).

In order to generate the next triple where $a = 1$, we must know the next value of b, thus the equation (3.2) when *a* is fixed and equal 1 becomes

$$x^2 - 2bx - (b^2 + 2b) = 0 \qquad (3.3)$$

and this equation has a solution if $A(b) = \sqrt{4b^2 + 4(b^2 + 2b)}$ is an integer.

So, the following table contains some values of b that show when the equation (3.3) is reducible or not.

Table 3.1: The value of discriminate $\sqrt{4b^2 + 4(b^2 + 2b)}$ from b=2 to b =10

$\boldsymbol{b}$	$\sqrt{\boldsymbol{4b^2 + 4(b^2 + 2b)}}$
2	$\sqrt{48}$
3	$\sqrt{96}$
4	$\sqrt{160}$
5	$\sqrt{240}$
6	$\sqrt{336}$
7	$\sqrt{448}$
8	$\sqrt{576} = 24$
9	$\sqrt{720}$
10	$\sqrt{880}$

We note that from the above table when $b = 8$ there is an integer solution of (3.3).

And, when $b = 2,3,4,5,7$ and 7 the equation (3.3) is irreducible.

Now let $b_2 = 8$, so the equation (3.3) becomes

$$x^2 - 16x - 80 = (x - 20)(x + 4) = 0$$

Take $x_2 = 20$ and substitute in the triple (3.1), we get the triple (20, 21, 29). By depending on two these triples (3,4,5) and (20, 21, 29) we get the relationship,

$$b_2 = x_1 + z_1$$

and thus we can generalize it as

$$b_{n+1} = x_n + z_n.$$

For all $n \geq 1$, which suggested itself as a possibility to be explored for determining subsequent values of b where x_n, x_n, z_n represent the nth Pythagorean triple of a sequence. In fact, the relation $b_{n+1} = x_n + z_n$ is proved in [7].

The first four triples when $a = 1$ can be generated by using equation (3.3) as follows:

1) When $b_1 = 1$, then the equation(3.3) becomes,

$$x^2 - 2x - 3 = (x-3)(x+1) = 0$$

So, take $x = 3$, then by (3.1), the PT is (3, 4, 5).

2) When $b_2 = 3 + 5 = 8$, then the equation (3.3) becomes,

$$x^2 - 16x - 80 = (x-20)(x+4) = 0$$

So, take $x = 20$, then by (3.1), the PT is (20, 21, 29).

3) When $b_3 = 20 + 29 = 49$, then the equation (3.3) becomes,

$$x^2 - 98x - 2499 = (x-119)(x+21) = 0$$

So, take $x = 119$, then by (3.1), the PT is (119, 120, 169).

4) when $b_4 = 119 + 169 = 288$, then the equation (3.3) becomes,

$$x^2 - 576x - 83520 = (x-696)(x+120) = 0$$

So, take $x = 696$, then by (3.1), the PT is (696, 697, 985).

Case (2): If *b* is fixed

By the same way as above,

If $a = 1$ and $b = 1$, we get on the triple (3,4,5) thus let b be fixed, say $b = 1$ and change the values of a to generate the next triples.

Thus, the equation (3.2) when b is fixed becomes

$$x^2 - 2x - (1 + 2a) = 0 \qquad (3.4)$$

And the equation (3.4) has an integer solution if $A(a) = \sqrt{4 + 4(1 + 2a)} = \sqrt{8 + 8a}$ is an integer.

So, the following table contains some values of a that shows when the equation (3.4) is reducible or not.

Table 3.2**:** The value of discriminate $\sqrt{8+8a}$ from a = 2 to a = 10

$\boldsymbol{a}$	$\boldsymbol{\sqrt{8+8a}}$
2	$\sqrt{24}$
3	$\sqrt{32}$
4	$\sqrt{40}$
5	$\sqrt{48}$
6	$\sqrt{56}$
7	$\sqrt{64}=8$
8	$\sqrt{72}$
9	$\sqrt{80}$
10	$\sqrt{88}$

So, we note from the above table when $a=7$ there is an integer solution of (3.4). Also, when $a=2,3,4,5$ and 6 the equation (3.4) is irreducible. So, by depending on $a=7$ the equation (3.4) becomes,

$$x^2-2x-15=(x-5)(x+3)=0$$

Take $x=5$, and substitute in the triple (3.1) we get the triple (5,12,13) Thus, by depending on the first two triples (3,4,5) , (5,12,13), and letting $a_1=1$ and $a_2=7$, we get the relationship

$$a_2=x_1+y_1$$

and thus we can generalize it as

$$a_{n+1}=x_n+y_n.$$

For all $n\geq 1$, which suggested itself as a possibility to be explored for determining subsequent values of b where x_n, y_n, z_n represent the n^th^ Pythagorean triple of a sequence. In fact, this relation is proved in [7].

The first four triples when $b=1$ can be generated by using equation (3.4) as follows:

1) When $a_1 = 1$, then the equation (3.4) becomes,

$$x^2 - 2x - 3 = (x-3)(x+1) = 0$$

So, take $x = 3$, then by (3.1), the PT is $(3, 4, 5)$.

2) When $a_2 = 3 + 4 = 7$ then the equation (3.4) becomes,

$$x^2 - 2x - 15 = (x-5)(x+3) = 0$$

So, take $x = 5$, then by (3.1), the PT is $(5, 12, 13)$.

3) When $a_3 = 5 + 12 = 17$, then the equation (3.4) becomes,

$$x^2 - 2x - 35 = (x-7)(x+5) = 0$$

So, take $x = 7$, then by (3.1), the PT is $(7, 24, 25)$.

4) when $a_4 = 7 + 24 = 31$, then the equation (3.4) becomes,

$$x^2 - 2x - 63 = (x-9)(x+7) = 0$$

So, take $x = 9$, then by (3.1), the PT is $(9, 40, 41)$.

At the end of this section, we can use the Java Language to generate infinitely many Pythagorean triples (PTs) by using the above procedures in case 1 and in case 2 as the following: -

1. We basically depend on the equation (3.2) which is
$$x^2 - 2bx - (b^2 + 2ab) = 0$$
2. The equation (3.2) has a solution only when the discriminate is an integer, i.e.
$$\sqrt{4b^2 + 4(b^2 + 2ab)}$$
is an integer.
3. Now, by using the general formula of the quadratic equation, we can find the values of *x* as:

$$x = \frac{2b \pm \sqrt{4b^2 + 4(b^2 + 2ab)}}{2}$$

and therefore we can compute y and z by

$$y = \frac{x^2 - b^2}{2b} \qquad \text{and} \qquad z = \frac{x^2 + b^2}{2b}$$

4. The above values of x , y and z are form Pythagorean triples because $x^2 + y^2 = z^2$ satisfied.
5. We tried to run the program when a is fixed and change the values of b or b is fixed and change the values of a and enter any value and any range for a or b.
6. Finally, we have stored the values of a, b, x, y and z in tables in excel.

All above procedures are written in Java Language which existing in program (1) at page (74) at the end of the thesis.

Now, we want to give some examples which are applicable on the above program where the first nine examples when a is fixed and the others when b is fixed.

Example 3.1.1:

Table 3.3**:** For a equals 1 and the values of b are changed.

b	***x***	***y***	***z***
1	3	4	5
8	20	21	29
49	119	120	169
288	696	697	985
1681	4059	4060	5741
9800	23660	23661	33461

Example 3.1.2:

Table 3.4**:** For a equals 2 and the values of b are changed.

b	***x***	***y***	***z***
2	6	8	10
16	40	42	58
98	238	240	338
576	1392	1394	1970
3362	8118	8120	11482

Example 3.1.3:

Table 3.5**:** For *a* equals 5 and the values of b are changed.

b	***x***	***y***	***z***
5	15	20	25
40	100	105	145
245	595	600	845
1440	3480	3485	4925
8405	20295	20300	28705

Example 3.1.4:

Table 3.6**:** For *a* equals 11 and the values of *b* are changed.

b	***x***	***y***	***z***
11	33	44	55
88	220	231	319
539	1309	1320	1859
3168	7656	7667	10835

Example 3.1.5:

Table 3.7**:** For a equals 18 and the values of b are changed.

b	***x***	***y***	***z***
18	54	72	90
144	360	378	522
882	2142	2160	3042
5184	12528	12546	17730

Example 3.1.6:

Table 3.8**:** For a equals 36 and the values of b are changed.

b	***x***	***y***	***z***
36	108	144	180
288	720	756	1044
1764	4284	4320	6084
10368	25056	25092	35460

Example 3.1.7:

Table 3.9**:** For a equals 75 and the values of b are changed.

b	***x***	***y***	***z***
75	225	300	375
600	1500	1575	2175
3675	8925	9000	12675

Example 3.1.8:

Table 3.10**:** For a equals 100 and the values of b are changed.

b	***x***	***y***	***z***
100	300	400	500
800	2000	2100	2900
4900	11900	12000	16900

Example 3.1.9:

Table 3.11**:** For a equals 1000 and the values of b are changed.

b	***x***	***y***	***z***
1000	3000	4000	5000
8000	20000	21000	29000

Now the following examples when b is fixed.

Example 3.1.10:

Table 3.12**:** For b equals 1 and the values of a are changed.

b	***x***	***y***	***z***
1	3	4	5
7	5	12	13
17	7	24	25
31	9	40	41
49	11	60	61
71	13	84	85
97	15	112	113
127	17	144	145
161	19	180	181
199	21	220	221
241	23	264	265
287	25	312	313
337	27	364	365
391	29	420	421
449	31	480	481
511	33	544	545
577	35	612	613
647	37	684	685
721	39	760	761
799	41	840	841

Example 3.1.11:

Table 3.13**:** For *b* equals 3 and the values of *a* are changed.

b	*x*	*y*	*z*
3	9	12	15
21	15	36	39
51	21	72	75
93	27	120	123
147	33	180	183
213	39	252	255
291	45	336	339
381	51	432	435
483	57	540	543
597	63	660	663
723	69	792	795
861	75	936	939
1011	81	1092	1095
1173	87	1260	1263
1347	93	1440	1443
1533	99	1632	1635
1731	105	1836	1839
1941	111	2052	2055
2163	117	2280	2283
2397	123	2520	2523
2643	129	2772	2775
2901	135	3036	3039

Example 3.1.12:

Table 3.14**:** For b equals 19 and the values of a are changed.

a	***x***	***y***	***z***
19	57	76	95
133	95	228	247
323	133	456	475
589	171	760	779
931	209	1140	1159
1349	247	1596	1615
1843	285	2128	2147
2413	323	2736	2755
3059	361	3420	3439
3781	399	4180	4199
4579	437	5016	5035
5453	475	5928	5947
6403	513	6916	6935
7429	551	7980	7999
8531	589	9120	9139
9709	627	10336	10355
10963	665	11628	11647
12293	703	12996	13015
13699	741	14440	14459
15181	779	15960	15979
16739	817	17556	17575
18373	855	19228	19247
20083	893	20976	20995
21869	931	22800	22819
23731	969	24700	24719
25669	1007	26676	26695
27683	1045	28728	28747
29773	1083	30856	30875

Example 3.1.13:

Table 3.15**:** For *b* equals 55 and the values of a are changed.

b	***x***	***y***	***z***
55	165	220	275
385	275	660	715
935	385	1320	1375
1705	495	2200	2255
2695	605	3300	3355
3905	715	4620	4675
5335	825	6160	6215
6985	935	7920	7975
8855	1045	9900	9955
10945	1155	12100	12155
13255	1265	14520	14575
15785	1375	17160	17215
18535	1485	20020	20075
21505	1595	23100	23155
24695	1705	26400	26455
28105	1815	29920	29975
31735	1925	33660	33715
35585	2035	37620	37675
39655	2145	41800	41855
43945	2255	46200	46255
48455	2365	50820	50875
53185	2475	55660	55715
58135	2585	60720	60775
63305	2695	66000	66055
68695	2805	71500	71555
74305	2915	77220	77275
80135	3025	83160	83215
86185	3135	89320	89375

Example 3.1.14:

Table 3.16**:** For *b* equals 105 and the values of a are changed.

b	***x***	***y***	***z***
105	315	420	525
735	525	1260	1365
1785	735	2520	2625
3255	945	4200	4305
5145	1155	6300	6405
7455	1365	8820	8925
10185	1575	11760	11865
13335	1785	15120	15225
16905	1995	18900	19005
20895	2205	23100	23205
25305	2415	27720	27825
30135	2625	32760	32865
35385	2835	38220	38325
41055	3045	44100	44205
47145	3255	50400	50505
53655	3465	57120	57225
60585	3675	64260	64365
67935	3885	71820	71925
75705	4095	79800	79905
83895	4305	88200	88305
92505	4515	97020	97125
101535	4725	106260	106365
110985	4935	115920	116025
120855	5145	126000	126105
131145	5355	136500	136605
141855	5565	147420	147525
152985	5775	158760	158865
164535	5985	170520	170625

Example 3.1.15:

Table 3.17**:** For b equals 2553 and the values of a are changed.

b	*x*	*y*	*z*
2553	7659	10212	12765
17871	12765	30636	33189
43401	17871	61272	63825
79143	22977	102120	104673

3.2 A Note on generating almost Pythagorean triples

In this section we try to generate almost Pythagorean triples (APT) by depending on two theorems that we will discuss now. See [3]

Theorem 3.2.1:

If (a, b, c) is a primitive Pythagorean triple and if (p, q, r) is an almost Pythagorean triple, then the triples:

$$(x, y, z) = (at + p, bt + q, ct + r)$$

And

$$(x', y', z') = (at + p', bt + q', ct + r')$$

are almost Pythagorean triples for all positive integers and for unique integers p, p', q, q', r *and* r' that depends on the primitive Pythagorean triple (a, b, c) satisfying $p + p' = a$, $q + q' = b$, $r + r' = c$ and $ap + bq = cr$ .For more details, see [8].

Proof:

Suppose (a, b, c) is a PPT and (p, q, r) is an APT and for any positive integer t, let

$x = at + p$ and $y = bt + q$, then

$$x^2 + y^2 = (at + p)^2 + (bt + q)^2$$

$$= a^2t^2 + 2apt + p^2 + b^2t^2 + 2qbt + q^2$$

$$= (a^2+b^2)t^2 + 2(apt + qbt) + p^2 + q^2$$
$$= c^2t^2 + 2(apt + qbt) + r^2 + 1.$$

Since $ab + bq = cr$, we get
$$x^2 + y^2 = c^2t^2 + 2crt + r^2 + 1$$
$$= (ct + r)^2 + 1$$
thus if we take $z = ct + r$, then
$$x^2 + y^2 = z^2 + 1$$
so (x, y, z) is an APT.

By the same way as above, let
$$x' = at + p' = (at + a - p) \qquad \text{and} \qquad y' = bt + q' = (bt + b - q)$$
Now
$$(x')^2 + (y')^2 = (at + a - p)^2 + (bt + b - q)^2$$
$$= a^2t^2 + 2a(a - p)t + (a - p)^2 + b^2t^2 + 2b(b - q)t + (b - q)^2$$
$$= a^2t^2 + 2a^2t - 2apt + a^2 - 2ap$$
$$+p^2 + b^2t^2 + 2b^2t - 2bqt + b^2 - 2bq + q^2$$
$$= (a^2 + b^2)t^2 + 2(a^2 + b^2)t - 2(ap + bq)t$$
$$+ a^2 + b^2 - 2(ap + bq) + p^2 + q^2$$
$$= c^2t^2 + 2c^2t - 2(cr)t + c^2 - 2cr + r^2 + 1$$
$$= c^2t^2 + 2c(c - r)t + (c - r)^2 + 1$$
$$= (ct + (c - r))^2 + 1$$
$$= (ct + r')^2 + 1.$$
Let $z' = ct + r'$, then

$$(x')^2 + (y')^2 = (z')^2 + 1$$
and hence (x', y', z') is an APT.■

By depending on the previous theorem we conclude that the general solution of $x^2 + y^2 = z^2 + 1$ does not explicitly and readily give a particular almost Pythagorean triple, also for PPT (a, b, c) when the components of this triple is large, the process of finding integers $p, p', q, q', r, and\ r'$ seems to be not an easy task.

So, from this note, we can use another formula that generate the almost Pythagorean triples (APTs) even if the component of the PPT is large.

Before presenting the theorem that generates APT's we must recall the theorem 1.1.11 which describes all primitive Pythagorean triples which says that the primitive Pythagorean triples (x, y, z) take the form $x = k^2 - l^2$ $y = 2kl$ and $z = k^2 + l^2$ where $k > l > 0$, $gcd(k, l) = 1$ and $k \not\equiv l(mod 2)$.

Now, let i be any positive integer such that $i \geq 2$. Take $k = i$ and $l = i - 1$, then clearly they are relatively prime for all i and satisfying the incongruence relation $k \not\equiv l(mod2)$ and we get the form $(2i - 1, 2i^2 - 2i, 2i^2 - 2i + 1)$, which is a PPT.

Theorem 3.2.3:

Let $(a, b, c) = (2i - 1, 2i^2 - 2i, 2i^2 - 2i + 1)$, then

$$(x, y, z) = (at + 1, bt + (2i - 1), ct + (2i - 1))$$

and

$$(x', y', z') = (at + (2i - 2), bt + (2i^2 - 4i + 1), ct + (2i^2 - 4i + 2))$$

are almost Pythagorean triples for all $t \in Z^+$.

Proof:

To show that $(x, y, z) = (at + 1, bt + (2i - 1), ct + (2i - 1))$ is an APT. Let us compute the left hand side (LHS) and the right hand side (RHS) as:

$$\begin{aligned} x^2 + y^2 &= (at + 1)^2 + (bt + (2i - 1))^2 \\ &= a^2t^2 + 2at + 1 + b^2t^2 + (2i - 1)^2 + 2b(2i - 1)t \\ &= (a^2 + b^2)t^2 + (2i - 1)^2 + 1 + 2(a + b(2i - 1))t. \end{aligned}$$

Also,

$$\begin{aligned} z^2 + 1 &= (ct + (2i - 1))^2 + 1 \\ &= c^2t^2 + (2i - 1)^2 + 2c(2i - 1)t + 1. \end{aligned}$$

For (LFS) and (RHS) to be equal, we must show that $a + b(2i - 1) = c(2i - 1)$

But $a = 2i - 1$, $b = 2i^2 - 2i$ and $c = 2i^2 - 2i + 1$, hence

$$a + b(2i - 1) = (2i - 1) + (2i^2 - 2i)(2i - 1)$$

$$= (2i-1)(2i^2-2i+1)$$
$$= (2i-1)c.$$

So $(x, y, z) = (at + 1, bt + (2i - 1), ct + (2i - 1))$ is an APT.■

Also, we must prove that

$$(x', y', z') = (at + (2i - 2), bt + (2i^2 - 4i + 1), ct + (2i^2 - 4i + 2)),$$

is an APT. By the same procedure as above, let us compute the left hand side (LHS) and the right hand side (RHS) as:

$$(x')^2 + (y')^2 = (at + (2i-2))^2 + (bt + (2i^2 - 4i + 1))^2$$
$$= a^2t^2 + (2i-1)^2 + 2a(2i-1)t + b^2t^2$$
$$+(2i^2 - 4i + 1)^2 + 2b(2i^2 - 4i + 1)t$$
$$= (a^2 + b^2)t^2 + (2i-2)^2 + +(2i^2 - 4i + 1)^2$$
$$+2(a(2i-2) + b(2i^2 - 4i + 1))t$$
$$= (a^2 + b^2)t^2 + 4i^2 - 8i + 4 + 4i^4 - 16i^3 + 20i^2$$
$$-8i + 1 + 2\big(a(2i-2) + b(2i^2 - 4i + 1)\big)t$$
$$= (a^2 + b^2)t^2 + 4i^4 - 16i^3 + 24i^2 - 16i + 5$$
$$+2(a(2i-2) + b(2i^2 - 4i + 1))t$$
$$= c^2t^2 + 4i^4 - 16i^3 + 24i^2 - 16i + 5$$
$$+2\big(a(2i-2) + b(2i^2 - 4i + 1)\big)t.$$

Also,

$$z^2 + 1 = \big(ct + (2i^2 - 4i + 2)\big)^2 + 1$$
$$= c^2t^2 + (2i^2 - 4i + 2)2 + 2c(2i^2 - 4i + 2)t + 1$$
$$= c^2t^2 + 4i^4 - 8i^3 + 4i^2 - 8i^3 + 16i^2 - 8i$$
$$+4i^2 - 8i + 4 + 2c(2i^2 - 4i + 2))t + 1$$
$$= c^2t^2 + 4i^4 - 16i^3 + 24i^2 - 16i$$
$$+5 + 2c(2i^2 - 4i + 2))t.$$

For (LHS) and (RHS) to be equal, we must show that

$$a(2i-2) + b(2i^2 - 4i + 1) = c(2i^2 - 4i + 2),$$

but

$$a(2i-2) + b(2i^2 - 4i + 1) = (2i-1)(2i-2) + (2i^2 - 2i)(2i^2 - 4i + 1)$$
$$= (4i^2 - 4i - 2i + 2) + (2i^2 - 2i)(2i^2 - 4i + 1)$$

$$= 4i^2 - 6i + 2 \ + (4i^4 - 8i^3 + 2i^2 - 4i^3 + 8i^2 - 2i)$$
$$= 4i^4 - 12i^3 + 14i^2 - 2i + 2$$

and

$$c(2i^2 - 4i + 2) \ = \ (2i^2 - 2i + 1)\,(2i^2 - 4i + 2)$$
$$= \ 4i^4 - 8i^3 + 4i^2 - 4i^3 + 8i^2 - 2i^2 - 41 + 2$$
$$= \ 4i^4 - 12i^3 + 14i^2 - 8i^3 + 2.$$

So the two sides are equal and hence

$(x', y', z') = \left(at \ + \ (2i \ - 2), bt \ + \ (2i^2 - 4i \ + \ 1), ct \ + \ (2i^2 - 4i \ + 2)\right)$ is an APT.

■

Example 3.2.4:

Let $i = 4$ and $t_1 = 5$ and $t_2 = 6$, what are the almost Pythagorean triples (APT) that generated.

Solution:

By previous theorem we must know that PPT, so

$$(a, b, c) \ = (2i - \ 1 \ , 2i^2 - \ 2i, 2i^2 - \ 2i \ + \ 1)$$
$$= (2(4) - \ 1 \ , 2(4)^2 - \ 2(4), 2(4)^2 - \ 2(4) \ + \ 1)$$
$$= (7,24,25)$$

Now, when $i = 4$ and $t = 5$, the (APTs) are

$$(x, y, z) \ = (7(5) + 1 \ , 24(5) + 7 \ , 25(5) + 7$$
$$= (36,127,132)$$

and

$$(x', y', z') \ = (7(5) + 6 \ , 24(5) + 17 \ , 25(5) + 18)$$
$$= (41,137,143)$$

When $i = 4$ and $t = 6$, by the same way, the (APTs) are $(43,151,157)(48,161,168)$.

Example 3.2.5:

Let $i_1 = 10$ and $i_2 = 11$ and $t = 7$, what are the almost Pythagorean triples that are generated?

When $i = 10$ the PPT is (19,180,181) and when $i = 11$ the PPT is $(21,220,221)$.

Now, when $i = 10$ and $t = 7$ the APTs are

$$(134,1279,1286)$$

$$(151,1421,1429)$$

and, when $i = 11$ and $t = 7$ the APTs are

$$(148,1561,1568)$$

$$(167,1739,1747).$$

Example 3.2.6:

Let $i = 3120$ and $t = 25$, then the formulas in the above theorem yields the triples

$$(155976, 486570239, 486570264)$$

and

$$(162213, 506020321, 506020347)$$

and they are APT's which can easily be verified using and computer software.

To generate infinitely many almost Pythagorean triples easily and if i and t are large, we use Java language where the procedures that work the program are taken as follows:

1) Define a, b, c, x, y, z, x', y' and z' as the following:

   $a = 2i - 1$

   $b = 2i^2 - 2i$

   $c = 2i^2 - 2i + 1$

   $x = at + 1$

   $y = bt + (2i - 1)$

   $z = ct + (2i - 1)$

   $x' = at + (2i - 2)$

   $y' = bt + (2i^2 - 4i + 1)$

and $z' = ct + (2i^2 - 4i + 2)$

where all of them are integers.

2) Enter the values of $i \geq 2$ and $t \geq 1$ and take any range of t.
3) We have stored the values of $i, t, a, b, c, x, y, z, x', y'$ and z' in tables in Excel sheet where (a, b, c) , (x, y, z) and (x', y', z') written as triples.

And all of this procedures are written in Java Language which existing in program (2) page (78) at the end of the thesis.

The following table generate many PPT (a, b, c) and also APT $(x, y, z), (x', y', z')$, where $i \geq 2$ and $t \geq 1$.

i	***t***	***(a,b,c)***	***(x,y,z)***	***(x',y',z')***
2	1	(3,4,5)	(4,7,8)	(5,5,7)
2	2	(3,4,5)	(7,11,13)	(8,9,12)
2	3	(3,4,5)	(10,15,18)	(11,13,17)
2	4	(3,4,5)	(13,19,23)	(14,17,22)
2	5	(3,4,5)	(16,23,28)	(17,21,27)
2	6	(3,4,5)	(19,27,33)	(20,25,32)
2	7	(3,4,5)	(22,31,38)	(23,29,37)
2	20	(3,4,5)	(61,83,103)	(62,81,102)
44	86	(87,3784,3785)	(7483,325511,325597)	(7568,329121,329208)
44	87	(87,3784,3785)	(7570,329295,329382)	(7655,332905,332993)
44	88	(87,3784,3785)	(7657,333079,333167)	(7742,336689,336778)
97	35	(193,18624,18625)	(6756,652033,652068)	(6947,670271,670307)
97	36	(193,18624,18625)	(6949,670657,670693)	(7140,688895,688932)
97	37	(193,18624,18625)	(7142,689281,689318)	(7333,707519,707557)
97	38	(193,18624,18625)	(7335,707905,707943)	(7526,726143,726182)
97	39	(193,18624,18625)	(7528,726529,726568)	(7719,744767,744807)
97	40	(193,18624,18625)	(7721,745153,745193)	(7912,763391,763432)
97	41	(193,18624,18625)	(7914,763777,763818)	(8105,782015,782057)
97	42	(193,18624,18625)	(8107,782401,782443)	(8298,800639,800682)
97	43	(193,18624,18625)	(8300,801025,801068)	(8491,819263,819307)
97	44	(193,18624,18625)	(8493,819649,819693)	(8684,837887,837932)

At the end of this section, we successfully gave an explicit formula in generating almost Pythagorean triples. These formulas were started in theorem 3.2.1 and theorem 3.2.3 in this section, but the result in theorem 3.2.3 does not generate all almost Pythagorean triples, because we deal with one formula for the PPT which is $(2i-1, 2i^2-2i, 2i^2-2i+1)$.

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Appendices:

Program (1)

```
/*a
 * To change this license header, choose License Headers in Project Properties.
 * To change this template file, choose Tools | Templates
 * and open the template in the editor.
 */
package javaapplication1;

import java.io.File;
import java.io.FileNotFoundException;
import java.io.FileOutputStream;
import java.io.PrintStream;
import java.io.PrintWriter;
import java.io.UnsupportedEncodingException;
import java.util.Scanner;

/**
 *a
 * @author Fall
 */
public class JavaApplication1 {

    /**
     * @param args the command line arguments
     */
    public static void main(String[] args) throws FileNotFoundException,
UnsupportedEncodingException {

        //  int a = 155975;
        // for (int j = 1001000; j < 1500000; j+=2) {
//          XSSFWorkbook workbook = new XSSFWorkbook();
//         XSSFSheet sheet = workbook.createSheet("Java Books");
//         File f = new File("file.txt");
//if(f.exists() && !f.isDirectory()) {
```

```
//    // do something
//}else{
       PrintWriter writer = new PrintWriter("file.txt", "UTF-8");
       PrintWriter wexcel = new PrintWriter("file.csv", "UTF-8");

//}



        Scanner sc = new Scanner(System.in);
        do{
        System.out.println("please choose witch one is constant ,enter a or b ");

         String str = sc.next();

        if(str.charAt(0)!='a'&&str.charAt(0)!='b'){
          //  System.out.println("please choose witch one is constant ,enter a or b ");
          //  System.err.println(str.charAt(0)!='b');

        }else{



       if(str.charAt(0)=='a'){
         System.out.println("please insert the value of a");
             int a = sc.nextInt();
              wexcel.println("a,b,x,y,z,next b,old b = b");

              System.out.println("please enter the maximum range");

             int max = sc.nextInt();
      //  int a = 87;
       int a2 = 0,b2=0;
        for (int i = 1; i < max; i++) {



            int b = i;
            int q = (4 * b * b) + 4 * ((b * b) + 2 * a * b);
```

```
        double c1 = Math.sqrt(q);
        int c2 = (int) Math.sqrt(q);
        if (c1 == c2) {

            int x = (2 * b + c2) / 2;
            int y = (x * x - b * b) / (2 * b);
            int z = (x * x + b * b) / (2 * b);
            boolean flag = (((x * x) + (y * y)) == (z * z));
            if (flag && x > 0 && y > 0 && z > 0) {

              // if(b2!=0){
                //   System.out.println(b2);
                   if(b2==b){
                     System.out.println("a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z
);
                   String str2 ="a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z ;
               writer.println(str2);
                wexcel.println(a+","+b+","+x+","+y+","+z);
                   }  else{
                       System.out.println("a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z
);
                   String str2 ="a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z ;
               writer.println(str2);
              //  writer2.println("a,b,x,y,z,next b,old b = b");
                wexcel.println(a+","+b+","+x+","+y+","+z);
                   }
            //   }
                  //if(b2!=0){
                   b2 =x+z ;

              // }
            }
//              else {
//                  String str2 = "a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z + " no" ;
//                   writer.println(str2);
```

```
//                      System.out.println("a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z + "
no");
//                  }
            }
        }
    }else if(str.charAt(0)=='b'){
         wexcel.println("a,b,x,y,z,next a,old a = a");
           System.out.println("please insert the value of b");
            int b = sc.nextInt();
 System.out.println("please enter the maximum range");
                    int a2 = 0,b2=0;
            int max = sc.nextInt();

        for (int i = 1; i < max; i++) {

            int a = i;
            int q = (4 * b * b) + 4 * ((b * b) + 2 * a * b);
            double c1 = Math.sqrt(q);
            int c2 = (int) Math.sqrt(q);
            if (c1 == c2) {
                int x = (2 * b + c2) / 2;
                int y = (x * x - b * b) / (2 * b);
                int z = (x * x + b * b) / (2 * b);
                boolean flag = (((x * x) + (y * y)) == (z * z));
                if (flag && x > 0 && y > 0 && z > 0) {
                    if(a2==a){
                    System.out.println("a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z  );
                        String str2 ="a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z     ;
                    writer.println(str2);
                    wexcel.println(a+","+b+","+x+","+y+","+z);

                    }
                    a2 = x+y;
                }
//                  else {
//                      String str2 = "a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z + " no" ;
```

```
//                    writer.println(str2);
//                   System.out.println("a = " + a + " b = " + b + " x = " + x + " y = " + y + " z = " + z + "
no");
//               }
            }
        }
    }
        break;
        }


        }while(true);
        writer.close();
        wexcel.close();
    }

   // }
}
```

Program (2)

```
/*
 * To change this license header, choose License Headers in Project Properties.
 * To change this template file, choose Tools | Templates
 * and open the template in the editor.
 */
package javaapplication2;

import java.io.FileNotFoundException;
import java.io.PrintWriter;
import java.io.UnsupportedEncodingException;
import java.util.Scanner;

/**
 *
 * @author Fall
```

```
 */
public class JavaApplication2 {

    /**
     * @param args the command line arguments
     */
    public static void main(String[] args) throws FileNotFoundException,
UnsupportedEncodingException {

        //System.out.println("please");
       // Scanner sc = new Scanner(System.in);

    PrintWriter writer1 = new PrintWriter("abt.txt", "UTF-8");
    PrintWriter writer = new PrintWriter("file.csv", "UTF-8");
    writer.println("i,t,(a-b-c),(x-y-z),(x'-y'-z')");
        for (int i = 2; i < 100; i++) {

            for (int t = 1; t < 100; t++) {



            int a = 2*i-1;
            int b = 2*i*i -2*i;
            int c = 2*i*i-2*i+1;

            int x = a*t+1 ;
            int y = b*t+(2*i-1);
            int z = c*t +(2*i-1);

            int x1 = a*t+(2*i-2);
            int y1 = b*t +(2*i*i-4*i+1);
            int z1 = c*t +(2*i*i - 4*i +2);

            writer1.println(
                    i+","+
                    t+",("+
                            a+","+
```

```
                b+","+
                c+"),("+
                x+","+
                y+","+
                z+"),("+
                x1+","+
                y1+","+
                z1+")");
        writer.println(
            i+","+
            t+",("+
                a+"-"+
                b+"-"+
                c+"),("+
                x+"-"+
                y+"-"+
                z+"),("+
                x1+"-"+
                y1+"-"+
                z1+")");
//
        String str =
            " i : "+i+
            " t : "+t+
            " a : "+a+
            " b : "+b+
            " c : "+c+
            " x : "+x+
            " y : "+y+
            " z : "+z+
            " x' : "+x1+
            " y' : "+y1+
            " z' : "+z1 ;
//          writer.println(str);
        System.out.println(str);

//      }else{
```

```java
//            System.out.println("invalid value for i");
//        }

            }
    }
    }

}
```

تعميم الاعداد الفيثاغورية

الإعداد: أميرة جميل محمد صلاحات

الإشراف: الدكتور ابراهيم محمود الغروز

ملخص:

في هذه الرسالة قمنا بدراسة الاعداد الفيثاغورية والاعداد الفيثاغورية الأولية عندما يكون العامل المشترك بين هذه الاعداد الواحد الصحيح وأيضا قمنا بدراسة الأعداد القريبة من الاعداد الفيثاغورية وذلك عندما نضيف او نطرح واحد لطرف المعادلة $س^2 + ص^2 = ع^2$. وأيضا درسنا الاعداد القريبة من الاعداد الفيثاغورية عندما يكون الفرق بين x و y وقمنا بدراسة خصائص جميع هذه الاعداد وناقشنا العديد من النظريات والتعريفات لكل واحدة على حدى وقدمنا في نهاية هذه الرسالة برنامج باستخدام احدى لغات البرمجة "الجافا" ليسهل علينا تصنيف العديد من الاعداد الفيثاغورية و الأعداد الفيثاغورية القريبة باستخدام خطوات عديدة مبينة وموضحة في هذه الرسالة .