Deanship of Graduate Studies

Al-Quds University

Optimal Homotopy Asymptotic Method for Solving Fredholm Integral Equations of First and Second Kind

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M.Sc. Thesis

Jerusalem- Palestine

1436/2015

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A thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics at Al-Quds University

1436/2015

Al-Quds University

Deanship of Graduate Studies

Graduate Studies / Mathematics

Thesis Approval

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Master thesis submitted and accepted, Date : 12/05/2015.

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Dedication

I dedicate this thesis to my mother and my husband Nahed for their patience, understanding, support and a lot of love.

Declaration

The work provided in this thesis , unless otherwise referenced , is the researcher's own work , and has not been submitted elsewhere for any other degree or qualification.

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Acknowledgement

I am thankful to my supervisor Dr. Yousef Zahaykah for his encouragement, guidance and support from the initial to the final level enabled me to develop and understand the subject.

I acknowledge Al-Quds University for supporting this work, and I wish to pay my great appreciation to all respected teachers and staff in the Department of Mathematics.

Lastly, I offer my regards to all of those who supported me during the completion of this thesis.

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Abstract

Numerical Analysis is of great importance, for it is hard to find closed form solutions (exact solutions) for many applied engineering and scientific problems. Thus, it is natural to consider numerical procedures (algorithms) for obtaining approximate solutions of such problems.

In this thesis, a semi-analytic approximating method, namely Optimal Homotopy Asymptotic Method (OHAM) is used to find continuous approximate solutions for Linear Fredholm Integral Equations of First and Second Kind.

Within this work, the geometrical topological homotopy concept is used to construct algorithms for solving such integral equations. A homotopy equation, that depends on an embedding parameter belongs to interval $[0,1]$ is assumed. As the parameter varies from 0 to 1 the solution of the homotopy equation (which is assumed to be a power series of the embedding parameter) varies continuously from a solution, which is easy to find, to the exact solution. The approximate continuous solution is obtained by truncating the series and using a finite number of its terms. Least

Squares Method is used to determine the so-called control-convergence scalars appear in the approximate solution.

The derived algorithms are applied to solve several examples and the obtained solutions are compared with exact solutions. The results confirm the validity of OHAM and reveal that OHAM is effective, simple and explicit.

After the classification of the integral equations we investigate some analytical and numerical methods for solving Fredholm Integral Equations of First and Second Kind such as the Adomian Decomposition Method and the Homotopy Analysis Method (HAM) which we study its convergence.

Introduction

Integral equations are among the most important mathematical topic in both pure and applied mathematics. It plays a very important role in modern science such as modeling and solving numerous problems in engineering, mechanics, and mathematical physics [2], [14].

In fact, Integral equations are encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, and medicine [3] and [4]. Most of the boundary value problems involving differential equations can be converted into problems in integral equations that solved more effectively. They arise as representation formulas for the solutions of differential equations. Indeed, a differential equation can be replaced by an integral equation which incorporates its boundary conditions. As such, each solution of the integral equation automatically satisfies the correspondence boundary conditions. Integral equations also form one of the most useful tools in many branches of pure analysis, such as the theories of functional analysis and stochastic process [18], [2], [14].

Even though, there are certain problems which can be formulated only in terms of integral equations. Since often it is hard or even impossible to find the exact solution of integral equations we resort to approximation and numerical solutions. For the numerical techniques and its related theory see Atkinson [10], Kress [18] and [5]. A computational approach to the solution of integral equations is therefore, an essential branch of scientific inquiry.

Mathematically, an integral equation is an equation in which an unknown function appears under an integral sign. There is a close connection between differential and integral equations, and some problems may be formulated either way.

The most basic type of integral equation is called Fredholm equation of the first type;

$$f(x) = \int_a^b K(x,t)y(t)dt.$$

Here y is an unknown function, f is a known function and K is another known function of two variables; often called the kernel function (we take it to be square integrable function on $[a,b] \times [a,b]$). Note that the limits of integration are constants.

If the unknown function occurs both inside and outside of the integral, it is known as a Fredholm equation of the second type;

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt.$$

The parameter λ is an unknown factor, which plays the same role as the eigenvalue in linear algebra. Description of the theory of integral equations can be found in Kanwal [19], Kress [18] and [14].

As mentioned, it is worth noting that integral equations often do not have an analytical solution, and must be solved numerically. One method to solve integral equations numerically requires discretizing variables and replacing the integral by a quadrature rule

$$\sum_{j=1}^{n} w_j K(s_i, t_j) y(t_j) = f(s_i), \quad i = 1,2,\dots,n.$$

Where w_j are weights that depend on the quadrature rule and s_i, t_j belong to the interval $[a, b]$.

Then we have a system with n equations and n variables. By solving it we get the values of the n variables

$$y(t_1), y(t_2), \dots, y(t_n).$$

In recent years, much work has been carried out by researchers in mathematics and engineering in applying and analyzing novel numerical and semi analytical methods for obtaining solutions of integral equations, in particular of the first kind. Among these are the homotopy analysis method Liao [21] and [1], operational Tau method [12], homotopy perturbation method [22] , Adomian decomposition method [2], quadrature method [7],[17],[23] and [18], and automatic augmented Galerkin algorithms [20].

In this thesis we will present analytic approximate solutions of Fredholm integral equations of first and second kinds. Namely we will consider "Optimal Homotopy Asymptotic Method (OHAM) ". This method is characterized by its convergence criteria which are more flexible than other methods, therefore it is applied successfully to obtain the solution of currently important problems in science. Further, the obtained results had shown its effectiveness, generalization and reliability [13], [16], [26]. Note that, throughout this thesis we will assume that the considered integral equations possess unique solution.

The outline of the thesis is as follows:

Chapter One is a general introduction about integral equations, special kinds of kernels, linearity and homogeneity of integral equations, eigenvalues and eigenfunctions, spaces and operators, the Riemann alternative, and the method of regularization .

Chapter Two introduces three continuous approximate methods for solving linear integral equations of first and second kinds, namely Adomian decomposition method, the homotopy analysis method (HAM) and the optimal homotopy asymptotic method (OHAM). We present the description of the first two methods and apply them to solve some examples. We discuss the convergence of the HAM method and the chapter end with the basic formulation of the OHAM method.

Chapter Three is devoted to Optimal Homotopy Asymptotic Method (OHAM) for solving the linear Fredholm integral equations of the first kind. We formulate the method for general Fredholm integral equation of the first kind. Several examples are proposed to demonstrate the efficiency and the accuracy of this method.

In **Chapter Four** we applied optimal homotopy asymptotic method (OHAM) for solving the linear Fredholm integral equations of the second kind. As in chapter three, we formulate the method for general Fredholm integral equation of the second kind. Two examples are discussed to demonstrate the validity of this technique.

Chapter One

Basic Theory

An integral equation is an equation in which the unknown function $y(x)$ appears inside an integral sign. The most general form of the one dimensional linear integral equation reads

$$h(x)y(x) = f(x) + \lambda \int_{v(x)}^{u(x)} K(x,t)y(t)dt \qquad (1.1)$$

where $v(x)$ and $u(x)$ are the limits of integration, λ is a nonzero real or complex parameter, and $K(x,t)$ is a known function of two variables x and t; called the kernel or the nucleus of the integral equation. The unknown function $y(x)$ that will be determined appears inside the integral sign. In many cases, the unknown function $y(x)$ appears inside and outside the integration as given in Equation (1.1). The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $v(x)$ and $u(x)$ may be both variables, constants, or mixed Wazwaz [2].

An integral equation in which nonlinear operations are performed upon the unknown function is called nonlinear integral equation. If the unknown function depends on n variables, the integral equation is then an n-

dimensional integral equation. Analogously, one can also consider system of integral equations.

In this thesis we will consider only one- dimensional linear integral equations.

1.1 Classification of Integral Equations

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation as follows:

1.1.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Further, the unknown function $y(x)$ may appears only inside the integral equation as

$$f(x) = \lambda \int_a^b K(x,t)y(t)dt. \tag{1.2}$$

This integral equation is called Fredholm integral equation of first kind. However, for Fredholm integral equations of the second kind, the unknown function $y(x)$ appears inside and outside the integral sign and is given as

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt. \tag{1.3}$$

If $f(x) = 0$ then the integral equation is called homogeneous [19].

1.1.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equations, the unknown function $y(x)$ appears only inside the integral sign;

$$f(x) = \lambda \int_{a}^{x} K(x,t) y(t) dt.$$

However, Volterra integral equations of the second kind, the unknown function $y(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$y(x) = f(x) + \lambda \int_{a}^{x} K(x,t) y(t) dt.$$

Again if $f(x) = 0$ then the integral equation is called homogeneous [2].

1.1.3 Singular Integral Equations

When one of the limits of integration or both are infinite or when the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration, the integral equation is called singular integral equation.

For example, the integral equations

$$y(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-t|} y(t) dt,$$

$$f(x) = \lambda \int_{0}^{x} \frac{1}{g(x) - g(t)} y(t)dt \;,\;\; 0 < x < 1$$

where $g(t)$ is strictly monotonically increasing and differentiable function in some interval $0 < t < 1$ and $g'(t) \neq 0$ for every t in the interval, are singular integral equations [2].

1.2 Special Kinds of Kernels

1.2.1 Separable or Degenerate Kernel

A kernel $K(x,t)$ is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only, as

$$K(x,t) = \sum_{i=1}^{n} a_i(x) b_i(t)$$

where the functions $a_i(x)$ and $b_i(t)$are linearly independent function.

1.2.2 Symmetric Kernel

If the kernel satisfies $K(x,t) = K(t,x)$, then it is called symmetric kernel.

1.3 Linearity of Integral Equations

Linear integral equations are of the form

$$y(x) = f(x) + \lambda \int_{v(x)}^{u(x)} K(x,t)y(t)dt$$

where only linear operations are performed upon the unknown function inside the integral sign, that is, the exponent of the unknown inside the integral sign is one, for example

$$y(x) = \frac{1}{2}x - \frac{1}{2} + \int_0^1 (x-t)y(t)dt$$

here the unknown function y appears in a linear form.

If the equation contains nonlinear functions of $y(x)$, such as e^y, $sinh\ y$, $cos\ y$, $ln(1 + y)$, or the unknown function y under the integral sign has exponent other than one, the integral equation is called *nonlinear*, and they are of the form

$$y(x) = f(x) + \lambda \int_{v(x)}^{u(x)} K(x,t,y(t))dt$$

for example

$$y(x) = 1 + \int_0^x (1 + x - t)y^4(t)dt$$

It is important to point out that linear equations, except Fredholm integral equations of the first kind, give a unique solution if such a solution exists.

However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle [2].

1.4 Homogeneity of Integral Equations

Integral equations are classified as *homogeneous* or *inhomogeneous*, as stated before, if the function $f(x)$ in the second kind of Volterra or Fredholm integral equation is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only.

For example

$$y(x) = x + \int_0^1 (x-t)^2 y(t)dt$$

is inhomogeneous because $f(x) = x$, whereas the following equation

$$y(x) = \int_0^x (1 + x - t) y^4(t)dt$$

is homogeneous because $f(x) = 0$.

1.5 Eigenvalues and Eigenfunctions

The concepts of eigenvalues and eigenfunctions are central to the theory of integral equations.

If we write the homogeneous and linear Fredholm equation as

$$y(x) = \lambda \int_a^b K(x,t)y(t)dt,$$

we have the classical eigenvalue or characteristic value problem; λ is the eigenvalue and the nontrivial solution $y(x)$ is itself called the corresponding *eigenfunction* or characteristic function, and their pair is known as the *eigenpair* of the integral equation.

Some major results about positive kernels are as follows [17]:

(1) If the kernel $K(x,t)$ is continuous and positive for $0 \le x, t \le 1$, then the homogenous integral equation of the second kind has a characteristic value $\mu = 1/\lambda$ which is positive, simple, larger in modulus than any other characteristic value.

(2) If $K_1\ (x,t)$ and $K_2\ (x,t)$ are two distinct continuous and positive kernels for $0 \le x, t \le 1$ such that $K_1\ (x,t) \ge K_2\ (x,t)$, then their largest characteristic values satisfy $\mu_1 > \mu_2$.

(3) If $K(x,t)\ \neq 0$ is a continuous symmetric kernel, then it has a nonzero characteristic value.

1.6 Review of Spaces and Operators

Definition 1.1 ***Normed Space***

A normed space X is a vector space with a norm defined on it. *A norm* on a vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties

(i) $\|x\| \geq 0$,

(ii) $\|x\| = 0$ if and only if $x = 0$,

(iii) $\|\alpha x\| = |\alpha|\|x\|$,

(iv) $\|x + y\| \leq \|x\| + \|y\|$, (*Triangle inequality*)

here x and y are arbitrary vectors in X and α is any scalar.

Definition 1.2 ***Metric Space***

A metric space is a pair (X, d), where X is a set and d is a metric on X (a distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have :

(i) d is real-valued, finite and nonnegative.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$.

(iv) $d(x, y) \leq d(x, z) + d(z, y)$. (*Triangle inequality*)

Definition 1.3 ***Cauchy Sequence and Completeness***

A sequence (x_n) in a metric space $X = (X, d)$ is said to be *Cauchy* or *fundamental* if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n > N$.

The space X is said to be *complete* if every Cauchy sequence in X converges in X.

Definition 1.4 ***Banach Space***

A Banach space is a complete normed space .

Definition 1.5 ***Inner Product Space, Hilbert Space***

An inner product on a vector space X is a mapping of $X \times X$ into the scalar field K of X, that is , to every pair of vectors x and y it associates a scalar

$$\langle x, y \rangle,$$

called the *inner product* of x and y, such that for all vectors x, y, z and a scalar α we have

(i) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(iv) $\langle x, x \rangle \geq 0$, $\langle x, x \rangle = 0$ if and only if $x = 0$

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

and a metric on X given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

A vector space X together with an inner product defined on X is called an *inner product space.*

A Hilbert space is a complete inner product space. For example $\mathbb{C}^n$ with the inner product $\langle x,y\rangle = \sum_{j=1}^{n} x_j \overline{y_j}$ is a Hilbert space (over $K = \mathbb{C}$). (Here we mean that $x = (x_1, x_2 \dots, x_n)$ and $y = (y_1, y_2 \dots, y_n)$).We know that $\mathbb{C}^n$ is complete (in the standard norm, which is the one arising from the inner product just given) and so $\mathbb{C}^n$ is a Hilbert space [6].

Definition 1. 6 *L^2-Functions and L^2- Spaces*

The set of all functions $f(x)$ of the real variable x on an interval (a,b) , where $-\infty \leq a < b \leq \infty$, is called the *function space* $L^2(a,b)$, $f(x)$ is called an L^2-*function*, if

$$\int_a^b |f(x)|^2 dx < \infty$$

in the Lebesgue sense. We say that $f(x)$ is *square integrable* on an interval $[a,b]$ if $|f(x)|^2$ is integrable on $[a,b]$.

The set of L^2-functions forms a complete linear vector space , and with an appropriate norm and inner product the space L^2 is a Hilbert space .

The norm $\|f\|_2$ (the L^2 norm) of an L^2-*function* f is defined as

$$\|f\|_2 = \left[\int_a^b |f(x)|^2 dx\right]^{\frac{1}{2}}$$

Definition 1.7 *L^p-Space*

The set of L^p-functions (where $p \geq 1$) generalizes L^2 -space. Instead of square integrable, the measurable function f must be p-integrable, to be in L^p.

On a measurable space X, the L^p norm of a function f is

$$\|f\|_{L^p} = \left[\int_X |f(x)|^p dx\right]^{\frac{1}{p}}$$

The L^p-functions are the functions for which this integral converges.

For $p \neq 2$, the space of L^p-functions is ***a Banach space*** which is not a Hilbert space [17].

In the case where $p = \infty$, we have $L^\infty(D)$ defined as

$$L^\infty(D) = \{f : f \text{ is measurable in } D \text{ and } \|f\|_\infty < \infty\}$$

where

$$\|f\|_\infty = \inf[\sup\{|f(x)|: x \in S\}, S \subset D]$$

Definition 1.8 *Regularity Condition*

A two-dimensional kernel function $K(x,t)$ is an L^2-function if the following conditions are satisfied:

(i) for each set of values of x and t in the square $a \leq x, t \leq b$,

$$\int_a^b\int_a^b |K(x,t)|^2 dx dt < \infty$$

(ii) for each value of x in $a \leq x \leq b$,

$$\int_a^b |K(x,t)|^2 dt < \infty$$

(iii) for each value of t in $a \leq t \leq b$,

$$\int_a^b |K(x,t)|^2 dx < \infty$$

These are called the *regularity conditions* of the kernel $K(x,t)$ [19].

Definition 1.9 ***Operators***

An operator $A: X \to Y$ assigns to every function $f \in X$ a function $Af \in Y$. It is therefore a mapping between two function spaces.

There are many kinds of operators such as differential operator, integral operator, binary operator, Hermitian operator and identity operator.

Definition 1.10 ***Linear Operator*s**

A *linear operator* A is an operator such that

(i) the domain $\mathcal{D}(A)$ of A is a vector space and the range $\mathcal{R}(A)$ lies in a vector space over the same field,

(ii) for all $x, y \in \mathcal{D}(A)$ and a scalar α ,

$$A(x+y) = Ax + Ay,$$

$$A(\alpha x) = \alpha Ax.$$

Definition 1.11 ***Null Space***

The *null space* of a linear operator A is the set of all $x \in \mathcal{D}(A)$ such that $Ax = 0$.

Definition 1.12 ***Injective, Surjective and Bijective Operators***

An operator $A: \mathcal{D}(A) \to Y$ is said to be *injective* or *one-to-one* if different points in the domain have different images, that is, if for any $x_1, x_2 \in \mathcal{D}(A)$,

$$x_1 \neq x_2 \quad \text{implies} \quad Ax_1 \neq Ax_2$$

or equivalently,

$$Ax_1 = Ax_2 \quad \text{implies} \quad x_1 = x_2.$$

An operator A is *surjective* or *onto* provided $A\big(\mathcal{D}(A)\big) = Y$; in other words, if for each $y \in Y,\ y = Ax$ for some $x \in \mathcal{D}(A)$.

An operator A is said to be *bijective* or *bijection* if it is both injective and surjective, see [6].

Definition 1.13 ***Inverse Operators***

Let X and Y be Banach spaces $A: \mathcal{D}(A) \to Y$ an injective linear operator and $\mathcal{D}(A) \subset X$, then there exists a mapping

$$A^{-1}: \mathcal{R}(A) \to \mathcal{D}(A)$$

given by

$$A^{-1}(y) = x$$

$\mathcal{R}(A) \subset Y$; which maps every $y \in \mathcal{R}(A)$ onto that $x \in \mathcal{D}(A)$ for which $Ax = y$.

The mapping A^{-1} is called the inverse of A.

We clearly have

$$A^{-1}Ax = x \quad \text{for all } x \in \mathcal{D}(A)$$

$$A^{-1}Ay = y \quad \text{for all } y \in \mathcal{R}(A).$$

Definition 1.14 ***Continuity***

Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be metric spaces. An $A: X \to Y$ is said to be *continuous mapping at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\tilde{d}(Ax, Ax_0) < \varepsilon \qquad \text{for all } x \text{ satisfying } d(x, x_0) < \delta .$$

A is said to be *continuous mapping* if it is continuous at every point of X.

Assume that X and Y are normed spaces, and $f, f_n \in X$. An operator $A: X \to Y$ is said to be *continuous operator* if

$$\|f_n - f\|_X \to 0$$

implies

$$\|Af_n - Af\|_Y \to 0.$$

Definition 1.15 ***Bounded Linear Operators*** [6]

Let X and Y be normed spaces and $A: \mathcal{D}(A) \to Y$ be a linear operator, where $\mathcal{D}(A) \subset X$. The operator A is said to be *bounded* if there is a real number c such that for all $x \in \mathcal{D}(A)$,

$$\|Ax\| \leq c\|x\|.$$

Definition 1. 16 ***Coercive Operator***

Let H be a Hilbert space and let $B: H \to H$ be a bounded linear operator such that $\langle Bx, x\rangle \geq c\|x\|_H^2$ for all $x \in H$ where c be positive constant, then B called *a coercive operator*, see [18].

Theorem 1.1 A linear operator is continuous if and only if it is bounded .

Proof: Let $A : X \to Y$ be bounded and let (x_n) be a sequence in X with $x_n \to 0, n \to \infty$. Then from $\|Ax_n\| \leq C\|x_n\|$ it follows that $Ax_n \to 0, n \to \infty$. Thus, A is continuous at $x = 0$, and because of the theorem, a linear operator is continuous if it is continuous at one element, it is continuous everywhere in X.

Conversely, let A be continuous and assume there is no $C > 0$ such that $\|Ax\| \leq C\|x\|$ for all $x \in X$. Then there exists a sequence (x_n) in X with $\|x_n\| = 1$ and $\|Ax_n\| \geq n$. Consider the sequence $y_n := \|Ax_n\|^{-1}x_n$. Then $y_n \to 0, n \to \infty$, and since A is continuous, $Ay_n \to A(0) = 0$, $n \to \infty$. This is a contradiction to $\|Ay_n\| = 1$ for all n. Hence, A is bounded, [18].

Theorem 1.2 ***Finite Dimension*** **[6]**

If a normed space X is finite dimensional, then every linear operator on X is bounded .

Theorem 1.3 [6]

Every finite dimensional subspace Y of a normed space X is closed in X.

Proof: Y is a finite dimensional normed vector space. Hence Y is complete.

Let $(y_n) \subset Y$ such that $n \in N$, $\lim_{n\to\infty} y_n = y, y \in X$.

Since Y is complete then $y \in Y$. So Y is closed. □

Theorem 1.4 (***Inverse Operator*****) [6]**

Let X, Y be vector spaces, both real or complex. Let $A: \mathcal{D}(A) \to Y$ be a linear operator with domain $\mathcal{D}(A) \subset X$ and range $\mathcal{R}(A) \subset Y$. Then:

(i) The inverse $A^{-1}: \mathcal{R}(A) \to \mathcal{D}(A)$ exists if and only if

$Ax = 0$ implies $x = 0$.

(ii) If A^{-1} exists, it is a linear operator.

Theorem 1.5 (***Open Mapping Theorem*** **) [16]**

Let X and Y be Banach spaces, and let $A: X \to Y$ be a bijective bounded linear operator. Then a bijective $A^{-1}: Y \to X$ is a bounded linear operator.

Theorem 1.6 (***Geometric Series Theorem*****) [18]**

Let X be a Banach space, and let A be a bounded operator from X into X, with $\|A\| < 1$. Then $I - A: X \to X$ is a bijective operator, $(I - A)^{-1}$ is a bounded linear operator, and

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$$

The series

$$(I-A)^{-1} = \sum_{j=0}^{\infty} A^j$$

is called the *Neumann series*, under the assumption $\| A\| < 1$, it converges in the space of bounded operators from X to X, see [18].

Definition 1.17 ***Compactness***

A metric space X is said to be *compact* if every sequence in X has a convergent subsequence. A subset M of X is said to be *compact* if M is compact considered as a subset of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.

A set $M \subset X$ is called *relatively compact* if every sequence $(x_n) \subset M$ contains a convergent subsequence.

Theorem 1.7 (*Finite Dimensional Domain or Range*) [6]

Let X and Y be normed spaces and $A: X \to Y$ a linear operator. Then:

(a) If A is bounded and $\dim A(X) < \infty$, the operator A is compact.

(b) If $dim\ X < \infty$, the operator A is compact.

Proof : (a) let (x_n) be any bounded sequence in X. Then the inequality $\|Ax_n\| \leq \|A\|\|x_n\|$ shows that (Ax_n) is bounded. Hence (Ax_n) is relatively compact. It follows that (Ax_n) has a convergent subsequence. Since (x_n) was an arbitrary bounded sequence in X, the operator A is compact (A is compact if and only if it maps every bounded sequence (x_n) in X onto a sequence (Ax_n) in Y which has a convergent subsequence).

(b) follows from (a) by noting that $\dim X < \infty$ implies boundedness of A (if a normed space X is finite dimensional, then every linear operator on X is bounded) and $\dim A(X) < \dim X$ ($\dim X = n < \infty \Rightarrow \dim Y < n$).

We mention that an operator A with $\dim A < \infty$ is often called an operator of *finite rank*.

1.7 Ill-Posed Problems

For problems in mathematical physics, in particular for initial and boundary value problems for partial differential equations, Hadamard [9] postulated three properties :

(1) Existence of a solution.

(2) Uniqueness of the solution .

(3) Continuous dependence of the solution on the data .

The third postulate is motivated by the fact that in all applications the data will be measured quantities. Therefore, one wants to make sure that small errors in the data will cause only small errors in the solution. A problem satisfying all three requirements is called well-posed [18].

A Fredholm integral equation of the first kind has the form given in equation (1.2).

These equations are inherently ill-posed problems, meaning that the solution is generally unstable. This ill-posedness makes numerical solutions very difficult, as a small error can lead to an unbounded error [17].

In research of the solution of this class of integral equations, it has been found that the eigenvalues of continuous Fredholm integral operators form a sequence that converges to zero. The solutions of Fredholm integral equations can be expressed in terms of the singular values of integral operators, where the singular values are the reciprocals of the square roots of the eigenvalues. As a result, those singular values that correspond to zero eigenvalues will become arbitrarily large [17].

This is justified as follows: It is known that when $K(x,t)$ is continuous, symmetric, and nondegenerate, the eigenvalue problem

$$\int_a^b K(x,t)y(t)dt = \lambda y(x)$$

has an infinite number of real eigenvalues λ_i such that $\lim_{i\to\infty} \lambda_i = 0$, with associated eigenfunctions $y_i(x)$. The function $y_i(x)$ can be normalized to $\psi_i(x)$ such that $\langle \psi_i(x), \psi_i(x) \rangle = 1$. Thus,

$$\int_a^b K(x,t)\psi_i(t)dt = \lambda_i \psi_i(x) \tag{1.6}$$

Both $f(x)$ and $y(x)$ can be expanded in a series of $\psi_i(x)$ as

$$f(x) = \sum_{i=1}^{\infty} f_i\, \psi_i(x), \quad f_i = \langle f, \psi_i \rangle, \quad y(x) = \sum_{i=1}^{\infty} y_i\, \psi_i(x) \tag{1.7}$$

Substituting (1.6) and (1.7) into equation (1.2) and comparing coefficients of $\psi_i(x)$ on both sides, we get

$$f_i = \lambda_i y_i$$

Hence, $y(x)$ can be expressed as

$$y(x) = \sum_{i=1}^{\infty} \frac{f_i}{\lambda_i} \psi_i(x) \tag{1.8}$$

In practice, we use an approximation of the series (1.8) by truncating it to n terms; thus,

$$[y(x)]_n \approx \sum_{i=1}^{n} \frac{f_i}{\lambda_i} \psi_i(x)$$

Suppose that a small error ϵ_i is introduced in the evaluation of f_i, resulting in an error $\delta[y(x)]_n$ in the values of $[y(x)]_n$. Then

$$[y(x)]_n + \delta[y(x)]_n \approx \sum_{i=1}^{\infty} \frac{\left(f_i + \epsilon_i\right)}{\lambda_i} \psi_i(x), \quad \delta[y(x)]_n \approx \sum_{i=1}^{n} \frac{\epsilon_i}{\lambda_i} \psi_i(x)$$

which in view of the orthonormality of $\psi_i(x)$ gives

$$\langle \delta[y(x)]_n, \delta[y(x)]_n \rangle = \sum_{i=1}^{n} \frac{\epsilon_i^2}{\lambda_i^2}$$

Thus, no matter how small the errors ϵ_i are, the squared error in $y_i(x)$ will grow.

Another source of ill-posedness comes from the Riemann-Lebesgue lemma [24], which states that if any function $f(x)$ is square-integrable, then

$$\lim_{n\to\infty}\int_a^b f(x)\sin\frac{n\pi(x-a)}{b}dx = 0$$

This result implies that if $y(x)$ is a solution of equation (1.2) and $K(x,t)$ is square-integrable, then

$$y(x) + C\sin\frac{n\pi(x-a)}{b} \text{ for } n\to\infty$$

will satisfy equation (1.2) for any value of C.

Another aspect of the problem is that if

$$\int_a^b K(x,t)g(t)dt = 0$$

for any $g \in C[a,b]$ and $y(x)$ is any solution of equation (1.2), then $y(x) + g(x)$ is also a solution of equation (1.2). Moreover, if both $y(x)$ and $K(x,t)$ are continuous but $f(x)$ is not continuous, then equation (1.2) is not solvable [17].

Definition 1.18

Let $A : U \to V$ be an operator from a subset U of a normed space X into a subset V of a normed space Y. The equation

$$A\varphi = f \tag{1.9}$$

is called *well-posed* or *properly posed* if A is bijective and the inverse operator $A^{-1}: V \to U$ is continuous. Otherwise, the equation is called *ill-posed* or improperly posed.

According to this definition we may distinguish three types of ill-posed-ness . If A is not surjective, then equation (1.9) is not solvable for all $f \in V$. If A is not injective, then equation (1.9) may have more than one solution . Finally, if $A^{-1}: V \rightarrow U$ exists but is not continuous, then the solution φ of Equation (1.9) does not depend continuously on the data f. The latter case of instability is the one of primary interest in the study of ill-posed problems. We note that the three properties, in general, are not independent.

For a long time the research on improperly posed problems was neglected, since they were not considered relevant to the proper treatment of applied problems.

Note that the well-posedness of a problem is a property of the operator A together with the solution space X and the data space Y including the norms on X and Y. Therefore, if an equation is ill-posed one could try to restore stability by changing the spaces X and Y and their norms. But, in general, this approach is inadequate, since the spaces X and Y, including their norms are determined by practical needs. In particular, the space Y and its norm must be suitable to describe the measured data and, especially the measurement errors [18].

Example1.1 By the Riemann-Lebesgue lemma, we know

$$\int_0^{\pi} K(x,t)\sin(nt)dt \to 0, \qquad as\ n \to \infty$$

which means that high frequency noise in a solution may be screened out by the integral operator, or it is equivalent to say that a very small change in the data f (in Equation(1.2)) may lead to a large change in the solution y (in Equation(1.2)).

Thus, we know the problem of Fredholm integral equation of first kind is ill-posed.

Theorem 1.8

Let X and Y be normed spaces and let $A : X \to Y$ be a compact linear operator. Then the equation of the first kind $A\varphi = f$ is improperly posed if X is not of finite dimension [18] .

Proof : We prove the theorem by contradiction.

To this end let X be of infinite dimension and assume that A^{-1} exists and continuous. That is $A\varphi = f$ is properly posed. Then A^{-1} is bounded. Since A is compact then $I = AA^{-1} : X \to X$ is compact. Hence X is of finite dimension, contradiction. Therefore $A\varphi = f$ is improperly posed if X is not of finite dimension. □

1.8 The Fredholm Alternative

In the early 1900s, Ivar Fredholm gave necessary and sufficient conditions for the solvability of a large class of Fredholm integral equations of the second kind; and with these results, he then was able to give much more general existence theorems for the solution of boundary value problems.

The basic theorems of the general theory of integral equations, which were first presented by Fredholm, correspond to the basic theorems of linear algebraic systems [10].

In this section we state and prove the most important result of Fredholm.

Theorem 1.9, [17], If λ is regular value , then both (1.3), and its transposed equation

$$y(x) = f(x) + \lambda \int_a^b K(t,x)y(t)dt$$

are solvable for any free term $f(x)$, and both equations have unique solutions. The associated homogeneous equation, with $f(x) = 0$, has only the trivial solution.

Theorem 1.10, [17], The nonhomogeneous Fredholm integral equation of the second kind (1.3) is solvable if and only if the free term $f(x)$ satisfies the condition

$$\int_a^b f(x)y_j^*(x)dx = 0, \quad j = 1,2,\ldots,n$$

where $\{y_j^*(x)\}$ denotes the complete set of linearly independent solutions of the associated transposed equation .

Theorem 1.11, [17], If λ is an eigenvalue, then both the homogeneous Fredholm integral equation of the second kind (1.3) and the transposed equation have nontrivial finitely many solution.

Theorem 1.12, [17], The Fredholm integral equation of the second kind (1.3) has at most countably many eigenvalues whose only possible accumulation point is the point at infinity.

For more details see [17], [19].

Theorem 1.13 (*Fredholm Alternative*), [10]

Let X be a Banach space, and let $K: X \to X$ be compact. Then the equation $(\lambda - K)x = y$, $\lambda \neq 0$ has a unique solution $x \in X$ if and only if the homogeneous equation $(\lambda - K)z = 0$ has only the trivial solution $z = 0$. In such a case, the bijective operator $\lambda - K: X \to X$ has a bounded inverse $(\lambda - K)^{-1}$.

Proof : We remark that the theorem is a generalization of the following standard result for finite dimensional vector spaces X: If A a matrix of order n, with $X = \mathbb{R}^n$ or $\mathbb{C}^n$, the linear system $Ax = y$ has a unique solution $x \in X$ for all $y \in X$ if and only if the homogeneous linear system $Ax = 0$ has only the zero solution $x = 0$.

(i) Let K be of finite-rank and bounded, and let $\{u_1, u_2, \dots, u_n\}$ be a basis for $Range\ (K)$. Rewrite the equation $(\lambda - K)x = y$ as

$$x = \frac{1}{\lambda}[y + Kx] \tag{1.10}$$

If this equation has a unique solution $x \in X$, then

$$x = \frac{1}{\lambda}[y + c_1 u_1 + c_2 u_2 + \cdots + c_n u_n] \tag{1.11}$$

For some uniquely determined set of constants $c_1, c_2, \dots, c_n$.

By substituting (1.11) into equation (1.10), we have

$$\lambda\left\{\frac{1}{\lambda}y + \frac{1}{\lambda}\sum_{i=1}^{n} c_i u_i\right\} - \frac{1}{\lambda}Ky - \frac{1}{\lambda}\sum_{j=1}^{n} c_j K u_j = y$$

Multiply by λ, and then simplify to obtain

$$\lambda \sum_{i=1}^{n} c_i u_i - \sum_{j=1}^{n} c_j K u_j = Ky \tag{1.12}$$

Using the basis $\{u_i\}$ for $Range(K)$, write

$$Ky = \sum_{i=1}^{n} \gamma_i u_i\,, \qquad K u_j = \sum_{i=1}^{n} a_{ij} u_i\,, \qquad 1 \le j \le n$$

The coefficients $\{\gamma_i\}$ and $\{a_{ij}\}$ are uniquely determined. Substituting into (1.12) and rearranging,

$$\sum_{i=1}^{n}\left\{\lambda c_i - \sum_{j=1}^{n} a_{ij}c_j\right\}u_i = \sum_{i=1}^{n}\gamma_i u_i$$

By the independence of the basis elements u_i, we obtain the linear system

$$\lambda c_i - \sum_{j=1}^{n} a_{ij}c_j = \gamma_i\,, \qquad 1 \le i \le n \tag{1.13}$$

Claim: This linear system and the equation $(\lambda - K)x = y$ are completely equivalent in their solvability with (1.11) furnishing a one-to-one correspondence between the solutions of the two of them.

We have shown above that if x is a solution of $(\lambda - K)x = y$, then $(c_1, c_2, \dots, c_n)$ is a solution of (1.13). In addition, suppose x_1 and x_2 are distinct solutions of $(\lambda - K)x = y$. Then

$$Kx_1 = \lambda x_1 - y \qquad \text{and} \qquad Kx_2 = \lambda x_2 - y\,, \qquad \lambda \neq 0\,,$$

are also distinct vectors in $Range(K)$, and thus the associated vectors of coordinates $\left(c_1^{(1)}, c_2^{(1)}, \dots, c_n^{(1)}\right)$ and $\left(c_1^{(2)}, c_2^{(2)}, \dots, c_n^{(2)}\right)$,

$$Kx_i = \sum_{k=1}^{n} c_k^{(i)} u_k\,, \ \ i = 1{,}2$$

must also be distinct.

For the converse statement, suppose $(c_1, c_2, \dots, c_n)$ is a solution of (1.13). Define a vector $x \in X$ by using (1.11), and then check whether this x satisfies the integral equation (1.10):

$$(\lambda - K)x = \lambda\left\{\frac{1}{\lambda}y + \frac{1}{\lambda}\sum_{i=1}^{n} c_i u_i\right\} - \frac{1}{\lambda}Ky - \frac{1}{\lambda}\sum_{j=1}^{n} c_j K u_j$$

$$= y + \frac{1}{\lambda}\left\{\lambda\sum_{i=1}^{n} c_i u_i - Ky - \sum_{j=1}^{n} c_j K u_j\right\}$$

$$= y + \frac{1}{\lambda}\left\{\sum_{i=1}^{n} \lambda c_i u_i - \sum_{i=1}^{n} \gamma_i u_i - \sum_{j=1}^{n} c_j \sum_{i=1}^{n} a_{ij} u_i\right\}$$

$$= y + \frac{1}{\lambda}\sum_{i=1}^{n}\left\{\lambda c_i - \gamma_i - \sum_{j=1}^{n} a_{ij} c_j\right\} u_i$$

$$= y$$

because $\lambda c_i - \gamma_i - \sum_{j=1}^{n} a_{ij} c_j = 0$ for $i = 1,2,\dots,n$

Since of the linear independence of the basis vectors $\{u_1, u_2, \dots, u_n\}$, distinct coordinate vectors $(c_1, c_2, \dots, c_n)$ lead to distinct solution vectors x in (1.11). This complete the proof of the above claim.

Now consider the Fredholm alternative theorem for $(\lambda - K)x = y$ with this finite rank K. Suppose the bijective operator $\lambda - K: X \to X$. Then, $Null(\lambda - K) = \{0\}$.

For the converse, assume $(\lambda - K)z = 0$ has only the solution $z = 0$, and we want to show that $(\lambda - K)x = y$ has a unique solution for every $y \in X$. Consider the associated linear system (1.13). It can be shown to have a unique solution for all right-hand sides $(\gamma_1, \gamma_2, \dots, \gamma_n)$ by showing that the homogeneous linear system has only the zero solution. The latter is done by means of the equivalence of the homogeneous linear system to the homogeneous equation $(\lambda - K)z = 0$, which implies $z = 0$. But since (1.13) has a unique solution, so must $(\lambda - K)x = y$, and is given by (1.11).

Note that $(\lambda - K)^{-1}$ is bounded by the Open Mapping Theorem .

(ii) Assume now that $\|K - K_n\| \to 0$, with K_n finite rank and bounded. Rewrite $(\lambda - K)x = y$ as

$$[\lambda - (K - K_n)]x = y + K_n x, \quad n \geq 1 \tag{1.14}$$

Pick an index $m > 0$ for which

$$\|K - K_m\| < |\lambda|$$

and fix it. By the Geometric Series Theorem

$$Q_m \equiv \left[\lambda - (K - K_m)\right]^{-1}$$

exists and is bounded, with

$$\|Q_m\| \leq \frac{1}{\left|\lambda\right| - \|K - K_m\|}$$

The equation (1.14) can now be written in the equivalent form

$$x - Q_m K_m x = Q_m y \qquad (1.15)$$

The operator $Q_m K_m$ is bounded and finite rank. The boundedness follows from that of Q_m and K_m. To show it is finite rank, let $Range(K_m) = Span\{u_1, u_2, \dots, u_m\}$. Then

$$Range(Q_m K_m) = Span\,\{Q_m u_1, Q_m u_2, \dots, Q_m u_m\}$$

a finite dimensional space.

Assume $\left(\lambda - K\right)z = 0$ implies $z = 0$. This yields

$$(I - Q_m K_m)z = 0 \text{ implies } z = 0$$

But from part (i), this says $(I - Q_m K_m)x = w$ has a unique solution x for every $w \in X$, and in particular, for $w = Q_m y$ as in (1.15). By the equivalence of (1.15) and $\left(\lambda - K\right)x = y$, we have $\left(\lambda - K\right)x = y$ is uniquely solvable for every $y \in X$. □

1.9 The Method of Regularization

The method of regularization consists of replacing ill-posed problem by well-posed problem. The method of regularization transforms the linear Fredholm integral equation of the first kind (Equation 1.2) to the approximation Fredholm integral equation

$$\mu y_\mu(x) = f(x) - \int_a^b K(x,t) y_\mu(t) dt, \qquad (1.16)$$

where μ is small positive parameter. It is clear that (1.16) is a Fredholm integral equation of the second kind that can be written

$$y_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu} \int_a^b K(x,t) y_\mu(t) dt, \qquad (1.17)$$

Moreover, the solution $y_\mu(x)$ of Equation (1.16) converges to the solution $y(x)$ of Equation (1.2) as $\mu \to 0$ according to the following lemma [2]:

Lemma 1.1

Suppose that the integral operator of (1.2) is continuous and coercive in the Hilbert space where $f(x), y(x)$, and $y_\mu(x)$ are defined, then:

(i) $|y_\mu|$ is bounded independently of μ, and

(ii) $|y_\mu(x) - y(x)| \to 0$ when $\mu \to 0$.

In summary, by combining the method of regularization with any of the methods used for solving Fredholm integral equation of the second kind, we can solve Fredholm integral equation of the first kind. The method of regularization transforms the first kind to a second kind.

The exact solution $y(x)$ of (1.2) can thus be obtain by

$$y(x) = \lim_{\mu \to 0} y_\mu(x).$$

Chapter Two

Continuous Approximation Methods for Solving Linear Fredholm Integral Equations of First and Second Kind

2.1 Introduction

A variety of analytic and numerical methods have been used to handle Fredholm integral equations. Among many traditional commonly used methods are: The direct computation method, the successive approximations method, and converting Fredholm equation to an equivalent boundary value methods.

In this chapter we study three of recently developed numerical methods that used to obtain continuous approximate solutions to Fredholm integral equations, namely the Adomian Decomposition Method (ADM) [2], the Homotopy Analysis Method (HAM) [1] and the Optimal Homotopy Analysis Method [25] (OHAM). Basically, each of these methods assumes that the solution is given as an infinite series, usually converges rapidly to the exact solution, whose components are functions that determined recursively. The continuous approximate solution is then obtained by truncating the infinite series. In the last section we reformulate the method of OHAM. The derivation is done for a general operator equation, in which the operator is decomposed into a sum of two parts; an easier part (linear), used to find an initial guess to the exact solution, and a nonlinear one. A homotopy equation is then applied, using the two parts of the operator, to derive various approximate functions (solutions in a sense to be explained) used as basis to construct the required approximate solution. In this chapter we also discuss the analysis of convergence of HAM ([1], [15]) and present several examples where we applied ADM and HAM methods. Examples using OHAM are discussed in Chapters 3 and 4. Much of the details presented in Sections 2, 3 and 4 come from [1] and [2].

2.2 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian [2]. The ADM is based on decomposing the unknown function $y(x)$ as an infinite series of components as

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \qquad (2.1)$$

where the components $y_n(x), n \geq 0$ will be determined recursively. The Adomian decomposition method concerns itself with finding the components $y_0(x), y_1(x), y_2(x), \ldots$ individually. The determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.

To set up the recurrence relation, we substitute (2.1) into the Fredholm integral equation (1.3) to get

$$\sum_{n=0}^{\infty} y_n(x) = f(x) + \lambda \int_a^b K(x,t)(\sum_{n=0}^{\infty} y_n(t))\, dt \qquad (2.2)$$

The zeroth component $y_0(x)$ is identified by all terms that are not included under the integral sign. This means that the components $y_j(x), j \geq 0$ of the unknown function $y(x)$ are completely determined by setting the recurrence relation

$$y_0(x) = f(x), \quad y_{n+1}(x) = \lambda \int_a^b K(x,t) y_n(t) dt\,, \;\; n \geq 0, \qquad (2.3)$$

In view of (2.3), the components $y_0(x), y_1(x), y_2(x), y_3(x), \ldots$ are completely determined. As a result, the solution $y(x)$ of the Fredholm integral equation (1.3) is readily obtained in a series form by using the series assumption in (2.1).

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown that if an exact solution exists for the problem, then the obtained series converges very rapidly to that exact solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series, see [2] and the references therein. However, for concrete problems, where an exact solution is not obtainable, a truncated number of terms are usually used for numerical purposes. The more components we use the higher accuracy we obtain.

Example 2.1

Solve the following Fredholm integral equation

$$y(x) = x + e^{x} - \frac{4}{3} + \int_{0}^{1} ty(t)dt \qquad (2.4)$$

Substituting the decomposition series (2.1) into both sides of (2.4) gives

$$\sum_{n=0}^{\infty} y_n(x) = x + e^{x} - \frac{4}{3} + \int_{0}^{1} t \sum_{n=0}^{\infty} y_n(t)\, dt$$

Proceeding as stated above, we arrive at the following recurrence relation:

$$y_0(x) = x + e^{x} - \frac{4}{3}\,, \qquad y_{k+1}(x) = \int_{0}^{1} ty_k(t)dt\,, \qquad k \geq 0.$$

Consequently, we obtain $y_1(x)$, $y_2(x)$, ... as

$$y_1(x) = \int_{0}^{1} ty_0(t)dt = \int_{0}^{1} t\left(t + e^{t} - \frac{4}{3}\right) dt = \frac{2}{3}\,,$$

$$y_2(x) = \int_0^1 ty_1(t)dt = \frac{2}{3}\int_0^1 tdt = \frac{1}{3},$$

$$y_3(x) = \int_0^1 ty_2(t)dt = \frac{1}{3}\int_0^1 tdt = \frac{1}{6},$$

$$y_4(x) = \int_0^1 ty_3(t)dt = \frac{1}{6}\int_0^1 tdt = \frac{1}{12},$$

and so on. Using (2.1) we get the series solution

$$y(x) = x + e^x - \frac{4}{3} + \frac{2}{3}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right). \qquad (2.5)$$

Notice that the sum of infinite geometric series at the right side is given by

$$S = \frac{1}{1-\frac{1}{2}} = 2.$$

Hence the series solution (2.5) converges to the exact solution

$$y(x) = x + e^x.$$

Example 2.2

Solve the following Fredholm integral equation

$$y(x) = e^x + 1 - e + \int_0^1 y(t)dt \qquad (2.6)$$

Substituting the decomposition series (2.1) into both sides of (2.6) gives

$$\sum_{n=0}^{\infty} y_n(x) = e^x + 1 - e + \int_0^1 \sum_{n=0}^{\infty} y_n(t)\,dt$$

Proceeding as stated above, we arrive at the following recurrence relation:

$$y_0(x) = e^x + 1 - e\,, \qquad y_{k+1}(x) = \int_0^1 y_k(t)dt\,, \qquad k \geq 0.$$

Consequently, we obtain $y_1(x), y_2(x), y_3(x), \ldots$ as

$$y_1(x) = 0\,, y_2(x) = 0, y_3(x) = 0,$$

and so on. Using (2.1) we get the series solution

$$y(x) = e^x + 1 - e + (0 + 0 + 0 + \cdots) \qquad (2.7)$$

Hence the series solution (2.7) is the exact solution

$$y(x) = e^x + 1 - e$$

2.3 The Homotopy Analysis Method (HAM)

In this method, as stated before, the solution is considered as the summation of an infinite series. The (HAM) and the (OHAM) that will be presented in the next section, are both based on a fundamental concept from topology and differential geometry, namely homotopy. Roughly speaking, by means of the (HAM), one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure the convergence of solution series. The method possesses great freedom in choosing initial approximations and auxiliary linear operators.

In this section, we present an iterative scheme based on the (HAM) for the first and second kind of linear Fredholm integral equations [1].

2.3.1 Description of HAM Method

We concern ourselves with an operator equation $N[y] = 0$,

where $y(x)$ is unknown real-valued function to be determined.

Let $y_0(x)$ denote an initial guess of the exact solution (x); $h \neq 0$, H $(such\ that\ H(x) \neq 0)$, and L $(such\ that\ L[0] = 0)$ denote auxiliary parameter, function and linear operator respectively. Then using $p \in [0, 1]$ as an embedding parameter, we construct such a homotopy

$$(1 - p)L[y(x,p) - y_0(x)] - phH(x)N[y(x,p)]$$
$$= \widehat{H}\ [y(x,p);\ y_0(x), H(x), h, p]. \qquad (2.8)$$

It should be emphasized that there is a great freedom in choosing $y_0(x)$, L, h, $H(x)$.

Enforcing the homotopy (2.8) to be zero, i.e.

$$\hat{H}\ [y(x,p);\ y_0(x), H(x), h, p] = 0,$$

Thus, we have the so-called zero-order deformation equation

$$(1 - p)L[y(x,p) - y_0(x)] = phH(x)N[y(x,p)]. \qquad (2.9)$$

When $p = 0$ the Equation (2.9) leads to

$$y(x,0) = y_0(x), \qquad (2.10)$$

and if $p = 1$, since $h \neq 0$ and $H(x) \neq 0$, it gives

$$y(x,1) = y(x). \qquad (2.11)$$

Thus, as the embedding parameter p increases from 0 to 1, $y(x,p)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$. Such continuous variation is called deformation in homotopy.

By Taylor's theorem, $y(x,p)$ can be expanded in a power series of p as

$$y(x,p) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)p^m\ , \qquad (2.12)$$

where

$$y_m(x) = \frac{1}{m!}\frac{\partial^{\mathrm{m}} y(x,p)}{\partial p^m}|_{p=0} \qquad (2.13)$$

If $y_0(x)$, L, h, and $H(x)$ are properly chosen so that the power series (2.12) of $y(x,p)$ converges at $p = 1$. Then, the power series converges to the exact solution. That is

$$y(x) = y(x,1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \qquad (2.14)$$

Define the vector

$$Y_n(x) = (y_0(x), y_1(x), y_2(x), \dots, y_n(x)). \qquad (2.15)$$

According to Equation (2.13), the governing equation of $y_m(x)$ can be derived from the zero-order deformation equation (2.9). Differentiating the zero-order deformation equation (2.9) m times with respective to p and then dividing by m! and finally setting $p = 0$, we have the so-called m^{th}-order deformation equation

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)\mathcal{R}_m\big(Y_m(x)\big),$$

$$y_m(0) = 0\,, \qquad (2.16)$$

where

$$\mathcal{R}_m\big(Y_{m-1}(x)\big) = \frac{1}{(m-1)!}\frac{\partial^{\mathrm{m}-1} N[y(x,p)]}{\partial p^{m-1}}|_{p=0}\,, \qquad (2.17)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

Note that the high-order deformation equation (2.16) is governing by the linear operator L, and the term $\mathcal{R}_m\big(Y_m(x)\big)$ can be expressed simply by (2.17) for any nonlinear operator N.

Therefore, $y_m(x)$ can be easily obtained, especially by means of computational software such as MATLAB. The solution $y(x)$ given by the above approach is dependent of $L, h, H(x)$, and $y_0(x)$. Thus, unlike all previous analytic techniques, the convergence region and rate of solution series given by the above approach might not be uniquely determined. If $\sum_{m=0}^{n} y_m(x)$ tends uniformly to a limit as $n \to \infty$, then this limit is the required solution.

It is worth to noting that the governing equation for the component $y_m(x)$, i.e. Equation (2.16), can be obtained directly by substituting from Equation (2.14) into Equation (2.9) using the power series expansion

$$N(y(x,p)) = N_0(y_0(x)) + \sum_{m=1}^{\infty} N_m(y_0(x), y_1(x), y_2(x), \dots, y_m(x))p^m$$

$$= N_0(y_0(x)) + \sum_{m=1}^{\infty} N_m(Y_m(x))p^m ,$$

and equating the coefficients of equal powers of p. The result is the establishment of the simple recursion formula

$$L[y_1(x)] = hH(x)N_0\big(y_0(x)\big),$$

$$L[y_m(x)] - L[y_{m-1}(x)] = hH(x)N_{m-1}\big(Y_{m-1}(x)\big), \quad m = 2,3,\ldots.$$

which is Equation (2.16).

2.3.2 Linear Integral Equations of the First Kind

Consider the linear integral equation (1.2). Let

$$N[y] = f(x) - \lambda \int_a^b K(x,t)y(t)dt = 0.$$

Equation (2.17) implies that

$$\mathcal{R}_m\big(Y_{m-1}(x)\big) = (1-\chi_m)f(x) - \lambda \int_a^b K(x,t)y_{m-1}(t)dt.$$

Thus the m^{th}-order deformation equation (2.16) reduces to

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)\left[(1-\chi_m)f(x) - \lambda \int_a^b K(x,t)y_{m-1}(t)dt\right] \quad (2.18)$$

Choose $Ly = y$, $y_0(x) = f(x)$, $h = 1$ and $H(x) = 1$ and substitute into (2.18) to get the following simple iteration formula for $y_m(x)$

$$y_0(x) = f(x) \tag{2.19}$$

$$y_m(x) = y_{m-1}(x) - \lambda \int_a^b K(x,t) y_{m-1}(t)dt, \qquad m = 1,2, \dots$$

Using the notation

$$K_1(x,t) = K(x,t), \qquad K_m(x,t) = \int_a^b K(x,\tau) K_{m-1}(\tau,t)dt, m = 2,3, \dots,$$

the solution $y(x)$ given by Equation (1.2) becomes

$$y(x) = f(x) + \sum_{m=1}^{\infty} y_m(x)$$

$$= f(x) + \sum_{m=1}^{\infty} \left(f(x) + \sum_{j=1}^{m} \binom{m}{j} (-\lambda)^j \int_a^b K_j(x,t) f(t)dt \right). \tag{2.20}$$

This series converges uniformly provided $||I - \lambda B||_\infty < 1$ where I is the identity operator, the operator B is defined by $B(y) = \int_a^b K(x,t)y(t)dt$ and $||.||$ is the maximum norm, see [1],[18].

To find an approximate continuous solution, we truncate the series solution in (2.14) and use only a finite number of terms. That is, the approximate solution will be $\mathrm{y}^{(m)} = \sum_{k=0}^{m} y_m(x)$.

Example 2.3

Consider the Fredholm integral equation of the first kind

$$\frac{1}{4}e^{-x} = \int_0^{\frac{1}{4}} e^{t-x} y(t)dt.$$

For which the exact solution is $y(x) = e^{-x}$. Let $f(x) = \frac{1}{4}e^{-x}$ and $K(x,t) = e^{t-x}$, we begin with

$$y_0(x) = \frac{1}{4}e^{-x}.$$

Its iteration formulation reads

$$y_m(x) = y_{m-1}(x) - \int_0^{\frac{1}{4}} e^{t-x} y_{m-1}(t)dt, \quad m = 1,2,\ldots$$

This in turn gives

$$y_1(x) = \frac{3}{16}e^{-x}, y_2(x) = \frac{9}{64}e^{-x}, y_3(x) = \frac{27}{256}e^{-x},$$

and so on. Consequently, the approximate solution is given by

$$y^{(m)}(x) = e^{-x}\left(\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + \cdots\right), m = 0,1,2,\ldots$$

that converges to the exact solution

$$y(x) = e^{-x}$$

2.3.3 Linear Integral Equations of the Second Kind

Here we consider the linear Fredholm integral equation of second kind (1.3) and we rewrite it in the form

$$N[y] = y(x) - f(x) - \lambda \int_a^b K(x,t)y(t)dt = 0.$$

Then, Equation (2.17) leads to

$$\mathcal{R}_m\big(Y_{m-1}(x)\big) = y_{m-1}(x) - (1-\chi_m)f(x) - \lambda \int_a^b K(x,t)y_{m-1}(t)dt,$$

and the m^{th}-order deformation equation (2.16) reduces to

$$L[y_m(x) - \chi_m y_{m-1}(x)]$$

$$= hH(x)\left[y_{m-1}(x) - (1-\chi_m)f(x) - \lambda \int_a^b K(x,t)y_{m-1}(t)dt\right] \quad (2.21)$$

Substituting $y_0(x) = f(x)$, $L(y) = y$, $h = -1$ and $H(x) = 1$, we obtain the following iterative formula

$$y_0(x) = f(x) \quad (2.22)$$

$$y_m(x) = \lambda \int_a^b K(x,t)y_{m-1}(t)dt\,.$$

Therefore the solution $y(x)$ becomes

$$y(x) = \sum_{m=0}^{\infty} y_m(x)$$

$$= f(x) + \lambda \int_a^b K_1(x,t)f(t)dt + \lambda^2 \int_a^b K_2(x,t)f(t)dt + \cdots$$

$$= f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,t)f(t)dt\,,$$

where

$$K_1(x,t) = K(x,t), K_m(x,t) = \int_a^b K(x,\tau)K_{m-1}(\tau,t)dt\,, m = 2,3,\ldots.$$

Note that this solution is the one we obtain when the method of successive approximations is used to solve (1.3). Again this series solution converges uniformly if $\|\lambda B\| < 1$ where the operator B is defined as in the previous subsection, See [1] and [18].

Example 2.4

Consider the linear Fredholm integral equation of the second kind

$$y(x) \ = \ cos(x) \ + \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\, y(t)dt$$

Note that $f(x) = cos(x)$ and $K(x,t) = sin(x)$.

Starting with $y_0(x) \ = \ cos(x)$. Its iteration formula reads

$$y_m(x) = \frac{1}{2}\int_0^{\frac{\pi}{2}} sin(x)\, y_{m-1}(t)dt\,, \qquad m = 1,2,\ldots$$

This in turn gives

$$y_1(x) = \frac{1}{2}sin(x)\,, y_2(x) = \frac{1}{4}sin(x)\,, y_3(x) = \frac{1}{8}sin(x),$$

$$y_4(x) = \frac{1}{16}sin(x).$$

and so on. Consequently, the approximate solution is given by

$$y^{(m)}(x) = \cos\ (x) + \sin(x)\,(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots)\,, \qquad m = 0,1,2,\ldots$$

that converges to the exact solution

$$y(x) = \cos(x) + sin(x)$$

2.4 Convergence of the Homotopy Analysis Method

In this section we state and prove two theorems about the convergence of the series solution of the Fredholm integral equations of first and second kind that we derived in the previous two sections using HAM method.

Theorem 2.1

As long as the series

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)$$

convergence, where $y_m(x)$ is governed by Equation

$$L[y_m(x) - \chi_m y_{m-1}(x)]$$
$$= hH(x)[(1 - \chi_m)f(x) - \lambda \int_a^b K(x,t) y_{m-1}(t)dt$$

it must be the exact solution of the integral Fredholm integral equation (1.2).

Proof. If the series (2.14) converges, we can write

$$S(x) = \sum_{m=0}^{\infty} y_m(x),$$

and it holds that

$$\lim_{m\to\infty} y_m(x) = 0.$$

We can verify that

$$\sum_{m=1}^{n}[y_m(x) - \chi_m y_{m-1}(x)] = y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1})$$

$$= y_n(x), \tag{2.23}$$

which gives us according to (2.23),

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}[y_m(x) - \chi_m y_{m-1}(x)] = \lim_{n\to\infty} y_n(x) = 0. \tag{2.24}$$

Furthermore, using (2.24) and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = L\sum_{m=1}^{\infty}[y_m(x) - \chi_m y_{m-1}(x)] = 0$$

$$= hH(x)\sum_{m=1}^{\infty}\mathcal{R}_{m-1}\big(Y_{m-1}(x)\big) = 0.$$

Since $h \neq 0$ and $H(x) \neq 0$ we have

$$\sum_{m=1}^{\infty}\mathcal{R}_{m-1}\big(Y_{m-1}(x)\big) = 0. \tag{2.25}$$

Now,

$$\sum_{m=1}^{\infty}\mathcal{R}_{m-1}\big(Y_{m-1}(x)\big) = \sum_{m=1}^{\infty}(1 - \chi_m)f(x) - \lambda\int_a^b K(x,t)y_{m-1}(t)dt$$

$$= f(x) - \lambda \int_a^b K(x,t)\left(\sum_{m=1}^{\infty} y_{m-1}(t)\right)dt$$

$$= f(x) - \lambda \int_a^b K(x,t)\left(\sum_{m=0}^{\infty} y_m(t)\right)dt$$

$$= f(x) - \lambda \int_a^b K(x,t)S(t)dt = 0.$$

Therefore

$$f(x) = \lambda \int_a^b K(x,t)S(t)dt,$$

and so, $S(x)$ must be the exact solution of Equation (1.2). □

Theorem 2.2

As long as the series

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x).$$

convergence, where $y_m(x)$ is governed by

$$L[y_m(x) - \chi_m y_{m-1}(x)]$$

$$= hH(x)\left[y_{m-1}(x) - (1-\chi_m)f(x) - \lambda \int_a^b K(x,t)y_{m-1}(t)dt\right]$$

it must be the exact solution of the integral Fredholm integral equation (1.3).

Proof : The proof is similar to the proof of Theorem (2.1), except now

$$\sum_{m=1}^{\infty} \mathcal{R}_{m-1}\big(Y_{m-1}(x)\big)$$

$$= \sum_{m=1}^{\infty} y_{m-1}(x) - (1-\chi_m)f(x) - \lambda \int_a^b K(x,t)y_{m-1}(t)dt$$

$$= \sum_{m=1}^{\infty} y_{m-1}(x) - f(x) - \lambda \int_a^b K(x,t)\left(\sum_{m=1}^{\infty} y_{m-1}(t)\right)dt$$

$$= \sum_{m=0}^{\infty} y_{m}(x) - f(x) - \lambda \int_a^b K(x,t)\left(\sum_{m=0}^{\infty} y_{m}(t)\right)dt$$

$$= S(x) - f(x) - \lambda \int_a^b K(x,t)S(t)dt = 0.$$

Therefore

$$S(x) = f(x) + \lambda \int_a^b K(x,t)S(t)dt,$$

and so, $S(x)$ must be the exact solution of Equation (1.3). □

2.5 Basic Formulation of Optimal Homotopy Asymptotic Method (OHAM)

Consider the operator equation of the form, [13],

$$A\big(y(x)\big) + f(x) = 0, \tag{2.26}$$

where A is an operator, $y(x)$ is unknown function, and $f(x)$ a known analytic function. Assume that A can be decomposed into two operators L (linear) and N (nonlinear) such that

$$A = L + N.$$

According to OHAM, one can construct an optimal homotopy map $y(x,p)\colon \Omega \times [0,1] \to \Re,$

that satisfies the homotopy equation

$$\hat{H}\,(y(x,p),p) = (1-p)\big\{L\big(y(x,p)\big) + f(x)\big\}$$

$$-H(p)\big\{A\big(y(x,p)\big) + f(x)\big\} = 0, \tag{2.27}$$

where the auxiliary $H(p)$ function is nonzero for $p \neq 0$; $H(0) = 0$

and $p \in [0,1]$ is an embedding parameter.

Equation (2.27) is called *optimal homotopy equation* or *zero-order homotopy equation*.

Note that if $p = 0$, we get $y(x, 0) = y_0(x)$, and when $p = 1$,we obtain $y(x, 1) = y(x)$; the exact solution. Thus, as p varies from 0 to 1, the solution $y(x, p)$ arrives from $y_0(x)$ at $y(x)$,where $y_0\,(x)$ is the solution of Equation (2.27) when we substitute $p = 0$, i.e $y_0\,(x)$ satisfies

$$L\big(y_0\,(x)\big) + f(x) = 0. \qquad (2.28)$$

Next, we choose the auxiliary function $H(p)$ to be the power series in p;

$H(p) = pc_1 + p^2 c_2 +$ ..., where c_i are constants for all $i, i = 1,2, \ldots$

To get an approximate solution, we expand $y(x, p, c_1, c_2, \ldots)$ by Taylor's series, [8], about p in the following manner:

$$y(x, p, c_1, c_2, \ldots) = y_0\,(x) + \sum_{k=1}^{\infty} y_k\,(x, c_1, \ldots, c_k) p^k, \qquad (2.29)$$

Substituting from Equation (2.29) into Equation (2.27) and equating the coefficients of like powers of p, we obtain the following $zeroth$ to the kth order problems governing equations of

$$y_0(x), y_1(x, c_1), \ldots, y_k(x, c_1, \ldots, c_k):$$

$$L\big(y_0(x)\big) + f(x) = 0,$$

$$L\big(y_1(x, c_1)\big) - L\big(y_0(x)\big) = c_1 N_0\big(y_0(x)\big),$$

$$L\big(y_2(x,c_1,c_2)\big) - L\big(y_1(x,c_1)\big)$$

$$= c_2 N_0\big(y_0(x)\big) + c_1\big[L\big(y_1(x,c_1)\big) + N_1\big(y_0(x), y_1(x,c_1)\big)\big],$$

$$L\big(y_k(x,c_1,\dots,c_k)\big) - L\big(y_{k-1}(x,c_1,\dots,c_{k-1})\big) =$$

$$c_k N_0\big(y_0(x)\big)$$

$$+\sum_{i=1}^{k-1} c_i[L\big(y_{k-i}(x,c_1,\dots,c_{k-i})\big)$$

$$+ N_{k-i}\big(y_0(x), y_1(x,c_1),\dots,y_{k-i}(x,c_1,\dots,c_{k-i})\big)] \qquad (2.30)$$

for $k = 2,3,\dots$ where N_{k-i} are the coefficient of p^{k-i} in the expansion of $N\big(y(x,p,c_1,c_2,\dots)\big)$ about the embedding parameter p;

$$N\big(y(x,p,c_1,c_2,\dots)\big)$$

$$= N_0\big(y_0(x)\big) + \sum_{k=1}^{\infty} N_k(y_0(x), y_1(x,c_1),\dots,y_k(x,c_1,\dots,c_k)p^k$$

Note that the governing equations are linear and can be easily solved for y_k , $k \geq 0$.

It has been observed that the convergence of the series in Equation (2.29) depends upon the auxiliary constants $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3, \dots$.

If it is convergent at $p = 1$, one get

$$y(x,1,\mathrm{c}_1,\mathrm{c}_2,\dots) = y_0(x) + \sum_{k=1}^{\infty} y_k(x,c_1,\dots,c_k). \qquad (2.31)$$

This equation is the source of the required *approximate solutions*.

Substituting from Equation (2.29) into

$$L(y(x)) + f(x) + N\big(y(x)\big) = 0,$$

leads to the following residual formula

$$R(x, c_1, c_2, \dots) = L\big(y(x, c_1, c_2, \dots)\big) + f(x) + N\big(y(x, c_1, c_2, \dots)\big).$$

If $R(x, c_1, c_2, \dots) = 0$ then $y(x, c_1, c_2, \dots)$ is the exact solution of the problem.

For the determination of auxiliary constants $c_i; \quad i = 1,2, \dots, m,$

there are different methods. One method is the Least Squares;

$$J(c_1, c_2, \dots, c_m) = \int_a^b R^2(x, c_1, c_2, \dots, c_m)dx$$

where $[a, b]$ is an interval depending on the given problem. The unknown constants c_i can be identified from the conditions

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_m} = 0. \qquad (2.32)$$

With these constants known, the approximate solution is well-determined as

$$y^{(m)}(x) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2) + \cdots + y_m(x, c_1, \dots, c_m).$$

Chapter Three

Optimal Homotopy Asymptotic Method (OHAM) for Solving the Linear Fredholm Integral Equations of the First Kind

3.1 Introduction

In this chapter we aim to reconstruct optimal homotopy asymptotic method (OHAM) for solving Fredholm integral equations of first kind. This method was proposed by Marinca et al. [25] and characterized by its convergence criteria which are more flexible than other methods.

We consider the following general form of the linear Fredholm integral equation of the first kind

$$f(x) = \lambda \int_a^b K(x,t)y(t)dt \qquad (3.1)$$

where *a* and *b* are constants, the functions $K(x,t)$, $f(x)$ are known functions and λ is a nonzero parameter .

We present and discuss several examples to demonstrate the ability of the method to solve linear Fredholm integral equations of first kind. The results show that the method is very effective and simple.

3.2 Application of OHAM to the Linear Fredholm Integral Equations of the First Kind

In this section we formulate the optimal homotopy asymptotic method (OHAM) for solving the linear Fredholm integral equations of the first kind considered in Equation (3.1). Using OHAM, we can obtain a family of homotopy equations as follows:

$$(1-p)L\big(y(x,p)\big)$$
$$= H(p)\big[L\big(y(x,p)\big) + f(x) + N\big(y(x,p)\big)\big] \qquad (3.2)$$

where $p \in [0\ ,1]$ is an embedding parameter $y(x,p)$ is the unknown function and $H(p)$ is a nonzero auxiliary function for

$p \neq 0$ with $H(0) = 0$.

The auxiliary function $H(p)$ is defined by means of a finite power series as

$$H(P) = \sum_{j=1}^{m} c_j p^j$$

where c_j are auxiliary constants with j= 1,2,… . As explained in Section 2.5, we define the operator L to be

$$L\big(y(x,p)\big) = y(x,p)$$

and take the operator N as

$$N\big(y(x,p)\big) = -y(x,p) - \lambda \int_a^b K(x,t)y(t,p)dt.$$

Note that $L\big(y(x)\big) + f(x) + N\big(y(x)\big) = 0$ reduced to Equation (3.1).

Upon substituting L and N in Equation (3.2) we get

$$(1-p)y(x,p) = H(p)\left[f(x) - \lambda \int_a^b K(x,t)y(t,p)dt\right]. \tag{3.3}$$

Setting $p = 0$ in (3.3) we obtain

$$L\big(y(x,0)\big) = 0, \tag{3.4}$$

and substituting $p = 1$,we get

$$H(1)\left[f(x) - \lambda \int_a^b K(x,t)y(t,1)dt\right] = 0.$$

Since $H(1) \neq 0$, we get

$$\left[f(x) - \lambda \int_a^b K(x,t)y(t,1)dt\right] = L\big(y(x,1)\big) + f(x) + N\big(y(x,1)\big)$$

$$= 0 \tag{3.5}$$

Let $y(x,0) = y_0(x)$ and $y(x,1) = y(x)$ be the solutions of (3.4), and (3.5), respectively.

Note that by construction, L is chosen such that $y_0(x)$ is easy to find, while $y(x)$ is the required solution that we need to determine.

To obtain an approximate solution to $y(x)$, we use Taylor series expansion (in powers of p) as follows:

$$y(x, p, c_1, c_2, \dots) = y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m)p^m, \qquad (3.6)$$

If the series (3.6) converges when $p = 1$, then one has

$$y(x, 1, c_1, c_2, \dots) = y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m) \qquad (3.7)$$

That is, it converges to the exact solution. In terms of auxiliary constants $c_1, c_2, \dots$ Equation (3.3) reads

$$(1-p)y(x, p, c_1, c_2, \dots) = H(p)\left[f(x) - \lambda \int_a^b K(x,t)y(t, p, c_1, c_2, \dots)dt\right].$$

Substitute from Equation (3.6) we obtain

$$(1-p)\left[y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m)p^m\right] =$$

$$\sum_{j=1}^{m} c_j p^j \left[f(x) - \lambda \int_a^b K(x,t)\left(y_0(t) + \sum_{m=1}^{\infty} y_m(t, c_1, \dots, c_m)p^m\right)dt\right] \qquad (3.8)$$

Equating the coefficients of like powers of p in Equation (3.8), as we did in Section (2.5), we get the following $zeroth$ to ith order approximations

$$y_0(x) = 0\ ,$$

$$y_1(x, c_1) = c_1 f(x) - c_1 \lambda \int_a^b K(x,t) y_0(t) dt,$$

$$y_2(x, c_1, c_2) = y_1(x, c_1) + c_2 f(x)$$

$$- c_1 \lambda \int_a^b K(x,t) y_1(t, c_1) dt - c_2 \lambda \int_a^b K(x,t) y_0(t) dt,$$

$$y_3(x, c_1, c_2, c_3) = y_2(x, c_1, c_2) + c_3 f(x) - c_1 \lambda \int_a^b K(x,t) y_2(t, c_1, c_2) dt$$

$$-c_2 \lambda \int_a^b K(x,t) y_1(t, c_1) dt - c_3 \lambda \int_a^b K(x,t) y_0(t) dt,$$

$$y_4(x, c_1, c_2, c_3, c_4) =$$

$$y_3(x, c_1, c_2, c_3) + c_4 f(x) - c_1 \lambda \int_a^b K(x,t) y_3(t, c_1, c_2, c_3) dt$$

$$-c_2 \lambda \int_a^b K(x,t) y_2(t, c_1, c_2) dt - c_3 \lambda \int_a^b K(x,t) y_1(t, c_1) dt,$$

$$y_i(x, c_1, \dots, c_i) = y_{i-1}(x, c_1, \dots, c_{i-1}) + c_i f(x)$$

$$-\sum_{k=1}^{i} c_k \lambda \int_a^b K(x,t) y_{i-k}(t, c_1, \dots, c_{i-k}) dt, \qquad i = 1,2, \dots.$$

Note that these approximations are given recursively and depends on the auxiliary constants c_j , $j = 1,2, \dots$. Further, these approximations are the solutions of the corresponding $zeroth$ to ith order problems given in Equation (2.28) in Section (2.5). Notice also that by $y_0(.)$ we mean $y_0(x)$ or $y_0(t)$.

Finding the constants $c_1, c_2, c_3, \dots$ leads to the mth-order approximation

$$y^{(m)}(x, c_1, \dots, c_m) = y_0(x) + \sum_{k=1}^{m} y_k(x, c_1, \dots, c_m) \tag{3.9}$$

Now, as mentioned before, the exact solution $y(x)$ satisfies the operator equation

$$L\big(y(x)\big) + N\big(y(x)\big) + f(x) = 0 \tag{3.10}$$

Hence substituting the approximation given by (3.9) into (3.10), we obtain the residual equation:

$$R(x, c_1, \dots, c_m) = f(x) - \lambda \int_a^b K(x,t) y^{(m)}(t, c_1, \dots, c_m) dt$$

It is important to note that if $R(x, c_1, \dots, c_m) = 0$, then $y^{(m)}(x, c_1, \dots, c_m)$ will be the exact solution, see [11].

For the determination of the auxiliary constants $c_1, c_2, \dots, c_m$, there are different methods (Galerkin's Method , Ritz Method , Least Squares Method and Collection Method, see [13]) that can be used.

As stated in section (2.5) we will apply Least Squares Method.

Consider the mapping

$$J(c_1, \dots, c_m) = \int_a^b R^2(x, c_1, \dots, c_m)dx. \qquad (3.11)$$

The required scalars $c_1, \dots, c_m$ are the one that minimize J. Thus we compute

$$\frac{\partial J}{\partial c_j} = 2\int_a^b R(x, c_1, \dots, c_m)\frac{\partial R}{\partial c_j}dx. \qquad (3.12)$$

Then the unknown constants $c_1, c_2, \dots, c_m$ can be identified by solving the algebraic system

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_m} = 0. \qquad (3.13)$$

With these constants known (namely convergence-control constants $c_1, \dots, c_m$, the approximate solution of order m (Equation (3.10)) is well-determined, see [25].

3.3 Numerical Examples and Discussion

In this section, we will present some examples of linear Fredholm integral equations of first kind. The aim is to demonstrate the efficiency of the algorithm based on OHAM technique when applied to approximate solutions of linear Fredholm integral equations of first kind.

Example 3.3.1

Consider the following linear Fredholm integral equation of first kind [11],

$$\frac{1}{2}\sin(x) = \int_0^{\frac{\pi}{2}} \frac{2}{\pi}\sin(x)\sin(t)\, y(t)dt \qquad (3.14)$$

Then $f(x) = \frac{1}{2}\, sin(x)$. To apply OHAM, we take

$$L\big(y(x,p)\big) = y(x,p),$$

$$N\big(y(x,p)\big) = -y(x,p) - \lambda\int_a^b K(x,t)y(t)dt$$

$$= -y(x,p) - \int_0^{\frac{\pi}{2}} \frac{2}{\pi}\sin(x)\sin(t)\, y(t)dt$$

In this case Equation (3.8) becomes

$$(1-p)\left[y_0(x)+\sum_{m=1}^{\infty} y_m(x,c_1,\dots,c_m)p^m\right]=$$

$$\sum_{j=1}^{m} c_j p^j\left[\frac{1}{2}sin(x)-\int_0^{\frac{\pi}{2}}\frac{2}{\pi}sin(x)\,sin(t)\left(y_0(t)+\sum_{m=1}^{\infty} y_m(t,c_1,\dots,c_m)p^m\right)dt\right]$$

….(3.15)

Equating corresponding powers of p in both sides leads to the following

$zero, second,$ and $third$ - order solutions, respectively:

$$y_0(x)=0,$$

$$y_1(x,c_1)=c_1f(x)-c_1\lambda\int_a^b K(x,t)y_0(t)dt$$

$$=\frac{1}{2}c_1sin(x)-c_1\int_0^{\frac{\pi}{2}}\frac{2}{\pi}sin(x)\,sin(t)\,y_0(t)dt\,,$$

$$y_2(x,c_1,c_2)=y_1(x,c_1)+c_2f(x)$$

$$-c_1\lambda\int_a^b K(x,t)y_1(t,c_1)dt-c_2\lambda\int_a^b K(x,t)y_0(t)dt$$

$$=\frac{1}{2}c_1sin(x)-c_1\int_0^{\frac{\pi}{2}}\frac{2}{\pi}sin(x)\,sin(t)\,y_0(t)dt+\frac{1}{2}c_2sin(x)$$

$$-c_1\int_0^{\frac{\pi}{2}}\frac{2}{\pi}sin(x)\,sin(t)\,y_1(t)dt-c_2\int_0^{\frac{\pi}{2}}\frac{2}{\pi}sin(x)\,sin(t)\,y_0(t)dt,$$

or we write

$$y_0(x) = 0,$$

$$y_1(x, c_1) = \frac{1}{2} c_1 \, sin(x),$$

$$y_2(x, c_1, c_2) = \frac{1}{2} c_1 \, sin(x) + \frac{1}{2} c_2 \, sin(x) - \frac{1}{4} c_1^2 \, sin(x).$$

With the aid of (3.7), we have

$$y(x, c_1, c_2, \dots) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2) + \cdots$$

or equivalently

$$y(x, c_1, c_2 \dots,) = c_1 \, sin(x) + \frac{1}{2} c_2 \, sin(x) - \frac{1}{4} c_1^2 \, sin(x) + \cdots \quad (3.16)$$

According to Equation (3.9), the second order approximation of (3.11) reads

$$y^{(2)}(x, c_1, c_2) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2)$$

$$= c_1 \, sin(x) + \frac{1}{2} c_2 \, sin(x) - \frac{1}{4} c_1^2 \, sin(x)$$

To find the constants c_1 and c_2 appear in this approximation we apply the Least Squares Method. First notice that the residual equation corresponding to this approximate solution is

$$R(x, c_1, c_2) = \frac{1}{2} sin(x)$$

$$- \int_0^{\frac{\pi}{2}} \frac{2}{\pi} sin(x) \, sin(t) \left(c_1 \, sin(t) + \frac{1}{2} c_2 \, sin(t) - \frac{1}{4} c_1^2 \, sin(t) \right) dt$$

$$= \frac{1}{2} sin(x) - \frac{1}{2} sin(x) \left(c_1 + \frac{1}{2} c_2 - \frac{1}{4} c_1^2 \right).$$

Therefore, the associated function defined by Equation (3.11) reads

$$J(c_1, c_2) = \int_0^{\frac{\pi}{2}} R^2(x, c_1, c_2) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} \sin(x) - \frac{1}{2} \sin(x) \left(c_1 + \frac{1}{2} c_2 - \frac{1}{4} {c_1}^2 \right) \right)^2 dx \, .$$

Hence the algebraic system corresponding to minimizing J;

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = 0$$

leads to $c_1 = 2$ and $c_2 = 0$. Therefore, the approximate solution becomes

$$y^{(2)}(x, c_1, c_2) = \sin(x).$$

It is the exact solution that we can find by applying the method of regularization as follows:

Equation (3.14) can be transformed into

$$y_\mu(x) = \frac{1}{2\mu}\sin(x) - \frac{1}{\mu}\int_0^{\pi/2} \frac{2}{\pi}\sin(x)\sin(t) y_\mu(t)\, dt \qquad (3.17)$$

This Fredholm integral equation is a linear of second kind and will be solved by the direct computation method. To achieve this we write Equation (3.17) as

$$y_\mu(x) = \left(\frac{1}{2\mu} - \frac{\alpha}{\mu}\right)\sin(x), \qquad (3.18)$$

where

$$\alpha = \int_0^{\pi/2} \frac{2}{\pi}\sin(t) y_\mu(t)\, dt. \qquad (3.19)$$

To determine α , we substitute (3.18) into (3.19) to get

$$\alpha = \int_0^{\pi/2} \frac{2}{\pi}\sin(t)\left(\frac{1}{2\mu} - \frac{\alpha}{\mu}\right)\sin(t)\, dt$$

$$= \frac{1}{4\mu + 2}.$$

Substituting instead of α in (3.18), we obtain

$$y_\mu(x) = \frac{2}{4\mu + 2} sin(x).$$

Now the exact solution $y(x)$ of Equation (3.17) is given as

$$y(x) = \lim_{\mu \to 0} y_\mu(x) = \lim_{\mu \to 0} \frac{1}{2\mu + 1} sin(x) = sin(x).$$

Example 3.3.2

In this example we consider the linear Fredholm integral equation of first kind

$$\frac{1}{4}x^2 = \int_0^1 \frac{5}{2}x^2 t^2 y(t)dt. \qquad (3.20)$$

First, we use regularization method to find the exact solution. According to regularization method, Equation (3.20) can be transformed into

$$y_\mu(x) = \frac{1}{4\mu}x^2 - \frac{1}{\mu}\int_0^1 \frac{5}{2}x^2 t^2\, y_\mu(t)\, dt \qquad (3.21)$$

or equivalently

$$y_\mu(x) = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)x^2 \qquad (3.22)$$

where

$$\alpha = \int_0^1 \frac{5}{2}t^2\, y_\mu(t)\, dt. \qquad (3.23)$$

To determine α, we substitute (3.22) into (3.23) to get

$$\alpha = \int_0^1 \frac{5}{2}t^2\left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)t^2 dt \ = \ \frac{1}{8\mu + 4}.$$

Use α in (3.22) gives $y_\mu(x) = \frac{1}{4\mu+2}x^2$.

Hence the exact solution is

$$y(x) = \lim_{\mu\to 0} y_\mu(x) = \lim_{\mu\to 0} \frac{1}{4\mu + 2}x^2 = \frac{1}{2}x^2.$$

Next, we apply OHAM method. Thus, let

$$f(x) = \frac{1}{4}x^2,$$

$$L\big(y(x,p)\big) = y(x,p),$$

$$N\big(y(x,p)\big) = -y(x,p) - \int_0^1 \frac{5}{2}x^2t^2y(t)dt.$$

Then Equation (3.8) reads

$$(1-p)\left[y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m)p^m\right] =$$

$$\sum_{j=1}^{m} c_j p^j \left[\frac{1}{4}x^2 - \int_0^1 \frac{5}{2}x^2t^2\left(y_0(t) + \sum_{m=1}^{\infty} y_m(t, c_1, \dots, c_m)p^m\right)dt\right]$$

Similar to the previous example, the corresponding $zero,$ first, $second,$ and $third$-order approximations are

$$y_0(x) = 0$$

$$y_1(x, c_1) = c_1 f(x) - c_1 \lambda \int_a^b K(x,t) y_0(t) dt$$

$$= \frac{1}{4} c_1 x^2,$$

$$y_2(x, c_1, c_2)$$

$$= y_1(x, c_1) + \frac{1}{4} c_2 x^2$$

$$- c_1 \int_0^1 \frac{5}{2} x^2 t^2 y_1(t, c_1) dt - c_2 \int_0^1 \frac{5}{2} x^2 t^2 y_0(t) dt$$

$$= \frac{1}{4} c_1 x^2 - \frac{1}{8} c_1^2 x^2 + \frac{1}{4} c_2 x^2,$$

$$y_3(x, c_1, c_2, c_3) = y_2(x, c_1, c_2) + \frac{1}{4} c_3 x^2$$

$$- \sum_{k=1}^{3} c_k \int_0^1 \frac{5}{2} x^2 t^2 y_{3-k}(t, c_1, \dots, c_{3-k}) dt$$

$$= \frac{1}{4} c_1 x^2 + \frac{1}{4} c_2 x^2 + \frac{1}{4} c_3 x^2 - \frac{1}{4} c_1^2 x^2 + \frac{1}{16} c_1^3 x^2 - \frac{1}{4} c_1 c_2 x^2.$$

With the aid of Equation (3.9) and using $y_0(x), y_1(x, c_1), y_2(x, c_1, c_2)$ and $y_3(x, c_1, c_2, c_3)$, the third approximate solution of (3.20) reads

$$y^{(3)}(x,c_1,c_2,c_3)=\frac{3}{4}c_1x^2+\frac{1}{2}c_2x^2+\frac{1}{4}c_3x^2-\frac{3}{8}c_1^2x^2+\frac{1}{16}c_1^3x^2$$

$$-\frac{1}{4}c_1c_2x^2. \qquad (3.24)$$

To calculate the constants c_1, c_2 and c_3, we use Least Squares Method. The corresponding residual equation is

$$R(x,c_1,c_2,c_3)$$

$$=L\left(y^{(3)}(x,c_1,c_2,c_3)\right)-y^{(3)}(x,c_1,c_2,c_3)+2f(x)$$

$$-\int_0^1\frac{5}{2}x^2t^2y^{(3)}(t,c_1,c_2,c_3)dt$$

$$=\frac{1}{4}x^2-\int_0^1\frac{5}{2}x^2t^2\left(\frac{3}{4}c_1t^2+\frac{1}{2}c_2t^2+\frac{1}{4}c_3t^2-\frac{3}{8}c_1^2t^2+\frac{1}{16}c_1^3t^2\right.$$

$$\left.-\frac{1}{4}c_1c_2t^2\right)dt$$

$$=\frac{1}{4}x^2-\frac{3}{8}c_1x^2-\frac{1}{4}c_2x^2-\frac{1}{8}c_3x^2+\frac{3}{16}c_1^2x^2-\frac{1}{32}c_1^3x^2+\frac{1}{8}c_1c_2x^2.$$

We compute

$$\frac{\partial R}{\partial c_1}=-\,\frac{3}{8}x^2+\frac{3}{8}c_1x^2-\frac{3}{32}c_1^2x^2+\frac{1}{8}c_2x^2,$$

$$\frac{\partial R}{\partial c_2}=-\frac{1}{4}x^2+\frac{1}{8}c_1x^2,$$

$$\frac{\partial R}{\partial c_3}=-\frac{1}{8}x^2.$$

Therefore, the partial derivatives of J are

$$\frac{\partial J}{\partial c_1} = -\frac{c_2 {c_1}^3}{160} + \frac{3{c_1}^5}{2560} + \frac{c_1 {c_2}^2}{160} - \frac{3{c_1}^4}{256} + \frac{3c_2{c_1}^2}{80} + \frac{3c_3{c_1}^2}{640} - \frac{c_2c_3}{160} + \frac{3{c_1}^3}{64}$$

$$-\frac{3c_1c_2}{40} - \frac{3c_1c_3}{160} - \frac{3{c_1}^2}{32} - \frac{{c_2}^2}{80} + \frac{c_2}{20} + \frac{3c_1}{32} + \frac{3c_3}{160} - \frac{3}{80},$$

$$\frac{\partial J}{\partial c_2} = \frac{c_2{c_1}^2}{160} - \frac{{c_1}^4}{640} - \frac{c_1c_2}{40} - \frac{c_1c_3}{160} + \frac{{c_1}^3}{80} - \frac{3{c_1}^2}{80} + \frac{c_1}{20} + \frac{c_2}{40} + \frac{c_3}{80}$$

$$-\frac{1}{40},$$

$$\frac{\partial J}{\partial c_3} = -\frac{c_1c_2}{160} + \frac{{c_1}^3}{640} - \frac{3{c_1}^2}{320} + \frac{c_2}{80} + \frac{c_3}{160} + \frac{3c_1}{160} - \frac{1}{80}.$$

Equating these equations with zero and solving the resulted system gives

$$c_1 = 3.0000960978587,$$

$$c_2 = 3.0000295823085,$$

$$c_3 = 2.7501626580529.$$

Hence, the final approximate solution reads

$$y^{(3)}(x, c_1, c_2, c_3) = 0.4999792149115\, x^2.$$

Note that the corresponding absolute error is

$$|y_{exact} - y_{OHAM}| = |0.5x^2 - 0.4999792149115\, x^2|$$

$$= 0.0000207850885\, x^2,$$

where $y_{OHAM} = y^{(3)}(x, c_1, c_2, c_3)$. Hence the maximum norm of error is

$$\|y_{exact} - y_{OHAM}\|_\infty = \max\{|(y_{exact} - y_{OHAM})(x)| : 0 \leqslant x \leqslant 1\}$$

$$= 0.2 \times 10^{-4}.$$

We compute also the L^2 norm. It is

$$\|y_{exact} - y_{OHAM}\|_{L^2} = \left[\int_0^1 |(y_{exact} - y_{OHAM})(x)|^2\right]^{\frac{1}{2}} = 0.9 \times 10^{-5}.$$

The following figure illustrates the exact solution and OHAM solution for this example.

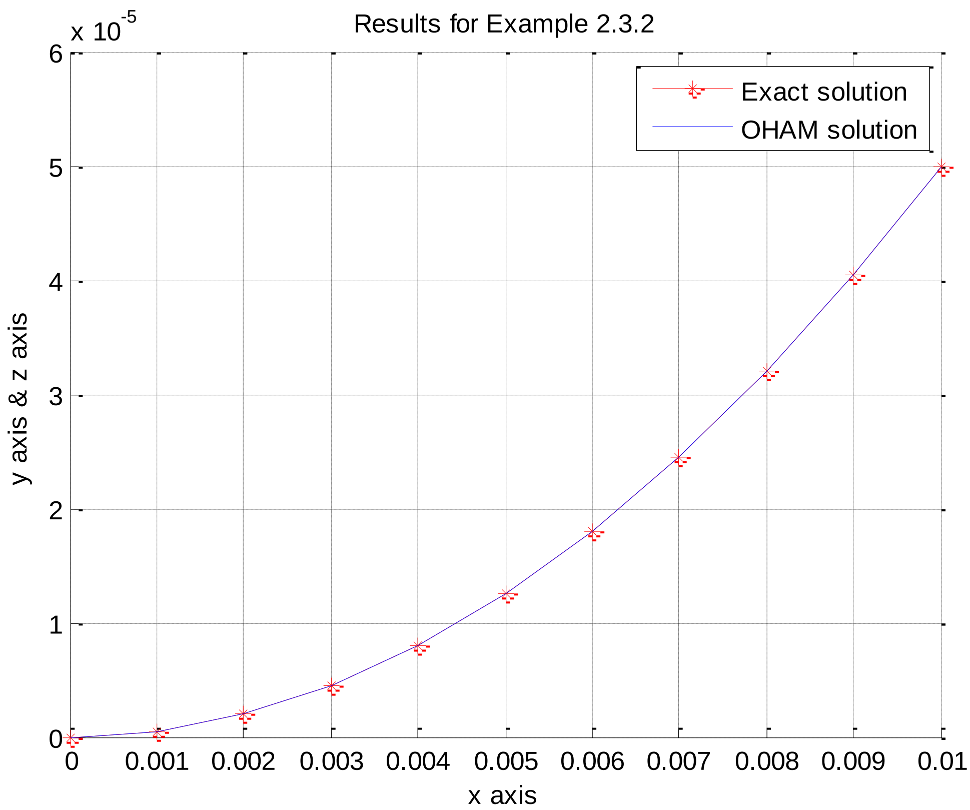

Figure 3.1

Now we compute the second order approximate solution of (3.20) $y^{(2)}(x, c_1, c_2)$. According to (3.9) we have

$$y^{(2)}(x, c_1, c_2) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2).$$

Hence

$$y^{(2)}(x, c_1, c_2) = \frac{1}{2}c_1x^2 + \frac{1}{4}c_2x^2 - \frac{1}{8}c_1^2x^2.$$

Again to determine the constants c_1 and c_2, we use the Least Squares Method. The details are as follows:

$$R(x, c_1, c_2) = f(x) - \int_0^1 \frac{5}{2}x^2t^2y^{(2)}(t, c_1, c_2)dt$$

$$= \frac{1}{4}x^2 - \int_0^1 \frac{5}{2}x^2t^2\left(\frac{1}{2}c_1t^2 + \frac{1}{4}c_2t^2 - \frac{1}{8}c_1^2t^2\right)dt$$

$$= \frac{1}{4}x^2 - \frac{1}{4}c_1x^2 + \frac{1}{16}c_1^2x^2 - \frac{1}{8}c_2x^2$$

Therefore

$$\frac{\partial R}{\partial c_1} = -\frac{1}{4}x^2 + \frac{1}{8}c_1x^2$$

$$\frac{\partial R}{\partial c_2} = \frac{1}{8}x^2$$

Hence the partial derivatives of J are

$$\frac{\partial J}{\partial c_1} = -\frac{1}{80} + \frac{3}{80}c_1 - \frac{3}{160}{c_1}^2 + \frac{1}{320}{c_1}^3 + \frac{1}{80}c_2 - \frac{1}{160}c_1c_{2,}$$

$$\frac{\partial J}{\partial c_2} = \frac{1}{80} - \frac{1}{80}c_1 + \frac{1}{320}{c_1}^2 - \frac{1}{160}c_2.$$

Equating these equations with zero and solving the resulting system gives $c_1 = 0$ and $c_2 = 2$. Therefore the approximate solution is

$$y^{(2)}(x, c_1, c_2, c_3) = \frac{1}{2}\, x^2.$$

It is the exact solution.

Example 3.3.3

In this example we consider the linear Fredholm integral equation of first kind

$$\frac{1}{2}x^2 = \int_0^1 2x^2 t y(t) dt. \tag{3.25}$$

Analogous to the previous examples, we use the method of regularization to find the exact solution of this equation. It is $y(x) = x^2$.

To apply OHAM method, note that

$f(x) = \frac{1}{2}x^2$, $\qquad L\big(y(x,p)\big) = y(x,p),$ and

$$N\big(y(x,p)\big) = -y(x,p) - \int_0^1 2x^2 t y(t)dt.$$

Proceeding as in the previous two examples, we get

$$y_0(x) = 0,$$

$$y_1(x, c_1) = \frac{1}{2} c_1 x^2,$$

$$y_2(x, c_1, c_2) = \frac{1}{2} c_1 x^2 + \frac{1}{2} c_2 x^2 - \frac{1}{4} c_1^2 x^2,$$

$$y_3(x, c_1, c_2, c_3)$$
$$= \frac{1}{2} c_1 x^2 + \frac{1}{2} c_2 x^2 + \frac{1}{2} c_3 x^2 - \frac{1}{2} c_1^2 x^2 + \frac{1}{8} c_1^3 x^2 - \frac{1}{2} c_1 c_2 x^2.$$

Hence the third order approximation reads

$$y^{(3)}(x, c_1, c_2, c_3) = \frac{3}{2} c_1 x^2 + c_2 x^2 + \frac{1}{2} c_3 x^2 - \frac{3}{4} c_1^2 x^2 + \frac{1}{8} c_1^3 x^2$$
$$- \frac{1}{2} c_1 c_2 x^2. \tag{3.26}$$

Applying Least Squares Method gives

$$c_1 \approx 3.0000295823085, c_2 \approx 3.0000960978587,$$
$$c_3 \approx 2.7501626580529.$$

Hence

$$y^{(3)}(x, c_1, c_2, c_3) = 0.9999584298229x^2.$$

Note that the corresponding absolute error is

$$|y_{exact} - y_{OHAM}| = |x^2 - 0.9999584298229x^2|$$

$$= 0.415701771 \times 10^{-4} x^2,$$

where again $y_{OHAM} = y^{(3)}(x, c_1, c_2, c_3)$. Hence the maximum norm of error is

$$\|y_{exact} - y_{OHAM}\|_\infty = \max\{|(y_{exact} - y_{OHAM})(x)|: 0 \leqslant x \leqslant 1\}$$
$$= 0.4 \times 10^{-4}.$$

We compute also the L^2 norm. It is

$$\|y_{exact} - y_{OHAM}\|_{L^2} = \left[\int_0^1 |(y_{exact} - y_{OHAM})(x)|^2\right]^{\frac{1}{2}} = 0.19 \times 10^{-4}.$$

The following figure illustrates the exact solution and OHAM solution of this example.

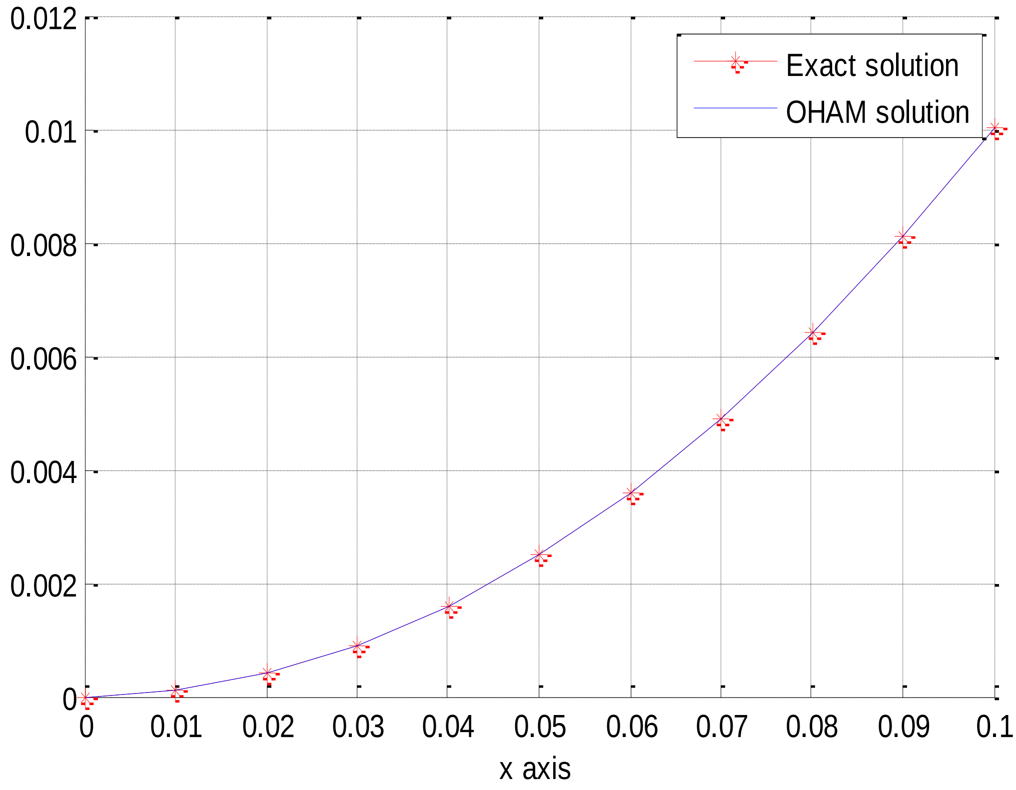

Figure 3.2

Example 3.3.4

In this example we consider the Fredholm integral equation

$$x = \int_0^1 xty(t)dt. \tag{3.27}$$

Note that; $f(x) = x, L\big(y(x,p)\big) = y(x,p)$, and

$$N\big(y(x,p)\big) = -y(x,p) - \lambda \int_0^1 K(x,t)y(t,p)dt.$$

So Equation (3.8) reads

$$(1-p)\left[y_0(x)+\sum_{m=1}^{\infty} y_m(x,c_1,\dots,c_m)p^m\right]=$$

$$\sum_{j=1}^{m} c_j p^j \left[x-\int_0^1 xt\left(y_0(t)+\sum_{m=1}^{\infty} y_m(t,c_1,\dots,c_m)p^m\right)dt\right]. \qquad (2.28)$$

By equating the coefficients of equal powers of p in Equation (2.28), we get to the following recurrence relation

$$y_0(x)=0,$$

$$y_i(x,c_1,\dots,c_i)=y_{i-1}(x,c_1,\dots,c_{i-1})+c_i x$$

$$-\sum_{k=1}^{i} c_k \int_a^b xt y_{i-k}(t,c_1,\dots,c_{i-k})dt, \qquad i=1,2,\dots.$$

Consequently, we obtain $y_0(x)=0$ and $y(x,c_1)= c_1 x$ and so on.

According to Equation (3.9), the first order approximate solution reads

$$y^{(1)}(x,c_1)=y_0(x)+y_1(x,c_1)=c_1 x.$$

Proceeding as before, we applied Least Squares Method to determine c_1. First we notice that the residual equation is

$$R(x,c_1)=x-\int_0^1 xty^{(1)}(t,c_1)dt=x-\frac{1}{3}c_1 x.$$

Therefore the associated function defined by Equation (3.11) reads

$$J(c_1) = \int_0^1 R^2(x, c_1)dx = \frac{1}{3} - \frac{2}{9}c_1 + \frac{1}{27}c_1^2$$

and

$$\frac{\partial J}{\partial c_1} = -\frac{2}{9} + \frac{2}{27}c_1.$$

Identifying this equation by zero, we get $c_1 = 3$. Hence $y^{(1)}(x, c_1) = 3x$.

This is the exact solution. We obtain it using regularization method as used in previous examples.

As a conclusion, we note that the proposed examples demonstrate the accuracy of the solutions obtained by OHAM when compared with exact solutions even with low order approximations. It reveals also that this method is a simple and an efficient method. Note also that a MATLAB procedure was used to determine the auxiliary constants in Examples 3 and 4.

Chapter Four

Optimal Homotopy Asymptotic Method (OHAM) for Solving the Linear Fredholm Integral Equations of the Second Kind

4.1 Introduction

In the previous chapter, we have reconstructed the optimal homotopy asymptotic method (OHAM) to solve Fredholm's linear integral equations of first kind. Motivated by the "good" results, we, in this chapter, derive an algorithm based on OHAM method to find approximate continuous solutions for Fredholm's linear integral equations of second kind. We will apply this algorithm on several examples and determine approximate continuous solutions to Fredholm's linear integral equations within it.

In this chapter we concern ourselves with Fredholm's linear integral equation of the second kind

$$y(x) = f(x) + \lambda \int_{a}^{b} K(x,t)y(t)dt, \qquad (4.1)$$

where a and b are constants, the functions $K(x,t)$, $f(x)$ are known and λ is a nonzero parameter.

4.2 Application of OHAM to the Linear Fredholm Integral Equations of the Second Kind

In the present section, we reformulate the optimal homotopy asymptotic method (OHAM) for solving Fredholm's integral equations of second kind. To this end consider Equation (4.1).

Applying OHAM to Equation (4.1), as discussed in Section (2.5), a family of homotopy equations is presented as:

$$(1-p)\left[L\big(y(x,p)\big)+f(x)\right]$$
$$=H(p)\left[L\big(y(x,p)\big)+f(x)+N\big(y(x,p)\big)\right] \qquad (4.2)$$

where, as stated in earlier sections, $p\in[0,1]$ is an embedding parameter,

$y(x,p)$ is unknown real-valued function and $H(p)$ is a nonzero auxiliary function for $p \neq 0$ with $H(0)=0$.

The auxiliary function $H(p)$ is defined by the finite sum (power series)

$$H(P)=\sum_{j=1}^{m} c_j p^j$$

where c_j are auxiliary constants with $j=1,2,\ldots$. The operators L and N are given by the formulas

$$L\big(y(x,p)\big)=-y(x,p),\ \ N\big(y(x,p)\big)=\lambda\int_a^b K(x,t)y(t)dt.$$

Then, equation (4.2) becomes

$$(1-p)[y(x,p)-f(x)] =$$

$$H(p)\left[y(x,p)-f(x)-\lambda\int_a^b K(x,t)y(t)dt\right]. \qquad (4.3)$$

Note that, by taking $p = 0$ in (4.3) we obtain

$$y(x,0)-f(x)=0, \qquad (4.4)$$

and when we set p equal to one, we get

$$H(1)\left[y(x,1)-f(x)-\lambda\int_a^b K(x,t)y(t)dt\right]=0.$$

Because $H(1) \neq 0$, we arrive at

$$\left[y(x,1)-f(x)-\lambda\int_a^b K(x,t)y(t)dt\right]=0. \qquad (4.5)$$

Let us denote the solutions $y(x,0)$ and $y(x,1)$ of Equations (4.4) and (4.5) by $y_0(x)$ and $y(x)$ respectively. It is clear from Equation (4.5) that $y(x,1)$ is the exact solution that we are looking for. Again, in sympathy with the philosophy that underlines OHAM idea, the operator (linear) L is chosen so that $y_0(x)$ is easy to find.

To reach our destination, the approximate solution of $y(x) = y(x,1)$, we appeal to Taylor series and expand $y(x)$ in a power series with powers in p as:

$$y(x,p,c_1,c_2 \dots) = y_0(x) + \sum_{m=1}^{\infty} y_m(x,c_1,\dots,c_m)p^m. \qquad (4.6)$$

Note that we introduce the scalars, $c_1, c_2, \dots$ to mark the dependence of our approximate solutions on the so-called convergence-control auxiliary scalars $c_1, c_2, \dots$. If the series (4.6) converges when $p = 1$, one gets

$$y(x,1,c_1,c_2,\dots) = y_0(x) + \sum_{m=1}^{\infty} y_m(x,c_1,\dots,c_m) \qquad (4.7)$$

That is, if (4.6) converges when $p = 1$; it converges to the solution of (4.1). Equation (4.7) is the source of our desired approximations. Using the auxiliary constants in Equation (4.3), we get

$$(1-p)[y(x,p,c_1,c_2,\dots) - f(x)] =$$

$$H(p)\left[y(x,p,c_1,c_2,\dots) - f(x) - \lambda \int_a^b K(x,t)y(t,p,c_1,c_2 \dots)dt\right] \ (4.8).$$

Substitute from (4.6), we obtain

$$(1-p)\left[y_0(x) + \sum_{m=1}^{\infty} y_m(x,c_1,\dots,c_m)p^m - f(x)\right] =$$

$$\sum_{j=1}^{m} c_j p^j \left[y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m) p^m - f(x) \right.$$

$$\left. - \lambda \int_a^b K(x,t) \left(y_0(t) + \sum_{m=1}^{\infty} y_m(t, c_1, \dots, c_m) p^m \right) dt \right] \quad (4.9)$$

Proceeding as before, we equate the coefficients of like powers of p in Equation (4.9). Consequently, we get the following $zeroth$ to ith -order approximations.

The $zeroth$ -order approximation: $y_0(x) = f(x)$.

The $first$ -order approximation:

$$y_1(x, c_1) = -c_1 \lambda \int_a^b K(x,t) y_0(t) dt.$$

The $second$ -order approximation:

$$y_2(x, c_1, c_2) = (1 + c_1) y_1(x, c_1)$$

$$- c_1 \lambda \int_a^b K(x,t) y_1(t, c_1) dt - c_2 \lambda \int_a^b K(x,t) y_0(t) dt.$$

The $third$ -order approximation:

$$y_3(x, c_1, c_2, c_3)$$

$$= (1 + c_1) y_2(x, c_1, c_2) + c_2 y_1(x, c_1) - c_1 \lambda \int_a^b K(x,t) y_2(t, c_1, c_2) dt$$

$$-c_2\lambda\int_a^b K(x,t)y_1(t,c_1)dt - c_3\lambda\int_a^b K(x,t)y_0(t)dt.$$

The $fourth$ -order approximation:

$$\begin{aligned} &y_4(x,c_1,c_2,c_3,c_4) \\ &\qquad = (1+c_1)y_3(x,c_1,c_2,c_3) + c_2y_2(x,c_1,c_2) + c_3y_1(x,c_1) \\ &\qquad -c_1\lambda\int_a^b K(x,t)y_3(t,c_1,c_2,c_3)dt - c_2\lambda\int_a^b K(x,t)y_2(t,c_1,c_2)dt \\ &\qquad -c_3\lambda\int_a^b K(x,t)y_1(t,c_1)dt - c_4\lambda\int_a^b K(x,t)y_0(t)dt. \end{aligned}$$

In general, ith -order approximation:

$$y_i(x,c_1,\dots,c_i) = (1+c_1)y_{i-1}(x,c_1,\dots,c_{i-1})$$

$$+\sum_{k=2}^{i-1} c_k y_{i-k}(x,c_1,\dots,c_{i-k}) - \sum_{k=1}^{i} c_k\lambda\int_a^b K(x,t)y_{i-k}(t,c_1,\dots,c_{i-1})dt,$$

$$i = 2,3,\dots\ . \tag{4.10}$$

Again, it is worth mention that these approximations are the solutions of the corresponding order problems given by Equation (2.28) in Section (2.5).

Note also, these approximations are given recursively and depend on the auxiliary constants c_j , $j = 1,2, \dots$.

Finding these scalars leads to the mth-order approximation:

$$y^{(m)}(x, c_1, \dots, c_m) = y_0(x) + \sum_{k=1}^{m} y_k(x, c_1, \dots, c_m). \qquad (4.11)$$

To determine $c_1, \dots, c_m$, note that Equation (4.1) is equivalent to the equation

$$L\big(y(x)\big) + N\big(y(x)\big) + f(x) = 0. \qquad (4.12)$$

Thus substituting $y^{(m)}(x, c_1, \dots, c_m)$ in (4.12) and using the definitions of L and N, we obtain the residual equation:

$$R(x, c_1, \dots, c_m)$$

$$= y^m(x, c_1, \dots, c_m) - f(x) - \lambda \int_a^b K(x,t) y^m(t, c_1, \dots, c_m) dt. \qquad (4.13)$$

Clearly, if $R(x, c_1, \dots, c_m) = 0$, then $y^{(m)}(x, c_1, \dots, c_m)$ will be the exact solution.

Now to determine the auxiliary constants $c_1, c_2, \dots, c_m$ we proceed as before and apply the Least Squares Method, i.e. we find $c_1, c_2, \dots, c_m$ that minimize the function

$$J(c_1, \dots, c_m) = \int_a^b R^2(x, c_1, \dots, c_m)dx. \qquad (4.14)$$

Thus, we compute

$$\frac{\partial J}{\partial c_j} = 2\int_a^b R(x, c_1, \dots, c_m)\frac{\partial R}{\partial c_j}dx \qquad (4.15)$$

and solve the algebraic system

$$\frac{\partial J}{\partial c_1} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_m} = 0 \qquad (4.16)$$

With these constants known, the continuous approximate solution of order m (Equation (4.12)) is well-determined, see [25].

4.3 Numerical Examples and Discussion

We now present some examples of Fredholm linear integral equations of the second kind to reveal the efficiency and reliability of the OHAM method.

Example 4.3.1

For the Fredholm integral equation of second kind

$$y(x) = e^x - x + \int_0^1 xty(t)dt, \qquad (4.17)$$

the exact solution is $y(x) = e^x$. To apply OHAM, note that

$L(y(x,p)) = -y(x,p)$, $N(y(x,p)) = \lambda \int_a^b K(x,t)y(t)dt = \int_0^1 xty(t)dt$

and $\quad f(x) = e^x - x$.

The OHAM method assumes that the solution $y(x)$ has the series form (4.6). Substituting the power series in both sides of the homotopy Equation (4.2) gives

$$(1-p)\left[y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m)p^m - e^x + x\right]$$

$$= \sum_{j=1}^{m} c_j p^j \left[y_0(x) + \sum_{m=1}^{\infty} y_m(x, c_1, \dots, c_m)p^m - e^x + x\right.$$

$$\left. - \int_0^1 xt\left(y_0(t) + \sum_{m=1}^{\infty} y_m(t, c_1, \dots, c_m)p^m\right)dt\right] \qquad (4.18)$$

We equate the like powers of p. Therefore, we obtain the following recurrence relation

$$y_0(x) = e^x - x,$$

$$y_1(x, c_1) = -c_1 \int_0^1 xty_0(t)dt$$

$$= -\frac{2}{3}c_1 x.$$

$$y_i(x, c_1, \dots, c_i)$$

$$= (1 + c_1)y_{i-1}(x, c_1, \dots, c_{i-1}) + \sum_{k=2}^{i-1} c_k y_{i-k}(x, c_1, \dots, c_{i-k})$$

$$- \sum_{k=1}^{i} c_k \int_a^b xt y_{i-k}(t, c_1, \dots, c_{i-k}) dt\,, \qquad i = 2,3, \dots.$$

Consequently, we obtain $y_2(x, c_1, c_2) = -\frac{2}{3} c_1 x - \frac{2}{3} c_2 x - \frac{4}{9} c_1^2 x,$

and so on . Using Equation (4.11) gives the second order approximation

$$y^{(2)}(x, c_1, c_2) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2)$$

$$= e^x - x - \frac{4}{3} c_1 x - \frac{2}{3} c_2 x - \frac{4}{9} c_1^2 x$$

To find the constants c_1 and c_2, we Proceed as before and apply the Least Squares Method. The residual equation (4.13) reads

$$R(x, c_1, c_2) = y^{(2)}(x, c_1, c_2) - f(x) - \int_a^b K(x, t) y^{(2)}(t, c_1, c_2) dt$$

$$= -\frac{8}{9} c_1 x - \frac{4}{9} c_2 x - \frac{8}{27} c_1^2 x - \frac{2}{3} x.$$

Thus

$$J(c_1, c_2) = \int_0^1 R^2(x, c_1, c_2) dx$$

Differentiating with respect to $c_1, and\ c_2$ and setting the derivatives equal to zero, we get the system

$$\frac{\partial J}{\partial c_1} = \frac{256{c_1}^3}{2187} + \frac{128c_1c_2}{729} + \frac{128{c_1}^2}{243} + \frac{64c_1}{81} + \frac{64c_2}{243} + \frac{32}{81} = 0,$$

$$\frac{\partial J}{\partial c_2} = \frac{64{c_1}^2}{729} + \frac{32c_2}{243} + \frac{64c_1}{243} + \frac{16}{81} = 0.$$

The solution of this system is $c_1 = -\frac{3}{2}$ and $c_2 = 0$. Therefore the second order approximation is $y^{(2)}(x, c_1, c_2) = e^x$. It is the exact solution.

Example 4.3.2

We use OHAM to find a second order approximate solution for the Fredholm integral equation

$$y(x) = sin(x) - x + \int_0^{\frac{\pi}{2}} xy(t)dt. \qquad (4.20)$$

The exact solution is $y(x) = sin\,x$. We take $L\big(y(x,p)\big) = -y(x,p)$, $N\big(y(x,p)\big) = \lambda \int_a^b K(x,t)y(t)dt = \int_0^{\pi/2} xy(t)dt$ and

$f(x) = \sin(x) - x$. Proceeding as in the previous example, we set the following iteration relation

$y_0(x) = \sin(x) - x,\ y_1(x, c_1) = -c_1 x(1 - \frac{\pi^2}{8})$

$y_i(x, c_1, \dots, c_i)$

$$= (1 + c_1)y_{i-1}(x, c_1, \dots, c_{i-1}) + \sum_{k=2}^{i-1} c_k y_{i-k}(x, c_1, \dots, c_{i-k})$$

$$- \sum_{k=1}^{i} c_k \int_a^b x y_{i-k}(t, c_1, \dots, c_{i-k}) dt\,, \qquad i = 2,3, \dots.$$

Consequently, we obtain $y_2(x, c_1, c_2) = -c_1 x - c_2 x - \left(1 - \frac{\pi^2}{8}\right) c_1^2 x.$

With the aid of Equation (4.11), the second order approximation reads

$$y^{(2)}(x, c_1, c_2) = y_0(x) + y_1(x, c_1) + y_2(x, c_1, c_2)$$

$$= sin(x) - \left[1 + \left(2 - \frac{\pi^2}{8}\right) c_1 + c_2 + \left(1 - \frac{\pi^2}{8}\right) c_1^2\right] x$$

Using the Least Squares Method gives $c_1 = 0$, then $c_2 = -1$. Hence

$y^{(2)}(x, c_1, c_2) = sin(x)$. Again it is the exact solution.

Conclusion

In this thesis we have presented a semi analytical method called the optimal homotopy asymptotic method (OHAM). Based on this technique we construct algorithms to find approximate continuous solutions for Fredholm linear integral equations of the first and the second kind. We apply these algorithms on several examples in chapter three and chapter four. The results show the high accuracy of the obtained solutions when compared with exact solutions. Besides the reliability of the OHAM, stands a remarkable feature of this method; its simplicity. It assumes a homotopy equation whose solution is assumed to be a power series of the embedding parameter. Upon substituting this series in the homotopy equation a recurrence relation is reached. The required approximate solution is a sum of finite numbers of (basis) functions generated by the recurrence relation. We utilize the Least Squares Method to determine the control scalars appear in the (basis) functions. In the first chapter we present the basic theory related to integral equations and in the second chapter we present the Adomian Decomposition Method as well as the Homotopy Analysis Method.

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طريقة (OHAM) لحل معادلات (Fredholm) التكاملية من النوع الأول والثاني

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ملخص

يعتبر حقل التحليل العددي من الحقول المعرفية الهامة جدا. إذ انه من الصعوبة ايجاد الحلول المغلقة (الحقيقية) لكثير من مسائل التطبيقات الهندسية والعلمية. لذلك تبرز الحاجة الطبيعية الى تطوير طرق عددية لوضع حلول تقريبية لهذه المسائل.

في هذا البحث استخدمنا الأسلوب المعروف (OHAM) لبناء طرق عددية تستخدم لإيجاد حلول تقريبية لمعادلات (Fredholm) التكاملية الخطية من النوعين الأول والثاني، وفي اطار هذا العمل تم استخدام المفهوم الهندسي التوبولوجي (Homotopy) لبناء تلك الطرق، حيث تم افتراض تحقق معادلة (Homotopy) محتوية على معلمة تضمينية (Embedding Parameter) تنتمي قيمها الى الفترة المغلقة بين الصفر والواحد، بحيث انه عند تغير قيمة المعلمة من الصفر الى الواحد يتغير حل تلك المعادلة (والذي هو عبارة عن متسلسلة قوة في المعلمة) بطريقة متصلة من حل يسهل ايجاده الى الحل المغلق (الحقيقي) المنشود. وللحصول على الحل التقريبي يتم قطع المتسلسلة واستخدام عدد محدود من حدودها. ولتحديد الثوابت التي تظهر في الحل التقريبي (التي تعرف بثوابت التحكم في التقارب) نستخدم طريقة المربعات الصغرى.

لقد تم في هذا البحث تطبيق الطرق المطورة على العديد من الأمثلة التي لها حلول مغلقة معروفة وأشارت النتائج الى أن هذه الطريقة تتصف بالدقة اضافة إلى كونها طريقة بسيطة وفعالة.

أيضا وبعدما صنفنا المعادلات التكاملية استخدمنا بعض الأساليب التحليلية التي تستخدم لحل معادلات (Fredholm) الخطية التكاملية من النوعين الأول والثاني وهي طريقة (Adomian) وأيضا طريقة (HAM).