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Trigonometric Approximation in Weighted L^p Spaces

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Dedication

To my parents

To my brothers

To my teachers

To my colleagues

To my best friends

Yahya Abu-Latifa

Declaration

I certify that this thesis submitted for the degree of Master, is the result of my own research, except where otherwise acknowledge, and that this thesis (or any part of the same) has not been submitted for the higher degree to any other university or institution.

Signed……………………..

Yahya Ahmed Hussein Abu-Latifa

Date:……………………….

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Abstract

In general, the word approximation means a representation of something that is not exact, but still close enough to be useful.

Approximations may be used because incomplete information prevents use of exact representations, since many problems in mathematics are either too complex to solve analytically, or impossible to solve using the available analytical tools. Thus, even when the exact representation is known, an approximation may yield a sufficiently accurate solution while reducing the complexity of the problem significantly; therefore, an approximate answer may be good enough? What exactly we mean here by a good enough solution? That depends on what are we working on.

As we know, in mathematics it is better for us to deal with simple functions, but taking into account the accuracy of the given solution in which it is the most significant thing in the whole work, for example the polynomials are very easy to handle since they have any property you may be looking for.

On the other hand, the trigonometrical functions are of the most smooth functions that are easy to handle too, but in the first place it depends on the way of approximation, kind of approximation and other things, for example to approximate a function of period 2π its more convenient to us to treat with the sine and cosine functions than the polynomials, its not significant reduction of the polynomials but it is the most appropriate.

In this thesis, we are dealing of that kind of functions, the periodic functions, so it is better for us to concentrate on the trigonometrical approximation methods.

Our investigation centered on approximating a periodic function in the weighted L^p spaces, and we will use among our work many methods of approximation, however, they all depend on the Fourier series of these functions, but the main topic we must focus on is the degree of approximation, and we denote here that the degree is at most $n^{-\alpha}, 0 < \alpha < 1$, and n is the degree of the mean of the Fourier series.

التقريب في فضاء L^p الموزون

اعداد: يحيى احمد حسين أبو لطيفة

اشراف: د. جميل جمال

ملخص:

التقريب وبشكل عام لا يعطي القيمة الحقيقية للإقترانات، ولكنها تكون قريبة من القيمة الحقيقية. ونظرا لوجود مسائل في الرياضيات تكون فيها الحسابات معقدة ويصعب حلها أو التعامل معها فإننا نلجأ وقتها لعملية التقريب وعادة ما يعطينا ذلك إجابة قريبة بشكل كاف.

من الأفضل في الرياضيات التعامل مع الإقترانات البسيطة، ولكن مع الأخذ بعين الاعتبار دقة التقريب، والتي تعتبر الأكثر أهمية في هذا المجال. على سبيل المثال كثيرات الحدود تعتبر إقترانات سهلة جدا ومميزة بحيث انها تملك خصائص جيدة كثيرة وبالتالي فانه يسهل التعامل معها.

في المقابل، الإقترانات المثلثية تعد من الإقترانات التي تملك خصائص مميزة أيضا، ولكن في المقام الأول هذا يعتمد على طريقة التقريب ونوع التقريب وعوامل أخرى. فعلى سبيل المثال لتقريب اقتران دوري ودورته 2π فإنه من الأفضل استخدام الإقترانات المثلثية.

في هذه الرسالة ما نقوم به هو تقريب الإقترانات الدورية في فضاء L^p وفضاء L^p الموزون، لذلك من الأفضل استخدام الإقترانات المثلثية لهذا الغرض، وسيتم استخدام اكثر من طريقة لتقريب تلك الإقترانات والتي تخضع لشروط معينة، ولكن كل الطرق تعتمد في المقام الأول على المتسلسلة الفورييه للاقتران المراد تقريبه، وما يجب التركيز عليه هنا هو درجة التقريب والتي لا تتجاوز $n^{-\alpha}$ بحيث ان $0 < \alpha \leq 1$ و n هو عدد طبيعي يمثل درجة المتسلسلة الفورييه.

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Introduction

Our aim in this research is discussing many methods of approximating any function in the weighted L^p spaces, so we first introduce some auxiliary information as a base for this thing.

In chapter one we submit a little helpful ideas about the Lebesgue integral as well as some of its valuable properties, where all of the integrals in this thesis is considered Lebesgue integrals, so it is remarkable that we discuss these ideas.

Chapter two deals with the most important subject that one needed to understand the idea of this thesis as our work is focusing on the trigonometrical series, in sections 1 and 2 we introduce the concept of the trigonometric series, section 3 talks about the modulus of continuity in which it is the most important here since as we will see later all the functions we approximate should gain the property that the modulus of continuity must be less than or equal to the bound $C\delta^{\alpha}, 0 < \alpha \leq 1$.

In chapter three we begin by the known definition of the vector spaces and normed vector spaces, then we study many critical ideas and some formulas that will be helpful in the sequel, also we give a brief but critical ideas about some classes of functions and we mention her the weighted L^p and the Muckenhoupt class which has in turn a huge importance in many fields in analysis.

To investigate the general case, I think we have be know a lot of information about the special case, that is what we see in chapter four where we concentrate on the approximation in the non-weighted L^p spaces, also we consider many methods of approximation that we will use in the proceeding chapter, in fact our work will be just a generalization of some theorems from the L^p spaces to the weighted L^p spaces.

At the end we do the task, that is, we develop the work in chapter four to more general class of function, i.e. the approximation in the weighted L^p spaces.

Chapter One

Lebesgue Integration

With a basic knowledge of the Lebesgue measure theory, for more details one can refer to [5], we now proceed to establish the Lebesgue integration theory.

In this chapter, unless otherwise stated, all sets considered will be assumed measurable.

1.1 Simple functions

Recall that the characteristic function $\mathcal{X}_A$of any set $A$is defined by

$$\mathcal{X}_A = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise} \end{cases}$$

A function $\varphi: E \to \mathcal{R}$is said to be simpleif there exists $a_1, a_2, \ldots, a_n \in \mathcal{R}$and $E_1, E_2, \ldots, E_n \subset E$ such that $\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{E_i}$. Note that here the E_i's are implicitly assumed to be measurable, so a simple function shall always be measurable.

Theorem1.1.1: A function $\varphi: E \to \mathcal{R}$is simple if and only if it takes only finitely many distinct values$a_1, a_2, \ldots, a_n$ and$\varphi^{-1}\{a_i\}$ is a measurable set for all $i = 1,2, \ldots, n$.

With the above proposition, we see that every simple function φcan be written uniquely in the form

$$\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{E_i}$$

Where the a_i's are all non-zero and distinct, and the E_i's are disjoint. (Simply take $E_i = \varphi^{-1}\{a_i\}\ \ for\ i = 1,2, \ldots, n$ where $a_1, a_2, \ldots, a_n$ are all the distinct values of φ. We say this is the canonical representation of φ.

Definition 1.1.2: A function $f : E \to \mathcal{R}$is said to vanish outside a set of finite measureif there exists a set $A$with $m(A) < \infty$ such that $f$vanishes outside A, i.e.

$$f = 0 \ \ on\ E \setminus A$$

Or equivalently $f(x) = 0$ for all $x \in E \setminus A$. We denote the set of all simple functions defined on E which vanish outside a set of finite measure by $S_0(E)$.

We are now ready for the definition of the Lebesgue integral of such functions.

Definition 1.1.3: For any $\varphi \in S_0(E)$ and any $A \subseteq E$, we define the Lebesgue integral of φover $A$by

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(E_i \cap A)$$

where$\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{E_i}$ is the canonical representation of φ (From now on we shall adopt the convention that 0·∞ = 0. We need this convention here because it may happen that one $a_i$is zero while the corresponding $E_i \cap A$has infinite measure. Also note that here $A$is implicitly assumed to be measurable so $m(E_i \cap A)$ makes sense. We shall never integrate over non-measurable sets.)

It follows readily from the above definition that

$$\int_A \varphi = \int_E \varphi \mathcal{X}_A$$

for any $\varphi \in S_0(E)$ and for any $A \subseteq E$.
We now establish some major properties of this integral (with monotonicity and linearity being probably the most important ones). We begin with the following lemma.

Lemma 1.1.4: Suppose $\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{E_i} \in S_0(E)$ where the $E_i's$ are disjoint, then for any $A \subseteq E$,

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(E_i \cap A)$$

holds even if the $a_i's$ are not necessarily distinct.

Theorem1.1.5.(Properties of the Lebesgue integral) Suppose $\varphi, \psi \in S_0(E)$. Then for any $A \subseteq E$,

(a) $\int_A (\varphi + \psi) = \int_A \varphi + \int_A \psi$. (Note that$\varphi + \psi \in S_0(E)$ too by the vector space structure of $S_0(E)$).

(b) $\int_A \alpha\varphi = \alpha \int_A \varphi$(for all $\alpha \in \mathcal{R}$. (Note $\alpha\varphi \in S_0(E)$ again)

(c) If$\varphi \leq \psi$a.e. on A then$\int_A \varphi \leq \int_A \psi$.

(d) If$\varphi = \psi$a.e. on$A$then$\int_A \varphi = \int_A \psi$.

(e) If $\varphi \geq 0$ a.e. on A and $\int_A \varphi = 0$, then $\varphi = 0$ a.e. on A.

(f) $\left|\int_A \varphi\right| \leq \int_A |\varphi|$. ($Note$ $|\varphi| \in S_0(E)$).

Remark. $(a)$and(b) are known as the linearity property of the integral, while (c) is known as the monotonicity property. Furthermore, Lemma 1.1.4 is now seen to hold by the linearity of the integral even without the disjointness assumption on the $E_i's$.

1.2 Bounded measurable functions

Resembling the construction of the Riemann integral (using simple functions in place of step functions), we define the upper and lower Lebesgue integrals.

Definition 1.2.1: Let $f : E \to \mathcal{R}$be a bounded function, which vanish outside a set of finite measure. For any $A \subseteq E$, we define the **upper integral** and the **lower integral**of f on $A$by

$$\int_A^{-} f = \inf\left\{\int_A \psi : f \le \psi \ on\ A, \psi \in S_0(E)\right\}$$

$$\int_{-A} f = \sup\left\{\int_A \varphi : f \ge \varphi \ on\ A, \varphi \in S_0(E)\right\}$$

If the two values agree, we denote the common value by $\int_A f$. (Again the set $A$is implicitly assumed to be measurable so that $\int_A \psi$and $\int_A \varphi$ make sense.)

Note that both the infimum and the supremum in the definitions of the upper and lower integrals exist because $f$is bounded and vanishes outside a set of finite measure. (This is why f has to be a bounded function here) It is evident that for the functions we investigated in Section 1 (namely simple functions vanishing outside a set of finite measure) have their upper and lower integrals both equal to their integral as defined in the last section. In other words, if $\varphi \in S_0(E)$ then $\int_A^{-} \varphi = \int_{-A} \varphi = \int_A \varphi$ .It is also clear that $-\infty < \int_{-A} f \le \int_A^{-} f < \infty$ whenever they are defined.

We investigate when $\int_{-A} f = \int_A^{-} f$.

Theorem1.2.2: Let f be as in the above definition. Then

$$\int_A^{-} f = \int_{-A} f \text{ for all } A \subseteq E$$

if and only if f is measurable.

Notation: We shall denote the set of all (real-valued) bounded measurable functions defined on E which vanishes outside a set of finite measure by$B_0(E)$.

So from now on for$f \in B_0(E)$, implies that

$$\int_A f = \inf\left\{\int_A \psi : f \le \psi \in S_0(E)\right\} = \sup\left\{\int_A \varphi : f \ge \varphi \in S_0(E)\right\}$$

for any $A \subseteq E$.

Theorem1.2.3:(**Properties of the Lebesgue integral**) Suppose$f, g \in B_0(E)$, then $f + g, \alpha f, |f| \in B_0(E)$, and for any$A \subseteq E$, we have

(a) $\int_A (f + g) = \int_A f + \int_A g$.

(b) $\int_A \alpha f = \alpha \int_A f$ for all $\alpha \in R$.

(c) $If\ B \subseteq A$ and $f \ge 0$ a.e. on A then $\int_B f \le \int_A f$.

(d) If $f \le g$ a.e. on A then $\int_A f \le \int_A g$.

(e) If $f = g$ a.e. on A then $\int_A f = \int_A g$.

(f) If $f \ge 0$ a.e. on A and $\int_A f = 0$, then $f = 0$ a.e. on A.

(g) $\left|\int_A f\right| \le \int_A |f|$.

Theorem 1.2.4:(Bounded Convergence Theorem) Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n| \le M$ for all n and for all x. If $f(x) = \lim_{n\to\infty} f_n(x)$ for each x in E, then

(5) $\lim_{n\to\infty} \int_E f_n = \int_E f$

1.3 Integration of non-negative measurable functions

We integrate non-negative measurable functions through approximation by bounded measurable functions vanishing outside a set of finite measure, which we studied in the last section.

Definition 1.3.1: For a non-negative measurable function $f : E \to [0,\infty]$ (where E is a set which may be of finite or infinite measure), we define

$$\int_A f = \sup\left\{\int_A \varphi : \varphi \le f \text{ } on \text{ } A, \varphi \in B_0(E)\right\}$$

for any $A \subseteq E$.

Note that for non-negative bounded measurable functions vanishing outside a set of finite measure, this definition agrees with the old one. Also, note that we allow the functions to take infinite value here.

Theorem 1.3.2: Suppose $f, g: E \to [0,\infty]$ are non-negative measurable and $A \subseteq E$.

(a) If $f \le g$ a.e. on A then $\int_A f \le \int_A g$.

(b) For $\alpha > 0, f + g$ and αf are non-negative measurable functions and

$$\int_A (f+g) = \int_A f + \int_A g$$

$$\int_A \alpha f = \alpha \int_A f$$

Theorem 1.3.3:(**Monotone Convergence Theorem**) If $\{f_n\}$ is an increasing sequence of non-negative measurable functions defined on E and $f_n \to f$ a.e. on E, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Corollary 1.3.4: Let u_n be a sequence of nonnegativemeasurable functions, and let $f = \sum_{n=1}^{\infty} u_n$, then

$$\int f = \sum_{n=1}^{\infty} \int u_n$$

1.4 Extended real-valued integrable functions

In the last section, we integrated non-negative measurable functions, and in this section, we wish to drop the non-negative requirement. Recall that it is a natural requirement that our integral be linear, and now we can integrate a general non-negative measurable function, so it is tempting to define the integral of a general (not necessarily non-negative) measurable function f to be $f^+ - f^-$where $f^+ = f \vee 0$, and $f^- = (-f) \vee 0$, since $f^+, f^-$are non-negative measurable and they sum up to f. But it turns out that we cannot always do that, because it may well happen that $f^+$and $f^-$are both infinite, in which case their difference would be meaningless (Remember that ∞−∞ is undefined.)

Definition 1.4.1: For any function $f: E \to [-\infty, \infty]$, denote $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Then f is said to be integrable if and only if both$\int_E f^+$ and$\int_E f^-$ are finite, in which case we define the integral of f by

$$\int_A f = \int_A f^+ - \int_A f^-$$

for any A ⊆ E.

Notation: We shall denote the class of all (extended real-valued) integrable functions defined on E by $\mathcal{L}(E)$.

Note that in the above definition, f^+ and $f^-$are both non-negative measurable, so for any set $A \subseteq E, \int_A f^+$ and $\int_A f^-$ are both defined. Furthermore, $\int_A f^+ \leq \int_E f^+ < \infty$, similarly $\int_A f^- < \infty$,so their difference makes sense now.

We provide an alternative characterization of integrable functions.

Theorem1.4.2:A measurable function f defined on E is integrable if and only if

$$\int_E |f| < \infty$$

Theorem 1.4.3: Let f, g be integrable functions over E, then

1. $\int_E (f + g) = \int_E f + \int_E g$
2. $\int_E \alpha f = \alpha \int_E f$.
3. Furthermore, if$f \leq g$a.e on E then $\int_E f \leq \int_E g$.

Chapter Two

Trigonometric Series

2.1 Introduction to Trigonometric Series

Definition 2.1.1. Trigonometrical series are series of the form

$$(1)\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

Where the coefficients $a_0, a_1, \dots, b_1, b_2, \dots$ are independent of the real variable x. It is convenient to provide the constant term of the trigonometrical series with the factor $1/2$. Since the terms of (1) are of period 2π, it is sufficient to study trigonometrical series in any interval of length 2π.

A finite trigonometric sum

$$T(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx),$$

is called a trigonometrical series of order n. Every $T(x)$ is a real part of an ordinary power polynomial

$$P(z) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k - ib_k)z^k .$$

of degree n, where $z = e^{ix}$. The fact that trigonometrical series are the real parts of power series facilitates in many cases finding the sum of the former.
For example, the series [3]

$$(2)P_r(x) = \frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos kx , \qquad Q_r(x) = \sum_{k=1}^{\infty} r^k \sin kx,$$

where$0 \le r < 1$, are the real and imaginary parts of the series

$$\frac{1}{2} + z + z^2 + \cdots = \frac{1}{2}\frac{1+z}{1-z},$$

where $z = re^{ix}$, and we obtained by simple calculations the two relations

$$(3)P_r(x) = \frac{1}{2}\frac{1-r^2}{1-2r\cos x + r^2}, \qquad Q_r(x) = \frac{r \sin x}{1-2r\cos x + r^2}.$$

If we denote the nth-partial sums of (3) as $D_n(x), \widetilde{D}_n(x), n = 0,1,2 \dots$ of the series (2) we obtain with $r = 1$ by the same argument that

$$(4)D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}, \qquad \widetilde{D}_n(x) = \frac{\cos\frac{1}{2}x - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}$$

From (4) we see that $D_n(x), \widetilde{D}_n(x)$are uniformly boundedon any interval $0 < \varepsilon \le x \le 2\pi - \varepsilon$.

Lemma 2.1.2: [3] let u_k, v_k be any two sequences in $\mathcal{R}$ then for $0 \leq m \leq n$, the formula

$$\sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k(v_k - v_{k+1}) - U_{m-1} \cdot v_m + U_n v_n,$$

is valid for any $k \geq 0$, where $U_k = u_0 + u_1 + \cdots + u_k$, and $U_{-1} \triangleq 0$. This relation is called Abel's transformation or summation by parts which can be easily verified and it is very useful tool in the general theory of series.

Definition 2.1.3: We say that a sequence $v = (v_0, v_1, \dots, v_n, \dots)$ is of bounded variation if the series.

$$\sum_{k=1}^{\infty} |v_k - v_{k-1}| \leq M.$$

Since the previous series is absolutely convergent, then it is convergent series, so $\sum_{k=1}^{\infty}(v_k - v_{k-1})$ is converges to some constant c, thus we have.

$$c = \lim_{n\to\infty} \sum_{k=1}^{n} (v_k - v_{k-1}) = \lim_{n\to\infty} [(v_1 - v_0) + \cdots + (v_n - v_{n-1})]$$
$$= \lim_{n\to\infty} (v_n - v_0)$$

Therefore, any sequence of bounded variation is convergent.

Lemma 2.1.4. [3]

a) If a series $u_0(x) + u_1(x) + \cdots$ converges uniformly, and$\{v_k\}$ is of bounded variation,the series $u_0 v_0 + u_1 v_1 + \cdots$ converges uniformly.
b) If $u_0(x) + u_1(x) + \cdots$ has its partial sums uniformly bounded, $\{v_k\}$ is of bounded variation, and $v_k \to 0$, the series $u_0 v_0 + u_1 v_1 + \cdots$ converges uniformly.

2.2 The trigonometrical system

Note: The integral we used here is the Lebesgue integral and we introduced the concept of integral in the first chapter. In addition, we assume that f is a periodic function of period 2π.

A system of real functions $g_0, g_1, \dots \dots, g_n, \dots$ defined in an interval $(a, b)$is said to be orthogonal in this interval if for some $\tau \in \mathcal{R}$.

$$(5) \int_a^b g_n(x) g_m(x) dx = \begin{cases} 0, & m \neq n \\ \tau, & m = n \end{cases} \quad m, n = 0,1, \dots$$

The importance of the orthogonal systems is based on the following fact. Suppose that a series $c_0 g_0(x) + c_1 g_1(x) + \cdots$, where $c_0, c_1, \dots$ are constants, and converges to a function $f(x)$ in(a, b). Then by multiplying each side of the formula

(6) $$f(x) = c_0\, g_0(x) + c_1 g_1(x) + \cdots c_n\, g_n(x) + \cdots$$

by$g_n(x)$ and integrating over the interval (a, b), we find, by means of (5), that

(7)$$c_n = \frac{1}{\tau}\int_a^b f(x) g_n(x) dx \qquad n = 0,1, \dots$$

We call the numbers c_n the Fourier coefficients of$f$, and the relation (6) the Fourier series of$f$ with respect to the system $\{g_n\}$.

Not that the system of functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$,that is, the trigonometrical system is orthogonal in $(-\pi, \pi)$.

In fact, let $I_{m,n} = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx$, and let $I'_{m,n} = \int_{-\pi}^{\pi} \cos mx \sin nx\, dx, I''_{m,n} = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx$.Integrating the formula

$$2 \sin mx \sin nx = \cos(m - n)x - \cos(m + n)x$$

and taking into account the periodicity of trigonometrical functions, we find that

$$I_{m,n} = 0 \text{ when } m \neq n,$$

Moreover, $I'_{m,n} = 0 = I''_{m,n}$for any $m, n = 0,1, \dots,$ so we may write (6) in means of the trigonometrical system as

(8) $$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

In addition, we define

(9)$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx,$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx,$$

In virtue of relation (9), we see that the problems of the theory of Fourier series are closely connected with the notation of integrals; in the last relation, we assumed that $f(x) \cos kx, f(x) \sin kx$ were integrable.

Every integrable function $f(x)$ $(0 \leq x \leq 2\pi)$ has its Fourier series as it is defined in (8). Two functions f and g which are equal a.e have the same Fourier series and we call them equivalent $g \equiv f$ and do not distinguish between them.

Notation: The partial sum of the Fourier series of any function, say$f$, denoted by $s_n(f)$ and given by the formula

$$s_n(f) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

while we shall denote by $\mathfrak{S}[f]$ to the Fourier series of f.

This series of partial sum can be written as the following form

$$\begin{aligned}
s_n(f(x)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left(\cos kx \int_{-\pi}^{\pi} f(t)\cos kt\, dt + \sin kx \int_{-\pi}^{\pi} f(t)\sin kt\, dt\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} f(t)\cos kx \cos kt\, dt + \int_{-\pi}^{\pi} f(t)\sin kx \sin kt\, dt\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} f(t)[\cos kx \cos kt + \sin kx \sin kt]dt\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} f(t)[\cos k(t-x)]dt\right) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right]dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(t-x)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)D_n(t)dt,
\end{aligned}$$

where

$$D_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{1}{2}x}.$$

The function$D_n(x)$is called the Drichlet's function.
Let $s_n^* = s_n - \left(\frac{a_n \cos nx + b_n \sin nx}{2}\right)$ be the modified partial sum of the Fourier series of f, now the difference $s_n^* - s_n$ tends uniformly to 0 so it is slightly more convenient to consider the modification expression.

Note 2.2.1: [3] Let f be measurable function that belongs to the L^p space, $p > 1$, and $s_n(f)$ is the partial sum of it is Fourier series then

$$\|s_n^*(f)\|_p \le 2A\|f\|_p$$

where$s_n^*(f)$ is the modified partial sum of $s_n\ (f)$.

Consider any trigonometric polynomial, say t_n, then we may write

$$s_n(f) - t_n = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) - \frac{a_0'}{2} - \sum_{k=1}^{n}(a_k' \cos kx + b_k' \sin kx)$$
$$= \frac{a_0 - a_0'}{2} + \sum_{k=1}^{n}([a_k - a_k'] \cos kx + [b_k - b_k'] \sin kx)$$
$$= s_n(f - t_n)$$

also we may write

$$s_n(f - t_n)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(u + x) - t_n(u + x)]D_n(u)du$$

if we set $k = u + 2\pi$ on the interval $(-\pi, 0)$, and $k = u$ on the interval $(0, \pi)$ and noting that f is 2π periodic, . Then

$$s_n(f - t_n)(x) = \frac{1}{\pi}\left(\int_{-\pi}^{0}[f(u + x) - t_n(u + x)]D_n(u)du + \int_{0}^{\pi}[f(u + x) - t_n(u + x)]D_n(u)du\right)$$
$$= \frac{1}{\pi}\left(\int_{\pi}^{2\pi}[f(k + x) - t_n(k + x)]D_n(u)du + \int_{0}^{\pi}[f(k + x) - t_n(k + x)]D_n(k)dk\right)$$
$$= \frac{1}{\pi}\left(\int_{0}^{2\pi}[f(k + x) - t_n(k + x)]D_n(k)dk\right).$$

Theorem 2.2.2: [3] If f and g have the same Fourier series then $f \equiv g$.

Since if they have the same Fourier series then the difference between these functions will cancel all coefficients in the Fourier series for which the difference will be equivalent to zero and so they are equivalent.

Theorem 2.2.3: [3]Let f be continuous function, if $\mathfrak{S}[f]$ converges uniformly then it converges to f.

Noting that the convergent will be to the images of f at the points of continuity and to the average value of the left-right limit of the point in which the function f is discontinuous.

Suppose that $f(x)$ is an integral function i.e. is absolutely continuous. Therefore, it is Fourier series given by(8).Integrating the first formula in (9) by parts, we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx = \frac{1}{\pi k}\int_{-\pi}^{\pi} f'(x)\sin kx\, dx = \frac{1}{k} b'_n$$

Therefore, $b'_k = ka_k$, the same manner we deduce that $a'_k = -kb_k$.

Since f is periodic then $a'_0 = 0$, so we have

$$f'(x) = \sum_{k=1}^{\infty}(a'_k \cos kx + b'_k \sin kx)$$
$$= \sum_{k=1}^{\infty} k(a_k \sin kx - b_k \cos kx), \quad k = 1,2,\ldots$$

In other words, if $\mathfrak{S}[f]$ is the Fourier series of f, and $\mathfrak{S}'[f]$ is the resulting of differentiating $\mathfrak{S}[f]$term by term then we have $\mathfrak{S}'[f] = \mathfrak{S}[f']$. With the same argument, we see that if f is a $k{\sim}th$ integral, then $\mathfrak{S}^k[f] = \mathfrak{S}[f^k]$.

Theorem 2.2.4: [3] Let f be periodic and F is the integral of f. since

$$F(x+2\pi) - F(x) = \int_{x}^{x+2\pi} f(t)dt,$$

then a necessary and sufficient condition for the periodicity of F is that the constant term of $\mathfrak{S}[f]$ should vanishes.

2.3 Modulus of continuity

Definition 2.3.1: [3] Let $f(x)$ be a function defined for $a \le x \le b$, then $\forall x, y \in (a,b)$ such that $|x-y| \le \delta$, we define the function
$\omega(\delta) = \omega(\delta; f) = \max|f(x) - f(y)|$to be the modulus of continuity of $f(x)$.

Example 2.3.2: Consider the function $f(x) = x^2, x \in (0,3)$, then the modulus of continuity of f is

$$\omega(\delta) = \max|x^2 - y^2| = \max|(x-y)(x+y)| \le 6 \cdot \delta.$$

Theorem 2.3.3: [3] A function f is continuous iff $\omega(\delta) \to 0 \; as \; \delta \to 0$.

Definition 2.3.4: With the same notation above, if $\omega(\delta) < C\delta^\alpha, 0 < \alpha \le 1$, and C denotes a number independent of δ, then we say that f satisfies Lipschitz condition of order α, or $f \in Lip(\alpha)$in(a,b).

For simplicity, we suppose that (a,b) is the interval $(0,2\pi)$, since we are dealing with a trigonometrical system that is of period 2π. Moreover, any interval of period 2π will be sufficient.

Definition 2.3.5: Let $\omega_1(\delta) = \omega_1(\delta; f) = \max \int_0^{2\pi} |f(x+h) - f(x)|\, dx$, for all $0 < h \leq \delta$.The function $\omega_1(\delta)$ will be called the integral modulus of continuity of f.

Theorem 2.3.6: [3] For any integrable function $f(x)$,

$$\lim_{\delta \to 0} \omega_1(\delta) = 0.$$

In addition, if for any $\varepsilon > 0$ we have $f = f_1 + f_2$, where $\omega_1(\delta; f_1) \to 0 \; as \; \delta \to 0$, and $I(f_2) \triangleq \int |f_2| < \varepsilon$, then $\omega_1(\delta; f) \to 0$.

In fact:

$$\omega_1(\delta; f) \leq \omega_1(\delta; f_1) + \omega_1(\delta; f_2) \leq \omega_1(\delta; f_1) + 2I(f_2) < 3\varepsilon,$$

if$0 < \delta \leq \delta_0(\varepsilon)$.

2.4 "Big O" notation and Test of convergence

The object of this section is to establish some conditions for the convergence of the Fourier series. It will be convenient to collect here a few elementary theorems on series concerning the Big$O$ notation, which will be used in the sequel.

"Big " notation describes the limiting behavior of a function, when the argument tends toward a particular value or infinity, usually in terms of simpler functions.

Big O notation characterizes functions according to their growth rates, different functions with the same growth rate may be represented using the same O notation. The letter $O$is used because the growth rate of a function is also referred to as order of the function. A description of a function in terms of big O notation usually only provides an upper bound on the growth rate of the function.

Definition 2.4.1: Let $f(x)$ and $g(x)$ be two functions defined for some $x \geq x_0$ in addition, let $g(x) \neq 0$ there. The symbol

$$f(x) = O(g(x))$$

means that $f(x)/g(x)$ is bounded for x large enough, the same notation is used when x tends to a finite limit, or to $-\infty$. i.e. $f(x) = O(g(x))$ if and only if there exists a positive real number M and a real number x_0 such that

$$f(x) \leq M|g(x)| \quad for \; all \; x > x_0$$

In particular $O(1)$ means that a function is bounded.

Chapter Three

Classes of Functions

3.1 Vector space

Definition 3.1.1. Let V be a set with two operations, the operation "addition", denoted by "+", which maps each pair$(x\ ,y)$in $V \times V$ into V, and the operation "scalar multiplication", denoted by a dot " $\cdot$ ", which maps each pair $(c, x)\ in\ \mathcal{R} \times V$ into V. Thus, a scalar is a real or complex number. The set V is called a real **vector space** if the addition and multiplication operations involved satisfy the following rules, for all$x, y\ and\ z\ in\ V$, and for all scalars a , b, and c in $\mathcal{R}$:

(a) $x + y = y + x$
(b) $x + (y + z) = (x + y) + z$
(c) There is a unique zero vector 0 in V such that $x + 0 = x$
(d) For each x there exists a unique vector $-x$ in V such that $x + (-x) = 0$
(e) $1 \cdot x = x$
(f) $(ab) \cdot x = a \cdot (b \cdot x)$
(g) $a \cdot (x + y) = a \cdot x + a \cdot y$
(h) $(a + b) \cdot x = a \cdot x + b \cdot x$

It is trivial to verify that the Euclidean space $\mathcal{R}^n$ is a real vector space. However, the notion of a vector space is much more general. For example, let V be the space of all continuous functions on $\mathcal{R}$, with pointwise addition and scalar multiplication defined the same way as for real numbers. Then it is easy to see that this space is a real vector space.

Another example of a vector space is the space V of positive real numbers with the "addition" operation $x + y = x \cdot y$ and the "scalar multiplication" $c \cdot x = x^c$. In this case the zero vector 0 is the number 1, and $-x\ = \frac{1}{x}$.

Now if we consider the set $F = \{f | f: \mathcal{R} \to \mathcal{R}\}$ of all real valued functions of one variable then F is a vector space under the operations:

$$(f + g)(x) = f(x) + g(x)$$

$$(a \cdot f)(x) = a \cdot f(x)$$

One more example we need to show here is that of the l^p space of sequences, let $p \geq 1$ be fixes real number, by definition each element in this space is a sequence $x = (x_n)_1^\infty = (x_1, x_2, \dots)$ of numbers such that $\sum_{i=1}^{\infty} |x_i|^p$ converges, and the addition and scalar multiplication are defined as

$$x + y = (x_1 + y_1, x_2 + y_2, \dots)$$
$$a \cdot x = (ax_1, ax_2, \dots)$$

for any $x, y \in l^p$.

Note that a vector space must have at least the zero vector, thus, the one element vector space is the smallest one possible.

It is not difficult to see that these properties yield other fundamental properties of vector addition and scalar multiplication of position vectors. For example, $a \cdot 0 = 0$ for any real number a, also we state another which can be deduced from the last definition easily.

i. $0 \cdot x = 0$
ii. $If\ a \cdot x = 0,$ then either $a = 0\ or\ x = 0,$ or both
iii. $a \cdot (-x) = -a \cdot x$
iv. $(-a) \cdot x = -a \cdot x$
v. $(a - b) \cdot x = a \cdot x - b \cdot x$
vi. $a \cdot (x - y) = a \cdot x - a \cdot y$

Definition 3.1.2: A subspace W of a vector space V is a non-empty subset of$V$, which satisfies the following two requirements:

(a) For any pair x, y in$W$, $x + y$ is in W.
(b) For any x in W and any scalar a in the field , $a \cdot x$ is in W.

Thus, a subspace W of a vector space $V$is closed under linear combinations in W.

3.2 Normed vector space

Definition 3.2.1: (Normed space, Banach space). A normed space V is a vector space with a norm defined on its elements. A Banach space is a complete normed space. Here a norm on a vector space V is a real valued function whose values at any element $x \in V$is defined as $\|x\|$ and which satisfies the properties.

(A1) $\|x\| \geq 0$
(A2) $\|x\| = 0$ iff $x = 0$
(A3) $\|a \cdot x\| = |a|\|x\|$
(A4) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)

Here x and y are arbitrary vectors in V and a is any scalar in the field $\mathcal{R}$.

Example 3.2.2: Depending on the definition above, deduce the following inequality.

$$|\|y\| - \|x\|| \leq \|y - x\|$$

Solution: Using (A4) above, we may write

$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$

Thus,

$\|x\| - \|y\| \leq \|x - y\| = \|y - x\|$

by the same way one can deduce that

$\|y\| - \|x\| \le \|y - x\|$

so by the last two inequalities we have

$-\|y - x\| \le \|y\| - \|x\| \le \|y - x\|$

and the inequality follows. ■

Another critical property of the norm is its continuity, which can be seen from the previous example, that is, $x \to \|x\|$ is a continuous function on the normed space $(V, \|\cdot\|)$into $\mathcal{R}$.

Examples 3.2.3:

1. The space $\mathcal{R}^n$ and the unitary space C^n are normed spaces with the norm defined for both by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

2. The space l^p is a normed space with the norm

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$

3. Norms on the space $C[a,b]$ of continuous real valued function, for $1 \le p < \infty$

$$\|f\|_p = \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}}, \|f\|_\infty = \sup_{x\in[a,b]} (|f|)$$

4. Other norms on $\mathcal{R}^n$ can be constructed; for example

$$\|x\| = 2|x_1| + \sqrt{3|x_1|^2 + \max(|x_3|, 2|x_4|)^2}$$

is a norm on $\mathcal{R}^4$.

5. The norm on the l^∞space, of all bounded sequences of complex numbers, that is every element in the space is of the form $x = (x_1, x_2, \dots), x_i \in C$, such that for each $i = 1,2,3 \dots$. we have $|x_i| \le M_x$ where M_x is a bound that depends only on the sequence x, the norm is given as

$$\|x\|_\infty = \sup|x_i|$$

3.3 The L^p space

The L^p spaces are function spaces defined using a natural generalization of the p-norm for finite-dimensional vector spaces, we will assume all functions in this space to be of period 2π, but it is not always the case, since we approximate these functions by cosine and sine functions so it is preferable to periodic of period 2π.

Definition 3.3.1: Let p be a positive real number, then the set of all measurable functions defined on a fixed interval $[a, b]$ such that the integral

$$\int_a^b |f(x)|^p < \infty$$

are said to belong the L^p space. Thus, the L^1 space consists precisely of the Lebesgue integrable function on the interval $[a, b]$.

Since$|f + g|^p \leq 2^p(|f|^p + |g|^p)$, will show that later, we say that the sum of two functions in L^p is a gain in L^p whenever f and g are. In addition, we point here that $af \in L^p$ whenever f is. Thus we have that $af + bg \in L^p$ whenever f and g are. The last statement ensures that the L^p space is a vector space.

Note that since L^p space is a vector space, so we can define a norm on it. Here we give the norm of any function $f \in L^p$ by

$$\|f\| = \|f\|_p = \left[\int_a^b |f|^p\right]^{\frac{1}{p}}$$

It is clear that $\|f\| = 0$ iff$f = 0$ a.e, if a is a constant then$\|af\| = |a|\|f\|$, we derive two inequalities, the first of which state that.

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Unfortunately, norms for the L^p spaces do not satisfy the second requirement (A2) of being a norm, for from $\|f\| = 0$ we only conclude that$f = 0$ a.e. We shall however, consider two measurable functions to be equivalent if they are equal almost everywhere; and, if we do not distinguish between equivalent functions, then the L^p spaces are normed vector spaces.

It is convenient to denote by L^∞ the space of all bounded measurable functions on $[a, b]$ (or rather all measurable functions, which are bounded except possibly on a subset of measure zero). Then L^∞ is a vector space, and it becomes a normed vector space if we define

$$\|f\| = \|f\|_\infty = ess\,\sup|f(t)|$$

Where $ess\,\sup|f(t)|$ is the infimum of $sup\ g(t)$ as g ranges over all functions which are equal to f almost everywhere. Thus

$$ess\,\sup|f(t)| = \inf\{M \in \mathcal{R}: m\{t: f(t) > M\} = 0\}$$

Example 3.3.2: Show that inL^∞, the relation

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

is valid.

Proof: If $|f(t)| \leq M_1$ a.e and $|g(t)| \leq M_2$ a.e, then

$$|f(t) + g(t)| \leq M_1 + M_2 \text{ a.e.}$$

So $||f + g||_\infty \leq M_1 + M_2$. Note that $|f(t)| \leq ||f||_\infty$ a.e. and $|g(t)| \leq ||g||_\infty$ a.e. Thus $||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$. ■

Example 3.3.3: If $f \in L^1$ and $g \in L^\infty$, then

$$\int |fg| \leq \|f\|_1 . \|g\|_\infty$$

Solution: Suppose $f \in L_1$ and $g \in L_\infty$. Then since $|g| \leq \|g\|_\infty$ we see that

$$\int |fg| \leq \int |f| \|g\|_\infty = ||g||_\infty \int |f| = ||f||_1 . ||g||_\infty$$ ■

Theorem 3.3.4: [3] **(Minkowski Inequality)** If f and g in L^p with $1 \leq p \leq \infty$, then so is $f + g$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

If $1 < p < \infty$, then equality can hold only if there are nonnegative constants a and b such that $bf = ag$.

Proof : The case when $p = \infty$ is elementary (see example 3.3.2), as are the cases when $\|f\| = 0$ or $\|g\| = 0$. Thus, we assume that $1 \leq p < \infty$, and $\|f\| = a \neq 0$, $\|g\| = b \neq 0$. Then there are functions f_0 and g_0 such that $|f| = af_0, |g| = bg_0$, and $\|f_0\| = \|g_0\| = 1$. Set $\rho = a/(a + b)$. Then $(1 - \rho) = b/(a + b)$, and we have

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p = [af_0(x) + bg_0(x)]^p$$

$$= (a + b)^p [\rho f_0(x) + (1 - \rho)g_0(x)]^p$$

$$\leq (a + b)^p [\rho f_0(x)^p + (1 - \rho)g_0(x)^p]$$

by the convexity of the function $\theta(t) = t^p$ on $[0, \infty]$ for $1 \leq p < \infty$, if $1 < p < \infty$, this inequality is strict unless $f_0(x) = g_0(x)$ and $sgn\, f(x) = sgn\, g(x)$. Integrating both sides of this inequality gives

$$\|f + g\|^p \leq (a + b)^p [\rho\|f_0\|^p + (1 - \rho)\|g_0\|^p] \leq (a + b)^p = (\|f\| + \|g\|)^p$$

Taking p-th roots gives

$$\|f + g\| \leq \|f\| + \|g\|.$$

If $1 < p < \infty$, the inequality is strict unless $f_0 = g_0$ $a.e.$ And $sgn(f) = sgn(g)$ $a.e.$ But this means that the inequality is strict unless $bf = ag$ ■.

Lemma 3.3.5: [5] Let $1 \leq p < \infty$. Then for a, b, t nonnegative we have

$$(a + tb)^p \geq a^p + ptba^{p-1}$$

For the proof, see [5].

Theorem 3.3.6: [5] **(Holder inequality)** If p and q are nonnegative extended real numbers such that

$$\frac{1}{p}+\frac{1}{q}=1,$$

and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q.$$

Equality holds if and only if for some constant a and b, not both zero, we have $a|f|^p = b|g|^q$ a.e.

For the proof, see [5].

3.4 The weighted L^p space

We assume throughout that our functions $f(x)$ are measurable periodic with the period 2π, that is $f(x) = f(x + 2\pi)$, unless otherwise stated.

Definition 3.4.1: Let $I \subseteq \mathcal{R}$ be an open interval, and $f: I \to \mathcal{R}$ be a measurable function, if the function f on I satisfies

$$\int_{I'} |f| < \infty$$

i.e. its Lebesgue integral is finite, for all compact subsets $I' \subseteq I$, then f is locally integrable

Definition 3.4.2: A weight function w is an almost everywhere positive function that is locally integrable. In other words, it is a measurable function $w: \mathcal{R} \to [0, \infty]$ such that the set $w^{-1}(\{0, \infty\})$ has Lebesgue measure zero.

Example 3.4.3: Consider the function e^x, for any $x \in \mathcal{R}$, it is positive everywhere (so we can assume that it is a.e positive since the Lebesgue measure of the empty set is zero) and it is locally integrable on any compact interval in $\mathcal{R}$.

Definition 3.4.4: The weighted L^p space is the space of all measurable 2π-periodic function f, for which it is denoted by $L_w^p[0,2\pi]$, where $1 \leq p < \infty$, and w is a weight function.

The norm defined on $L_w^p = L_w^p[0,2\pi]$ is given by

$$\|f\|_{p,w} = \left(\int_0^{2\pi} |f(x)|^p w(x) dx\right)^{1/p} < \infty$$

Note that Minkowski and Holder inequalities hold here, since for any functions $f, g \in L_w^p$, and knowing that w is positive then

$$\|f+g\|_{p,w} = \left(\int_0^{2\pi} |f(x)+g(x)|^p w(x)dx\right)^{1/p}$$
$$= \left(\int_0^{2\pi} |f(x)+g(x)|^p \left(w^{1/p}(x)\right)^p dx\right)^{1/p}$$
$$= \left(\int_0^{2\pi} \left(\left|f(x)w^{1/p}(x)+g(x)w^{1/p}(x)\right|\right)^p dx\right)^{1/p}$$
$$= \left\|fw^{1/p}+gw^{1/p}\right\|_p$$
$$\leq \left\|fw^{1/p}\right\|_p + \left\|gw^{1/p}\right\|_p$$
$$= \|f\|_{p,w} + \|g\|_{p,w}.$$

Also noting that $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\int |fg|wdx = \int \left|fw^{1/p}gw^{1/q}\right|dx \leq \left\|fw^{1/p}\right\|_p \cdot \left\|gw^{1/q}\right\|_p = \|f\|_p \cdot \|g\|_q$$

3.5 The Muckenhoupt weight $\mathcal{A}_p$

Definition 3.5.1: Let f be a locally integrable function, which is defined on the interval $[0,2\pi]$, then for any $x \in I \subseteq [0,2\pi]$, the Hardy-Littlewood maximal operator M for any function$f$ is given by

$$M(f)(x) = \sup \frac{1}{|I|}\int_I |f(t)|dt$$

and the supremum is taken over all subintervals $I$of $[0,2\pi]$.

The class of Muckenhoupt weights $\mathcal{A}_p$ consists of those weights w for which the Hardy-Littlewood maximal operator is bounded onL_w^p. That is; $\mathcal{A}_p$ is the class of all positive, locally integrable weighted functions such that there is a constant K with

$$\|M(f)\|_{p,w} \leq K\|f\|_{p,w}$$

Equivalently,

$$\int |M(f)|^p\, w(x)dx \leq K \int |f|^p w(x)dx.$$

Proposition 3.5.2: [9]If $w \in \mathcal{A}_p$, then it is necessary and sufficient condition that w satisfied the inequality

$$\sup\left(\frac{1}{|I|}\int_I \; w(x)dx\right)\left(\frac{1}{|I|}\int_I \; [w(x)]^{-1/p-1}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I with length $|I| \leq 2\pi$.

When we assume the domain of w to be any subset of $\mathcal{R}^n$ then the condition on w will be such that for any point x

$$\sup\left(\frac{1}{|B|}\int_B \; w(x)dx\right)\left(\frac{1}{|B|}\int_B \; [w(x)]^{-1/p-1}dx\right)^{p-1} < \infty,$$

for all balls $B(x,r) \subseteq \mathcal{R}^n$ where $r > 0$, while $|B|$ means the measure of the ball B.

Example 3.5.3: One of the most examples of an$\mathcal{A}_{\boldsymbol{p}}$weight is given by

$$w_\alpha(x) = |x|^\alpha, \;\; x \in \mathcal{R}^n, \;\; -n < \alpha < n(p-1).$$

3.6 The class L_Φ^*

Definition 3.6.1: Let$\varphi(x), \psi(x), x \geq 0$be two functions, continuous, vanishes at the origin, strictly increasing, tending to infinity, and inverse to each other, then we say that $\varphi(x), \;\psi(x)$ are Young's functions.

For all $a, b \geq 0$, we have the inequality due to Young

(1) $$ab \leq \Phi(a) + \Psi(b) \text{where} \Phi(a) = \int_0^a \varphi(t)dt \;, \Psi(b) = \int_0^b \psi(t)dt.$$

Note that the equality in (1) holds if and only if$b = \varphi(a)$.The functions $\Phi(x), \Psi(x)$ will be called complementary functions. If we set,

$$\varphi(u) = u^\alpha, \psi(v) = v^{1/\alpha} (\alpha > 0), \qquad p = 1 + \alpha, \qquad p' = 1 + 1/\alpha,$$

we get the inequality

(2) $$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \,(a, b \geq 0),$$

where the complementary exponentsp, p' both exceed 1 and they connected by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

If $p = p' = 2$, (2) reduces to the familiar inequality $2 \cdot a \cdot b \leq a^2 + b^2$.If$p \to 1$, then$p' \to \infty$,and conversely.

Definition 3.6.2: A function $f(x)$ defined on an open interval $(a, b)$is said to be convex if for each $x, y \in (a, b)$ and each $\lambda,\ 0 \leq \lambda \leq 1$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

In other words, if for each point on the chord between the $\big(x, f(x)\big)$, and $(y, f(y))$ is above the graph of$f$.

For example $f(x) = x^p, p \geq 1$ is convex function on $(0, \infty)$.

As a consequence of the last definition, for any set of points $p_1, p_2, \dots, p_n$, and for any set of points $x_1, x_2, \dots, x_n$ in (a, b). We have

$$f\left(\frac{p_1x_1 + p_2x_2 + \cdots + p_nx_n}{p_1 + p_2 + \cdots + p_n}\right) \leq \frac{p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n)}{p_1 + p_2 + \cdots + p_n}.$$

This inequality is called Jensen's inequality. For $n = 2$, the inequality implies the definition, and for $n > 2$, it follows by induction.

Note: By the last inequality, let p_1, p_2, p_3be three ordered points on the convex curve $f(x)$, in the order indicated. Since p_2 is below or on the chord p_1p_3, the slope of $p_1p_2$does not exceed that ofp_1p_3. Hence if a point p approaches p_1 from the right then the slope ofp_1p is non-increasing. Thus, the right-hand side derivative exists for any point $a \leq x \leq b$ and is less than ∞.Also, there are many properties of convex functions, which are very useful in many fields in mathematics, and we here introduce few of them.

Theorem 3.6.3: [5] If $f(x)$ is convex on (a, b), then $f(x)$ is absolutely continuous on each closed subinterval of (a, b). The right (left) hand side derivatives of f exists at each point of (a, b) and are equal to each other except on a countable set. The left (right) hand derivatives are monotone increasing functions, and at each point the left-hand derivative is less than or equal to the right –hand.

Theorem 3.6.4:[5]Iff is a continuous function on (a, b) and if one derivative of f (the left or right) is non-decreasing, then f is convex.

Corollary 3.6.5:[5] Let f have a second derivative at each point of (a, b). Then f is convex on (a, b) if and only if $f''(x) \geq 0$ for each $x \in (a, b)$.

Theorem 3.6.6: [5] **(Jensen's Inequality for integrals)** Let φ be a convex function on $(-\infty, \infty)$ and$f$an integrable function on [0,1]. then

$$\int \varphi(f(t))dt \geq \varphi\left[\int f(t)dt\right].$$

Antoni Zygmund [3] states in his book the same theorem with slightly different conditions. That is, if $\varphi(x)$ is convex in an interval $a \leq x \leq b$, and $a \leq f(x) \leq b$ at each point

$\alpha \leq x \leq \beta, p(x)$ is non-negative and not identically zero, and that all integrals in the next inequality exist. Then

$$\varphi\left[\frac{\int_\alpha^\beta f(x)p(x)dx}{\int_\alpha^\beta p(x)\,dx}\right] \leq \frac{\int_\alpha^\beta \varphi[f(x)]p(x)dx}{\int_\alpha^\beta p(x)\,dx},$$

Theorem 3.6.7: [3] A necessary and sufficient condition that $\varphi(x), a \leq x \leq b$, should be convex is that it should be the integral of a non-decreasing function.

Let now $\varphi(x),\ x \geq 0$, be an arbitrary function, non-negative, non-decreasing, vanishes at $x = 0$ and tending to $+\infty$ with x, the curve $y = \varphi(x)$ may possess discontinuities and stretches of constancy, if at each point x_0 of discontinuity of φ we adjoin to the curve $y = \varphi(x)$ the vertical segment $x = x_0, \varphi(x_0 - 0) \leq y \leq \varphi(x_0 + 0)$, obtain a continuous curve, and we may define a function $\psi(y)$ inverse to $\varphi(x)$ by defining$\psi(y_0)(0 < y_0 < \infty)$ to be any x_0 such that the point (x_0, y_0) is continuous curve, The stretches of constancy of φ then correspond to discontinuities of ψ, and conversely. The function $\psi(y)$ is defined uniquely except for the $y\,'s$which correspond to the stretches of constancy of φ.But since the set of such stretches is denumerable, our choice of $\psi(y)$ has no influence upon the integral $\Psi(y)$of $\psi(y)$,and it is easy to see that Young's inequality is valid in this slightly more general case.

From 3.6.7 it follows that every function $\Phi(x),\ x \geq 0$,which is non-negative, convex, and satisfies the relation $\Phi(0) = 0$and$\Phi(x)/x \to \infty$,may be considered as a Young's function.

More precisely to every such function corresponds another function $\Psi(x)$ with similar properties such that

$$ab \leq \Phi(a) + \Psi(b)$$

For every $a \geq 0, b \geq 0$. it is sufficient to take for $\Psi(y)$ the integral over $(0, y)$ of the function $\psi(x)$ inverse to $\varphi(x) = D^+\Phi(x)$. since $\Phi(x)\ /x \to \infty$with x. It is easy to see that $\varphi(x)$ and $\psi(x)$ also tend to $+\infty$ with x. We have $ab = \Phi(a) + \Psi(b)$if and only if the point (a, b) is on the continuous curve obtained from the function $y = \varphi(x)$.

Definition 3.6.8: Let $\Phi(u) \geq 0\ for\ u \geq 0$.We say that a measurable function $f(x), 0 \leq x \leq 2\pi$, belongs to the class$L_\Phi(0,2\pi)$ if the function $\Phi(|f|)$ is integrable over $(0,2\pi)$. That is, the class $L_\Phi(0,2\pi) = L_\Phi$ is the set of all measurable functions such that

$$\int_0^{2\pi} \Phi(|f(x)|)dx < \infty$$

This class may fail to be vector space; it may fail to be closed under scalar multiplication, due to the function $\Phi(u)$.

Example 3.6.9: If we set $\Phi(u) = u^p$, then$\Psi(v) = v^{1/p}(p > 0)$.So, the class $L_\Phi(0,2\pi)$ is identical with the L^p space.

Integrating the inequality

$$|fg| \leq \Phi(|f|) + \Psi(|g|)$$

over $a \leq x \leq b$, we get that fg is integrable over (a, b) if $f \in L_\Phi(a, b), g \in L_\Psi(a, b)$.

Throughout the text, we will write $\Phi|u| + \Psi|u|$, for $\Phi(|u|) + \Psi(|u|)$ for simplicity.

Definition 3.6.10: If $f(x)$ is measurable and such that $\int_0^{2\pi} \Phi|f|dx$ exists, $f(x)$ is said to belong to the space $L_\Phi(0, 2\pi)$. If $f(x)$ is such that the product $f(x)g(x)$ is integrable for every $g(x) \in L_\Psi$, then $f(x) \in L_\Phi^*$.

For this space, the norm is given by

$$\|f\|_\Phi = sup \left|\int_0^{2\pi} f(x)g(x)\, dx\right|$$

for all measurable $g(x)$ with $\rho_g \equiv \int_0^{2\pi} \Psi|g|dx \leq 1$. This space is a vector space, and also complete [5]. If $f(x) \in L_\Phi^*$, we put for $\delta > 0$,

$$\omega_\Phi(\delta; f) = \sup\|f(x+h) - f(x)\|_\Phi \text{ for } 0 < |h| \leq \delta.$$

When $p > 1$, then L^p is a class L_Φ^*.

In $L^p, p \geq 1$,

$$\begin{aligned} \omega_p(\delta; f) &= \sup\|f(x + h) - f(x)\|_p \\ &= \sup\left[\int_0^{2\pi} |f(x + h) - f(x)|^p dx\right]^{1/p} \end{aligned}$$

If $\omega_p(\delta; f) = O(\delta^a)$, $\delta \to 0$, $f(x)$ is said to belong to the class $Lip(\alpha, p)$.

The limiting case of $Lip(\alpha, p)$, denoted $Lip(\alpha, \infty)$ is identical with $Lip(\alpha)$.For brevity, we shall write $\|f\|$ for $\|f\|_\Phi$ whenever it will not lead to confusion.

Chapter Four

Trigonometric Approximation in the L^p Spaces

4.1 Introduction

So far, we introduced many concepts, classes of functions, Trigonometric series, and Lebesgue integral, in which they are the basic building blocks in the field of approximation by trigonometrical functions; on the other hand, we will not cover the whole branch of approximation in the L^pspace and our argument in this subject will be a basic theorems and lemmas that will qualify us to study the approximation in the weighted L^p space.

Throughout this chapter, we will assume any function to be periodic of period 2π. Also we define $s_n(f;x)$ to be the n-th partial sum of the Fourier series of the function $f(x)$.

We already defined the Lipschitz class for $0 < \alpha \leq 1$ to be the class of all functions such that $\omega(\delta) < C\delta^\alpha$, for some constant $C. i. e$

$$\omega(\delta; f) = O(\delta^\alpha)$$

Let $1 < p < \infty, w \in \mathcal{A}_p, f \in L^p$and $0 < \alpha \leq 1$, we define the Lipschitz class$Lip(\alpha, p)$ as

$$Lip(\alpha, p) = \{f \in L^p: \omega_p(f; \delta) = O(\delta^\alpha), \delta > 0\}.$$

Definition 4.1.1: **(Nörlund method)** [10].Each sequence $p_0, p_1, \ldots$ of real or complex constants for which $P_n = p_0 + p_1 + \cdots + p_n \neq 0$ for each n defines a Nörlund method (transformation) of summability by means by the formula

$$N_n(f; x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k(f; x).$$

The class of Nörlund transformation is identical with the class of triangular matrix transformations

$$\rho_n = \sum_{k=0}^{n} a_{n,k} s_k$$

for which

$$a_{n,k} \geq 0, for\ k \leq n, a_{n,k} = 0, for\ k > n\ (k, n = 0,1, \ldots)$$

and

$$\sum_{k=0}^{n} a_{n,k} = 1 \qquad\qquad n = 0,1,2, \ldots$$

where

$$A = [a_{i,j}]_{i,j=1}^{\infty}$$

is a lower triangular infinite matrix of real numbers.

Definition 4.1.2: (Riesz method) [10] Each sequence$p_0,\ p_1,$ ... for which $P_n = p_0 + p_1 + \cdots + p_n \neq 0$, determines a Riesz transformation as

$$R_n(f;x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k(f;x).$$

Definition 4.1.3 :(Ces`aro method) [10] If we set$p_n = 1$for each$n$ then both of the Nörlund and of Riesz transformations coincide with the Ces'aro transformation

$$\sigma_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(f;x).$$

4.2 Trigonometrical approximation in the mean

In this section, the investigation will be in approximating functions in the L^p spaces, especially for the class $Lip\ (a,p)$ where the functions has the property that the modulus of continuity is less than $C\delta^\alpha$, and the approximation will be done here in two methods, the first will involve the Nörlund means and the second is more general method of matrix transformation in which it implies the previous one, we note that we are concerning with a degree of error to be $O(\delta^\alpha)$.

The following theorem is stated without proof by G. H. Hardy and J. E. Littlewood.

Theorem 4.2.1: [4] The class $Lip\ (\alpha,p)$ is identical with the class of functions$f(x)$ approximable in the mean $p\sim th$ power, with error $O(n^{-\alpha})$,by trigonometrical polynomials of degree n.

In the following $s_n = s_n(f) = s_n(x;\ f)$ denotes the n-th partial sum of the Fourier series of $f(x)$ and $\sigma_n = \sigma_n(x;f)$ denotes the Ces'aro mean for the function f,i.e.

$$\sigma_n(x;f) = \frac{1}{n+1}\sum_{k=0}^{n} s_n(f(x)).$$

Theorem 4.2.2: [4] If $f \in L^*_\Phi$ possesses a derivative of order r, say $f^{(r)}(x)$, in L^*_Φ, where r is appositive integer or zero, then, for any positive integer $n, f(x)$ may be approximated in L^*_Φ by a trigonometrical polynomial $t_n(x)$, of order n at most, such that

$$\|f - t_n(x)\|_\Phi = O\left(n^{-r}\omega_\Phi\left(\frac{1}{n}; f^{(r)}\right)\right).$$

Proof: Let $\lambda = \left[\frac{n}{2}\right]$ and $\mu(t) = \sum_{k=0}^{\lambda} a_k f\left(x + \frac{2t}{2^k}\right)$,
The $\lambda + 1$ constants $a_k,\ k = 0,1,\ldots\lambda$ being so determined that $\mu(0) = f(x), \mu^{(2s)}(0) = 0$, $s = 1,2,\ldots,\lambda$.The trigonometrical polynomial $t_n(x)$ is defined by the equation

$$t_n(x) = \frac{1}{\tau(\lambda+2)} \int_{-\infty}^{\infty} \mu\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt$$

where

$$\tau(\lambda+2) = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^{2\lambda+4} dt$$

The order n of $t_n(x)$ is $(\lambda+2)2^{\lambda}m-1$.Since$\mu(0)=f(x)$, we may write

$$t_n(x)-f(x) = \frac{1}{\tau(\lambda+2)} \int_{-\infty}^{\infty} \mu\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt - f(x)$$

$$= \frac{1}{\tau(\lambda+2)}\left[\int_{-\infty}^{0} \mu\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt + \int_{0}^{\infty} \mu\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt\right] - \mu(0)$$

$$= \frac{1}{\tau(\lambda+2)}\left[\int_{0}^{\infty} \mu\left(\frac{-t}{m}\right)\left(\frac{\sin -t}{-t}\right)^{2\lambda+4} dt + \int_{0}^{\infty} \mu\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt\right]$$

$$-\mu(0)\frac{1}{\tau(\lambda+2)} \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^{2\lambda+4} dt$$

$$= \frac{1}{\tau(\lambda+2)}\left[\int_{0}^{\infty} \mu\left(\frac{-t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt + \int_{0}^{\infty} \mu\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt\right]$$

$$-\mu(0)\frac{2}{\tau(\lambda+2)} \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^{2\lambda+4} dt$$

$$= \frac{1}{\tau(\lambda+2)}\left[\int_{0}^{\infty} \left(\mu\left(\frac{-t}{m}\right)+\mu\left(\frac{t}{m}\right)\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} - 2\mu(0)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt\right]$$

$$= \frac{1}{\tau(\lambda+2)}\left[\int_{0}^{\infty} \left(\mu\left(\frac{-t}{m}\right)+\mu\left(\frac{t}{m}\right)-2\mu(0)\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt\right]$$

$$= \frac{1}{\tau(\lambda+2)}\left[\int_{0}^{\infty} F\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4} dt\right]$$

where

$$F\left(\frac{t}{m}\right) = \mu\left(\frac{-t}{m}\right)+\mu\left(\frac{t}{m}\right)-2\mu(0).$$

with these definitions

$$\mu^{(r)}(t) = \sum_{k=0}^{\lambda} a_k\left(\frac{1}{2^{(k-1)r}}\right) f^{(r)}\left(x+\frac{2t}{2^k}\right)$$

and

$$F^{(r)}(t) = \mu^{(r)}(t)+\mu^{(r)}(-t), \qquad r \text{ is even}$$

$$F^{(r)}(t) = \mu^{(r)}(t)-\mu^{(r)}(-t)-2\mu^{(r)}(0), \qquad r \text{ is odd}$$

Consequently if r is odd then

$$\left\|F^{(r)}(t)\right\| = \left\|\mu^{(r)}(t) - \mu^{(r)}(-t) - 2\mu^{(r)}(0)\right\|$$

$$= \left\|\sum_{k=0}^{\lambda} a_k\left(\frac{1}{2^{(k-1)r}}\right) f^{(r)}\left(x + \frac{2t}{2^k}\right) - \sum_{k=0}^{\lambda} a_k\left(\frac{1}{2^{(k-1)r}}\right) f^{(r)}\left(x - \frac{2t}{2^k}\right)\right.$$

$$\left. - 2\sum_{k=0}^{\lambda} a_k\left(\frac{1}{2^{(k-1)r}}\right) f^{(r)}(x)\right\|$$

$$\leq \sum_{k=0}^{\lambda} \frac{|a_k|}{2^{(k-1)r}} \left\|f^{(r)}\left(x + \frac{2t}{2^k}\right) - f^{(r)}\left(x - \frac{2t}{2^k}\right) - 2f^{(r)}(x)\right\|$$

$$\leq \sum_{k=0}^{\lambda} \frac{|a_k|}{2^{(k-1)r}} \left(\left\|f^{(r)}\left(x + \frac{2t}{2^k}\right) - f^{(r)}(x)\right\| + \left\|f^{(r)}\left(x - \frac{2t}{2^k}\right) - f^{(r)}(x)\right\|\right)$$

by the same way when r is even

$$\left\|F^{(r)}(t)\right\| \leq \sum_{k=0}^{\lambda} \frac{|a_k|}{2^{(k-1)r}} \left\|f^{(r)}\left(x + \frac{2t}{2^k}\right) - f^{(r)}\left(x - \frac{2t}{2^k}\right)\right\|$$

$$= \sum_{k=0}^{\lambda} \frac{|a_k|}{2^{(k-1)r}} \left\|f^{(r)}\left(x + \frac{2t}{2^k}\right) - f^{(r)}(x) + f^{(r)}(x) - f^{(r)}\left(x - \frac{2t}{2^k}\right)\right\|$$

$$\leq \sum_{k=0}^{\lambda} \frac{|a_k|}{2^{(k-1)r}} \left(\left\|f^{(r)}\left(x + \frac{2t}{2^k}\right) - f^{(r)}(x)\right\| + \left\|f^{(r)}\left(x - \frac{2t}{2^k}\right) - f^{(r)}(x)\right\|\right)$$

but the expressions in the norms above are the modulus of continuity of f so we may write

$$\left\|F^{(r)}(t)\right\| \leq \sum_{k=0}^{\lambda} \frac{|a_k|}{2^{(k-1)r}} \left(2\omega_\Phi\left(\frac{2t}{2^k}; f^{(r)}\right)\right) = O\left(\omega_\Phi\left(\frac{2t}{2^k}; f^{(r)}\right)\right) = O\left(\omega_\Phi\left(2t; f^{(r)}\right)\right)$$

also we have, for $r \geq 1$,

$$F(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{r-1}} \|F^r(u)\| du dt_{r-1} \dots dt_2 dt_1$$

Therefor,

$$\|F(t)\| \leq \int_0^t \int_0^{t_1} \dots \int_0^{t_{r-1}} F^r(u) du dt_{r-1} \dots dt_2 dt_1$$

$$= O\left(\int_0^t \int_0^{t_1} \dots \int_0^{t_{r-1}} \omega_\Phi\left(2u; f^{(r)}\right) du dt_{r-1} \dots dt_2 dt_1\right)$$

Thus we have

$$\left\|F\left(\frac{t}{m}\right)\right\| = O\left(\frac{1}{m^r} \int_0^t \int_0^{t_1} \dots \int_0^{t_{r-1}} \omega_\Phi\left(\frac{2u}{m}; f^{(r)}\right) du dt_{r-1} \dots dt_2 dt_1\right)$$

Now since $f^{(r)}(x)$ is periodic, then

$$\omega_\Phi\left(\frac{2u}{m}; f^{(r)}\right) \leq (2u + 1)\omega_\Phi\left(\frac{1}{m}; f^{(r)}\right)$$

Thus we have

$$\left\|F\left(\frac{t}{m}\right)\right\| = O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)\int_0^t\int_0^{t_1}\dots\int_0^{t_{r-1}}(2u+1)dudt_{r-1}\dots dt_2dt_1\right)$$
$$= O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)\int_0^t\int_0^{t_1}\dots\int_0^{t_{r-1}}u\,dudt_{r-1}\dots dt_2dt_1\right)$$
$$= O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)t^r\right)$$

Now we see that

$$\|t_n - f\| = \left\|\frac{1}{\tau(\lambda+2)}\left[\int_0^\infty F\left(\frac{t}{m}\right)\left(\frac{\sin t}{t}\right)^{2\lambda+4}dt\right]\right\| \leq \frac{1}{\tau(\lambda+2)}\int_0^\infty \left\|F\left(\frac{t}{m}\right)\right\|\left(\frac{\sin t}{t}\right)^{2\lambda+4}dt$$
$$= O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)\right)\frac{1}{\tau(\lambda+2)}\int_0^\infty t^r\left(\frac{\sin t}{t}\right)^{2\lambda+4}dt$$
$$= O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)\right)\int_0^\infty \frac{t^r}{t^{2\lambda+4}}(\sin t)^{2\lambda+4}dt$$
$$= O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)\right)\int_0^\infty \frac{t^r}{t^{r+4}}(\sin t)^{2\lambda+4}dt$$
$$= O\left(\frac{1}{m^r}\omega_\Phi\left(\frac{1}{m};f^{(r)}\right)\right)$$

since$m = (n+1)(\lambda+2)^{-1}2^{-\lambda}$, we have

$$\|t_n - f\| = O\left(\frac{1}{n^r}\omega_\Phi\left(\frac{1}{n};f^{(r)}\right)\right) ■$$

Theorem 4.2.3: [4]If $f(x) \in Lip(\alpha,p)$, $p \geq 1, 0 < \alpha \leq 1$, then, for any positive integer $n, f(x)$ may be approximated in L^p by a trigonometrical polynomial, $t_n(x)$, of order n such that

$$\|f - t_n\|_p = O(n^{-\alpha}).$$

Proof: If we put $r = 0$ and $\omega_p\left(\frac{1}{n};f^{(r)}\right) = Mn^{-\alpha}$ in the last theorem (4.2.2) then all conditions of the previous theorem is satisfied and so the result holds.■

Lemma 4.2.4: [4] If $f(x) \in L^p$ and $t_n(x)$ is an arbitrary trigonometric polynomial of degree $n \geq 1$ at most, then
(i)if $p > 1, \|f - s_n\|_p \leq A\|f - t_n\|_p$,
(ii)if $p = 1, \|f - s_n\|_1 \leq A(1 + \log n)\|f - t_n\|_1$
where$A$ is independent of $f(x)$ and n.

Proof:Case I: We may write

$$\|f - s_n\| = \|f - t_n + t_n - s_n\| \leq \|f - t_n\| + \|t_n - s_n\|.$$

Hencewhen $p > 1$, we have

$$\|s_n(f)\| = \left\| s_n^* + \left(\frac{a_n \cos nx + b_n \sin nx}{2}\right) \right\| \leq \|s_n^*(f) + f(x)\| \leq \|s_n^*(f)\| + \|f(x)\|$$
using (2.2.1) we have

$$\|s_n(f)\|_p \leq \|s_n^*(f)\|_p + \|f(x)\|_p \leq 2K\|f(x)\|_p + \|f(x)\|_p = (2K+1)\|f(x)\|_p$$

Butthe trigonometric polynomial $s_n(f) - t_n(x) = s_n(f - t_n)$. Thus we have

$$\|s_n(f) - t_n\|_p = \|s_n(f - t_n)\|_p \leq (2K+1)\|f(x) - t_n\|_p \leq A\|f(x) - t_n\|_p$$

Case II: When $p = 1$, we have

$$\|s_n - t_n\|_1 = \frac{1}{\pi}\int_0^{2\pi} \left| \int_0^{2\pi} [f(x+u) - t_n(x+u)] D_n(u) du \right| dx,$$

where$D_n(u)$ is the Drichlet's kernel. Interchanging the order of integration, we have

$$\begin{aligned}\|s_n - t_n\|_1 &= \frac{1}{\pi}\left| \int_0^{2\pi} \left(\int_0^{2\pi} [f(x+u) - t_n(x+u)] dx \right) D_n(u) du \right| \\ &= \frac{1}{\pi}\|f - t_n\|_1 \left| \int_0^{2\pi} D_n(u) du \right| \\ &\leq \frac{1}{\pi}\|f - t_n\|_1 \int_0^{2\pi} |D_n(u)| du \leq A(1 + \log n)\|f - t_n\|_1\end{aligned}$$

Since [3]

$$\frac{1}{\pi}\int_0^{2\pi} |D_n(u)| du \cong \frac{4}{\pi^2}\log n + O(1). \blacksquare$$

Theorem 4.2.5: [4] If $f(x) \in Lip(a, p), 0 < \alpha \leq 1$, then

(i) when $p > 1, \|f - s_n\|_p = O(n^{-\alpha})$;

(ii) when $p = 1, \|f - s_n\|_1 = O(n^{-\alpha}\log n)$.

Proof: combining theorem 4.2.3 and 4.2.4 we have

Case I:

$$\|f - s_n\|_p \leq A\|f - t_n\|_p = O(n^{-\alpha})$$

Case II: By the same way, we see that,

$$\|f - s_n\|_1 \leq A(1 + \log n)\|f - t_n\|_1 = O(n^{-\alpha}\log n) \blacksquare$$

Theorem 4.2.6: [4] If $f(x) \in Lip(\alpha, p), 0 < \alpha \leq 1$, then

$(i) if\ p > 1\ or\ if\ p = 1, \alpha < 1, \|f - \sigma_n\|_p = O(n^{-\alpha})$;

$(ii)\ if\ p = \alpha = 1, \|f - \sigma_n\|_1 = O\left(\frac{\log n}{n}\right)$.

Lemma 4.2.7: [4] If $f \in Lip(1,p)(p > 1)$,then

$$\|\sigma_n(f) - s_n(f)\|_p = O(n^{-1}).$$

Lemma 4.2.8: [8] Let (p_n) be positive and non-increasing. Then, for $0 < \alpha < 1$,

$$\sum_{k=1}^{n} k^{-\alpha} p_{n-k} = O(n^{-\alpha} P_n).$$

Proof: Let r denote the integral part of $\frac{1}{2}n$.Then

$$\sum_{k=1}^{n} k^{-\alpha} p_{n-k} = \sum_{k=1}^{r} k^{-\alpha} p_{n-k} + \sum_{k=r+1}^{n} k^{-\alpha} p_{n-k} \leq p_{n-r} \sum_{k=1}^{n} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{n} p_{n-k}$$
$$= O(n^{1-\alpha}) p_{n-r} + O(n^{-\alpha}) P_n = O(n^{-\alpha}) P_n.$$

sincep_n is non-increasing.■

We shall use the notation $\Delta_k p_k = p_k - p_{k+1}$.

Lemma 4.2.9: Given any positive sequence (p_n), then for any function f, the Nörlund mean

$$N_n(f) = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k}\, u_k$$

whereu_k are the k-th element in the Fourier series of f, and $P_n = \sum_{k=0}^{n} p_k$.

Proof: The Nörlund mean is defined as

$$N_n(f) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k$$

Using Abel's transformation, we have the following

$$N_n(f) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k = \frac{1}{P_n} \left[s_n P_n + \sum_{k=0}^{n-1} (p_n + \cdots + p_{n-m})(s_k - s_{k+1}) \right]$$
$$= \frac{1}{P_n} \left[(u_0 + \cdots + u_n) P_n + \sum_{k=0}^{n-1} -u_{k+1} (p_n + \cdots + p_{n-m}) \right]$$
$$= \frac{1}{P_n} [u_0 P_n + \cdots + u_n P_n - u_1 p_n - u_2 (p_n + p_{n-1}) - \cdots - u_n (p_n + \cdots + p_1)]$$
$$= \frac{1}{P_n} [u_0 P_{n-0} + u_1 P_{n-1} + u_n P_{n-n}] = \frac{1}{P_n} \sum_{k=0}^{n} u_k P_{n-k} \;■$$

Lemma 4.2.10: Let $s_n(f)$ be the partial sum of the Fourier series of f, then

$$\sum_{k=1}^{n} k u_k(f) = (n+1)\big(s_n(f) - \sigma_n(f)\big)$$

Proof: By definition, the Caseros mean is given by

$$\sigma_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{n} s_k\,(f)$$

Thus

$$\sigma_n(f) - s_n(f) = \frac{1}{n+1}\sum_{k=0}^{n} s_k\,(f) - s_n(f) = \frac{1}{n+1}(s_0 + \cdots + s_n) - s_n(f)$$

$$= \frac{1}{n+1}(u_0 + \cdots + [u_0 + \cdots u_n]) - (u_0 + \cdots + u_n)$$

$$= \frac{1}{n+1}[(n+1)u_0 + \cdots + u_n] - (u_0 + \cdots + u_n)$$

$$= \frac{1}{n+1}[(n+1)u_0 + \cdots + u_n - (n+1)(u_0 + \cdots + u_n)]$$

$$= \frac{1}{n+1}[(n+1)u_0 + \cdots + u_n - (n+1)u_0 + \cdots + (n+1)u_n]$$

$$= -\frac{1u_1 + 2u_2 + \cdots + nu_n}{n+1} = -\frac{1}{n+1}\sum_{k=1}^{n} ku_k(f)$$

Therefore,

$$\sum_{k=1}^{n} ku_k(f) = (n+1)\big(s_n(f) - \sigma_n(f)\big) \blacksquare$$

Theorem 4.2.11: [8]Let $f \in Lip(\alpha, p)$ and let (p_n) be positive such that

(1) $$(n+1)p_n = O(P_n).$$

If either

(i) $p > 1$, $0 < \alpha \leq 1$ and$(ii)(p_n)$is monotonic

or

(i) $p = 1$, $0 < \alpha < 1$ and$(ii)(p_n)$is non decreasing

Then

(2) $$\|f - N_n(f)\|_p = O(n^{-\alpha})$$

Proof:Case I:$p > 1$ and $0 < \alpha < 1$.
since

$$f(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} f(x)$$

then

$$N_n(f;x) - f(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\,\{s_k(f;x) - f(x)\}$$

and hence we get

$$
\begin{aligned}
\|f-N_n(f)\|_p &\le \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\|f-s_k(f)\|_p \\
&= \frac{1}{P_n}\sum_{k=1}^{n} p_{n-k}\|f-s_k(f)\|_p + \frac{1}{P_n}p_n\|f-s_0(f)\|_p \\
&= \frac{1}{P_n}\sum_{k=1}^{n} p_{n-k}\,O(k^{-\alpha}) + \frac{p_n}{P_n}\cdot h \\
&= \frac{1}{P_n}O(P_n n^{-\alpha}) + O\left(\frac{p_n}{P_n}\right) \\
&= O(n^{-\alpha}) + O(n) = O(n^{-\alpha}) + O(n^{-\alpha}) = O(n^{-\alpha}),
\end{aligned}
$$

where $\|f-s_0(f)\|_p \le h$ =constant, and using (1) and Lemma 4.2.8 and theorem 4.2.6.

Case II: $p > 1$ and $\alpha = 1$.

By lemma 4.2.9 we have $N_n(f) = \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\,u_k$

where

$$
s_n(f;x) = \sum_{k=0}^{n} u_k(f;x) = \frac{1}{P_n}\sum_{k=0}^{n} P_n u_k\,(f;x)
$$

hence

$$
s_n(f;x) - N_n(f;x) = \frac{1}{P_n}\sum_{k=1}^{n}(P_n - P_{n-k})u_k(f;x) = \frac{1}{p_n}\sum_{k=1}^{n}\left(\frac{P_n - P_{n-k}}{k}\right) k u_k(f;x)
$$

by Abel's transformation and convention $P_{-1} = 0$, we deduce that

$$
\begin{aligned}
s_n(f;x) - N_n(f;x) &= \frac{1}{P_n}\left[\frac{P_n - P_0}{n}\sum_{k=1}^{n} k u_k\,(f) + \sum_{k=1}^{n-1}\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\sum_{m=1}^{k} m u_m(f)\right] \\
&= \frac{1}{P_n}\left[\frac{P_n - P_0}{n}\sum_{k=1}^{n} k u_k(f) - \frac{P_n - P_{-1}}{n+1}\sum_{k=1}^{n} k u_k(f) + \frac{P_n - P_{-1}}{n+1}(f)\right. \\
&\qquad \left. + \sum_{k=1}^{n-1}\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\sum_{m=1}^{k} m u_m\,(f)\right] \\
&= \frac{1}{P_n}\left[\frac{P_n}{n+1}\sum_{k=1}^{n} k u_k + \sum_{k=1}^{n}\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\sum_{m=1}^{k} m u_m\,(f)\right] \\
&= \frac{1}{n+1}\sum_{k=1}^{n} k u_k + \frac{1}{P_n}\sum_{k=1}^{n}\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\sum_{m=1}^{k} m u_m(f)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(3)\ \|s_n(f) - N_n(f)\|_p &\le \frac{1}{p_n}\sum_{k=1}^{n}\left|\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\right|\left\|\sum_{m=1}^{k} m u_m(f)\right\|_p \\
&\qquad + \frac{1}{n+1}\left\|\sum_{k=1}^{n} k u_k\,(f)\right\|_p
\end{aligned}
$$

And by lemma 4.2.10

$$(4)\sigma_n(f;x) - s_n(f;x) = -\frac{1}{n+1}\sum_{k=1}^{n} ku_k\,(f;x),$$

we have by Lemma 4.2.7,

$$(5)\left\|\sum_{k=1}^{n} ku_k\,(f)\right\|_p = (n+1)\|\sigma_n(f) - s_n(f)\|_p = O(1)$$

Now, combining (3) and (5), we get

$$(6)\|s_n(f) - N_n(f)\|_p = O\left(\frac{1}{p_n}\right)\sum_{k=1}^{n}\left|\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\right| + O(n^{-1}).$$

However,

$$\begin{aligned}(7)\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right) &= \frac{P_n - P_{n-k}}{k} - \frac{P_n - P_{n-k-1}}{k+1}\\ &= \frac{P_n}{k} - \frac{P_n}{k+1} - \frac{P_{n-k}}{k} + \frac{P_{n-k-1}}{k+1}\\ &= \frac{P_n}{k(k+1)} - \frac{P_{n-k}}{k} + \frac{P_{n-k-1}}{k} - \frac{P_{n-k-1}}{k} + \frac{P_{n-k-1}}{k+1}\\ &= \frac{P_n}{k(k+1)} - \frac{P_{n-k}}{k} + \frac{P_{n-k-1}}{k} - \frac{P_{n-k-1}}{k(k+1)}\\ &= \frac{P_n - P_{n-k-1}}{k(k+1)} + \frac{P_{n-k-1} - P_{n-k}}{k}\\ &= \frac{P_n - P_{n-k-1}}{k(k+1)} + \frac{-p_{n-k}}{k}\\ &= \frac{1}{k(k+1)}\{(P_n - P_{n-k-1}) - (k+1)p_{n-k}\}\\ &= \frac{1}{k(k+1)}\left\{\left(\sum_{k=n-m}^{n} p_k\right) - (k+1)p_{n-k}\right\},\end{aligned}$$

which is non-negative or non-positive whenever (p_n) is non-decreasing or non-increasing respectively. Hence

$$\left\{\frac{P_n - P_{n-m}}{m}\right\}_{m=1}^{n+1}$$

is monotonic whenever (p_n) is monotonic and this implies that

$$(8)\sum_{k=1}^{n}\left|\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\right| = \left|p_n - \frac{P_n}{n+1}\right|$$

by using convention $P_{-1} = 0$. Thus using (8) and (1) in (6), we get

$$(9)\|s_n(f) - N_n(f)\|_p = O(n^{-1})$$

Finally, by using (9) and Theorem 4.2.5, we get (2) with $\alpha = 1$.

Case III: $p = 1$ and $0 < \alpha < 1$.

By Abel's transformation and using convention $p_{-1} = 0$,we get

$$N_n(f;x) - f(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\{s_k(f) - f(x)\}$$
$$= \frac{1}{P_n}\left[p_0\sum_{k=0}^{n}\big(s_k(f) - f(x)\big) + \sum_{k=0}^{n-1}\Delta_k p_{n-k}\sum_{m=0}^{k}\{s_k(f) - f(x)\}\right]$$
$$= \frac{1}{P_n}\left[(p_0 + p_{-1})\sum_{k=0}^{n}\big(s_k(f) - f(x)\big) + \sum_{k=0}^{n-1}\Delta_k p_{n-k}\sum_{m=0}^{k}\{s_k(f) - f(x)\}\right]$$
$$= \frac{1}{P_n}\sum_{k=0}^{n}\Delta_k\, p_{n-k}\sum_{m=0}^{k}\{s_k(f) - f(x)\}$$
$$= \frac{1}{P_n}\sum_{k=0}^{n}\Delta_k\, p_{n-k}\left[\sum_{m=0}^{k} s_k(f) - (n+1)f(x)\right]$$
$$= \frac{1}{P_n}\sum_{k=0}^{n}(k+1)\Delta_k\, p_{n-k}\{\sigma_k(f;x) - f(x)\}$$

Hence, by theorem 4.2.6, we get

$$\|f(x) - N_n(f;x)\|_1 \le \frac{1}{P_n}\sum_{k=0}^{n}(k+1)|\Delta_k p_{n-k}|\|\sigma_k(f;x) - f(x)\|_1$$
$$= O\left(\frac{1}{P_n}\right)\sum_{k=0}^{n}(k+1)^{1-\alpha}|\Delta_k p_{n-k}|$$
$$= O\left(\frac{n^{1-\alpha}}{P_n}\right)\sum_{k=0}^{n}|\Delta_k p_{n-k}| = O\left(\frac{n^{1-\alpha}}{P_n}\right).(p_n)$$
$$= O\left(\frac{n^{1-\alpha}}{P_n}\right).O\left(\frac{P_n}{n}\right)$$
$$= O(n^{-\alpha}),$$

Since p_n non-decreasing and using (1)
This completes the proof.■

Theorem 4.2.12: [8] Let $f \in Lip(\alpha, 1), 0 < \alpha < 1$.And let (p_n) be positive non-decreasing sequence with$(n+1)p_n = O(P_n)$.Then

$$\|f - R_n(f)\|_1 = O(n^{-\alpha}).$$

Proof: For $p = 1$ and $0 < \alpha < 1$.We get by Abel's transformation

$$f - R_n(f) = \frac{1}{P_n}\sum_{k=0}^{n} p_k\big(f - s_k(f)\big) = \frac{1}{P_n}\sum_{k=0}^{n-1}\Delta p_k\left(\sum_{m=0}^{k} f - s_k(f)\right) + \frac{p_n}{P_n}\sum_{k=0}^{n}[f - s_k(f)]$$
$$= \frac{1}{P_n}\sum_{k=0}^{n-1}\Delta p_k\big((n+1)f - (n+1)\sigma_k(f)\big) + \frac{p_n}{P_n}\big((n+1)f - (n+1)\sigma_n(f)\big)$$
$$= \frac{1}{P_n}\sum_{k=0}^{n-1}\Delta p_k(n+1)\big(f - \sigma_k(f)\big) + \frac{p_n}{P_n}(n+1)\big(f - \sigma_n(f)\big)$$

Thus we have

$$\begin{aligned}\|f-R_n(f)\|_1 &= \left\|\frac{1}{P_n}\sum_{k=0}^{n-1}\Delta p_k(n+1)\big(f-\sigma_k(f)\big)+\frac{p_n}{P_n}(n+1)\big(f-\sigma_n(f)\big)\right\|_1 \\ &\le \frac{1}{P_n}\sum_{k=0}^{n-1}|\Delta p_k|(n+1)\|f-\sigma_k(f)\|_1+\frac{p_n}{P_n}(n+1)\|f-\sigma_n(f)\|_1 \\ &= \frac{1}{P_n}\sum_{k=0}^{n-1}|\Delta p_k|(n+1)O(n^{-\alpha})+O(1)\cdot O(n^{-\alpha}) \\ &= O(n^{-\alpha})+O(n^{-\alpha})=O(n^{-\alpha}).\ \blacksquare\end{aligned}$$

4.3 Approximation by general class of triangular matrices using trigonometrical polynomials

In this section, we shall weaken the conditions of monotonicity given by theorems (4.2.11) and (4.2.12); we see that these theorems assumed the sequence $p_n$to be monotonic. Here we will give a less strength conditions on p_n but keeping the degree of estimate. Before we do that, we introduce some concepts about sequences.

Definition 4.3.1: [6] A positive sequence $p:=(p_n)$ is called almost monotone decreasing (increasing) if there exists a constant $K:=K(p)$, depends only on p, such that for all $n\ge m$

$$p_n \le Kp_m(Kp_n \ge p_m).$$

Such sequences will be denoted as $p\in AMDS$ and $p\in AMIS$, respectively.

We shall also use the notation

$$\Delta g_n = g_n - g_{n+1}$$

An auxiliary lemma is needed to proof the next theorem.

Lemma 4.3.2: [6]Let$\{p_n\}\in$ AMDS, or let$\{p_n\}\in$ AMIS and satisfy(12). Then, for$0<\alpha<1$,

$$\sum_{k=1}^{n}k^{-\alpha}p_{n-k}=O(n^{-\alpha}P_n)$$

Proof: Let r denote the integral part of $n/2$.Then, if $\{p_n\}\in AMDS$,

$$\begin{aligned}\sum_{m=1}^{n}m^{-\alpha}p_{n-m} &\le Kp_{n-r}\sum_{m=1}^{r}m^{-\alpha}+(r+1)^{-\alpha}\sum_{m=r+1}^{n}p_{n-m} \\ &\le Kp_{n-r}\sum_{m=1}^{n}m^{-\alpha}+(r+1)^{-\alpha}\sum_{m=0}^{n}p_{n-m} \\ &= O(n^{1-\alpha})p_{n-r}+O(n^{-\alpha})P_n \\ &= O(n^{-\alpha})P_n+O(n^{-\alpha})P_n \\ &= O(n^{-\alpha}P_n).\end{aligned}$$

If$\{p_n\} \in AMDS$, and (1) is valid, then

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} \le K p_n \sum_{m=1}^{r} m^{-\alpha} + (r+1)^{-\alpha} \sum_{m=r+1}^{n} p_{n-m}$$
$$\le K p_n \sum_{m=1}^{n} m^{-\alpha} + (r+1)^{-\alpha} \sum_{m=0}^{n} p_{n-m}$$
$$= O(P_n/n) \sum_{m=1}^{n} m^{-\alpha} + O(n^{-\alpha}) P_n$$
$$= O(P_n/n) O(n^{1-\alpha}) + O(n^{-\alpha}) P_n$$
$$= O(n^{-\alpha} P_n).$$

The proof is complete.■

In the previous section, an approximation in the L^p space is established by conditions involving the monotonicity of the positive sequence (p_n), the next theorem will give a generalization of theorem 4.2.13 by weakened the conditions on the sequence (p_n), note that the non-increasing sequence is AMDS and the non-decreasing sequence is AMIS.

Theorem 4.3.3: [6] Let $f \in Lip\,(\alpha, p)$ and let $\{p_n\}$be positive. If one of the conditions

$(i)\;\; P > 1, 0 < \alpha < 1 \; and \; \{p_n\} \in \; AMDS,$

$(ii)\;\; p > 1, 0 < \alpha < 1 \; and \; \{p_n\} \in \; AMIS \; and \; (1) holds,$

$(iii)\; p > 1, \alpha = 1 \; and \; \sum_{k=0}^{n-1} k|\Delta p_k| = O(P_n),$

$(iv) p > 1, \alpha = 1, \sum_{k=0}^{n-1} |\Delta p_k| = O(P_n/n)$ and(1) holds

$(v)\;\; p = 1, 0 < \alpha < 1$ and $\sum_{k=-1}^{n-1} |\Delta p_k| = O(P_n/n)$

maintains, then

(10)$\|f - N_n(f)\|_p = O(n^{-\alpha}).$

Proof: We prove the cases (i) and (ii) together utilizing theorem 4.2.5 and lemma 4.3.2. Since

(11)$N_n(f;x) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}(s_k(f;x) - f(x))$

Thus

$$\|N_n(f;x) - f(x)\|_p \le \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \|s_k(f;x) - f(x)\|_p$$
$$= \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k} \|s_k(f;x) - f(x)\|_p + \frac{1}{P_n} p_n \|s_0(f;x) - f(x)\|_p$$
$$= \frac{1}{P_n} \sum_{k=1}^{n} p_{n-k} O(k^{-\alpha}) + O(p_n/P_n) = O(n^{-\alpha}).$$

Next, we consider the case (iv).By Lemma 4.2.9,

$$N_n(f;x)=\frac{1}{P_n}\sum_{k=0}^{n}P_{n-k}u_k(f),$$

and thus

$$s_n(f;x)-N_n(f;x)=\frac{1}{P_n}\sum_{k=1}^{n}(P_n-P_{n-k})u_k(f)$$

hence, again by Abel's transformation and $P_{-1}\ =\ 0$,

$$s_n(f;x)-N_n(f;x)=\frac{1}{P_n}\sum_{k=1}^{n}\Delta_k\left(\frac{P_n-P_{n-k}}{k}\right)\sum_{m=1}^{k}mu_m(f)+\frac{1}{n+1}\sum_{k=1}^{n}ku_k(f).$$

therefor,

$$(12)\|s_n(f)-N_n(f)\|_p$$
$$=\frac{1}{P_n}\sum_{k=1}^{n}\left|\Delta_k\left(\frac{P_n-P_{n-k}}{k}\right)\right|\left\|\sum_{m=1}^{k}mu_m(f)\right\|_p+\frac{1}{n+1}\left\|\sum_{k=1}^{n}ku_k(f)\right\|_p.$$

by Lemma 4.2.10 and Lemma 4.2.7 we have

$$(13)\left\|\sum_{k=1}^{n}ku_k(f;x)\right\|_p=(n+1)\|\sigma_n(f;x)-s_n(f;x)\|_p=O(1).$$

Combining (12) and (13), we obtain that

$$(14)\|s_n(f;x)-N_n(f;x)\|_p=O\left(\frac{1}{P_n}\right)\sum_{k=1}^{n}\left|\Delta_k\left(\frac{P_n-P_{n-k}}{k}\right)\right|+O(n^{-1}).$$

An elementary calculation yields that (see(7))

$$(15)\Delta_k\left(\frac{P_n-P_{n-k}}{k}\right)=\frac{1}{k(k+1)}\left\{\left[\sum_{m=n-k}^{n}p_m\right]-(k+1)p_{n-k}\right\}$$

Next we shall verify by induction that

$$(16)\left|\sum_{m=n-k}^{n}p_m-(k+1)p_{n-k}\right|\le\sum_{m=1}^{k}m|p_{n-m+1}-p_{n-m}|$$

If $k\ =1$, then

$$\left|\left(\sum_{m=n-1}^{n}p_m\right)-2p_{n-1}\right|=|p_n-p_{n-1}|.$$

Thus (16) holds.

Now let us assume that (16) is proved for $k\ =\ v$ and we verify it for $k\ =v\ +1(\le n)$. Since

$$\left|\left(\sum_{m=n-(v+1)}^{n}p_m\right)-(v+2)p_{n-(v+1)}\right|=\left|\left(\sum_{m=n-v}^{n}p_m\right)-(v+1)p_{n-(v+1)}\right|$$
$$\le\left|\left(\sum_{m=n-v}^{n}p_m\right)-(v+1)p_{n-v}\right|+|(v+1)p_{n-v}-(v+1)p_{n-(v+1)}|$$

$$\leq \sum_{m=1}^{v} m|p_{n-m+1} - p_{n-m}| + (v+1)\big|p_{n-v} - p_{n-(v+1)}\big|,$$

Thus (16) is proved for $m = \nu + 1$.
Using this and (15), we get that

$$\sum_{k=1}^{n}\big|\Delta_k\big(k^{-1}(P_n - P_{n-k})\big)\big| \leq \sum_{k=1}^{n}\frac{1}{k(k+1)}\sum_{m=1}^{k} m|p_{n-m+1} - p_{n-m}|$$
$$\leq \sum_{k=1}^{n} k|p_{n-k+1} - p_{n-k}|\sum_{m=k}^{\infty}\frac{1}{m(m+1)}$$
$$= \sum_{k=0}^{n-1}|\Delta p_k|.$$

Now combining this, the assumption

$$\sum_{k=0}^{n-1}|\Delta p_k| = O(P_n/n)$$

and (14), we get

$$\|s_n(f;x) - N_n(f;x)\|_p = O(n^{-1})$$

this and Theorem 4.2.5 with $\alpha = 1$ yield (10). Here with the case (iv) is proved.

In the proof of the case (iii), we first verify that the condition

$$\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$$

implies that

$$(17) B_n := \sum_{k=0}^{n-1}\left|\Delta_k\left(\frac{P_n - P_{n-k}}{k}\right)\right| = O(P_n/n).$$

For simplicity we shall write Δp_{n-k} instead of $p_{n-k} - p_{n-k+1}$, by (15) and (16)

$$B_n \leq \sum_{k=1}^{n}\frac{1}{k(k+1)}\sum_{m=1}^{k} m|\Delta p_{n-m}|.$$

denote again by r the integer part of $n/2$. Then, we have

$$\sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k} m|\Delta p_{n-m}| \leq \sum_{k=1}^{r}|\Delta p_{n-k}| \leq \sum_{k=r-2}^{n-1}|\Delta p_k| = O(P_n/n).$$

at the last step we have used the condition

$$\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$$

On the other hand,

$$\sum_{k=r}^{n}\frac{1}{k(k+1)}\sum_{m=1}^{r} m|\Delta p_{n-m}| \leq \sum_{k=r}^{n}\frac{1}{k(k+1)}\left\{\sum_{m=1}^{r} m|\Delta p_{n-m}| + \sum_{m=r}^{r} m|\Delta p_{n-m}|\right\}$$
$$=: B_{n1} + B_{n2}$$

Furthermore, using again our assumption, we get

$$B_{n1} \leq \sum_{k=r}^{n} (k+1)^{-1} \sum_{m=r-2}^{n-1} |\Delta p_m| = O(P_n/n)$$

and

$$B_{n2} \leq \sum_{k=r}^{n} (k+1)^{-1} \sum_{m=r}^{k} |\Delta p_{n-m}| = O(n^{-1})\{|\Delta p_0| + 2|\Delta p_1| + \cdots + (r+1)|\Delta p_{r+1}|\}$$

$$= O(P_n/n).$$

summing up our partial results, we verified (17). Thus, (14) and theorem 4.2.5 again yield (10).

Finally, the prove of the case (v). Utilizing $(10), p_{-1} = 0$ and the Abel's transformation, we get

$$N_n(f;x) - f(x) = \frac{1}{P_n} \sum_{m=0}^{n} (\Delta_m p_{n-m}) \sum_{k=0}^{m} \big(s_k(f;x) - f(x)\big)$$

$$= \frac{1}{P_n} \sum_{m=0}^{n} (m+1)(\Delta_m p_{n-m})\{\sigma_m(f;x) - f(x)\}.$$

Hence, by theorem 4.2.6, we have that

$$\|f - N_n(f)\|_1 \leq \frac{1}{P_n} \sum_{m=0}^{n} (m+1)|\Delta_m p_{n-m}| \|f - \sigma_m(f)\|_1$$

$$= O\left(\frac{1}{P_n}\right) \sum_{m=0}^{n} (m+1)^{1-\alpha} |\Delta_m p_{n-m}|$$

$$= O\left(\frac{n^{1-\alpha}}{P_n}\right) \sum_{m=-1}^{n} |\Delta p_m| = O(n^{-\alpha}).$$

Here with the case (v) is also verified, and thus the proof is complete.■

Theorem 4.3.4: [6]Let $f \in Lip\,(\alpha, 1), 0 < \alpha < 1$.If the positive$\{p_n\}$ satisfies (1)and the condition

$$\sum_{k=0}^{n-1} |\Delta p_k| = O(P_n/n)$$

holds, then

$$(18) \|f - R_n(f)\|_1 = O(n^{-\alpha}).$$

Proof: Since

$$R_n(f) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k\{s_k(f) - f(x)\}$$

Thus following the consideration of the case (v) of Theorem 4.3.3, we get

$$\|f(x)-R_n(f)\|_1 = \frac{1}{P_n}\left\|\sum_{k=0}^{n} p_k\{f-s_k(f)\}\right\|_1$$
$$\leq \frac{1}{P_n}\sum_{k=0}^{n-1}(k+1)|\Delta p_k|\|f-\sigma_k(f;x)\|_1 + (n+1)p_n\frac{1}{P_n}\|f-\sigma_n(f;x)\|_1$$
$$= O\left(n^{1-\alpha}\frac{1}{P_n}\right)\sum_{k=0}^{n-1}|\Delta p_k| + O(n^{-\alpha}) \;= O(n^{-\alpha}).$$

This proves the theorem. ■

It is very easy to examine that all of the conditions in theorem 4.3.3 and 4.3.4 claim less than the requirements of Theorems 4.2.11 and 4.2.12. Since if we consider the first part of theorem 4.2.11, that is, for $p > 1, 0 < \alpha < 1, (p_n)$is non-decreasing then it is obvious that the sequence is AMIS, for which the condition ii of 4.3.3 is satisfied.

Also for $p > 1, 0 < \alpha < 1, (p_n)$ is non-increasing sequence, then(p_n) is AMDS and condition I of 4.2.3 holds. Moreover, if $p > 1, \alpha = 1, (p_n)$ is non-increasing then we may write

$$\sum_{k=1}^{n-1} k|\Delta p_k| = \sum_{k=1}^{n-1} k(p_k - p_{k+1}) = 1(p_1-p_2)+2(p_2-p_3)+\cdots+(n-1)(p_{n-1}-p_n)$$
$$= p_1+p_2+\cdots+p_{n-1}-(n-1)p_n$$
$$= p_1+p_2+\cdots+p_{n-1}+p_n-(n)p_n = P_n - np_n \leq P_n = O(P_n).$$

and that is condition III of 4.3.3.

Now if $p > 1, \alpha = 1, (p_n)$ is non-decreasing and satisfying condition 12 then we have

$$\sum_{k=0}^{n-1}|\Delta p_k| = \sum_{k=0}^{n-1}(p_{k+1}-p_k) = (p_1-p_0)+(p_2-p_1)+\cdots+(p_n-p_{n-1})$$
$$= -p_0 + p_n = p_n - p_0 \leq p_n = O\left({P_n}/{n}\right).$$

so condition IV of 4.3.3 is satisfied.

Finally, if the second condition in 4.2.11 holds, that is, if $p = 1,\ 0 < \alpha < 1$ and (p_n) is non-decreasing and 12 holds, then

$$\sum_{k=-1}^{n-1}|\Delta p_k| = \sum_{k=-1}^{n-1}(p_{k+1}-p_k) = (p_0-p_{-1})+(p_1-p_0)+\cdots+(p_n-p_{n-1})$$
$$= p_n = \left({P_n}/{n}\right)$$

Since by definition $p_{-1} = 0$, therefore, the last condition of 4.3.3 is satisfied.
By the same we can see that theorem 4.3.4 is more general that the corresponding theorem 4.2.12, for which if $p = 1, 0 < \alpha < 1,$ and (p_n) is non-decreasing with $(n+1)p_n = O(P_n)$.

Then

$$\sum_{k=0}^{n-1}|\Delta p_k| = \sum_{k=0}^{n-1}(p_{k+1} - p_k) = (p_1 - p_0) + (p_2 - p_1) + \cdots + (p_n - p_{n-1})$$

$$= p_n - p_0 = \left({}^{P_n}/_{n}\right)$$

In the last two theorems we obtain the same degree of approximation for any function $f \in Lip(\alpha, p)$, by weakened conditions, we now will treat the same theorems (4.2.11, 4.2.12) with a general class of triangular matrices, thus we can deduce these two theorems as a corollaries of our next theorems.

Let $A = (a_{n,k})$ be a lower triangular regular matrix with non-negative entries and row sumss_n^A. such a matrix A is said to have monotone rows if, for each $n, \{a_{nk}\}$ is either non-increasing or non-decreasing in $k, 0 \le k \le n$.Also we call the matrix $A = (a_{n,k})$ has almost monotone increasing (decreasing) rows if there exists a constant K, depending only on A, such that $a_{n,k} \le K a_{n,m} (a_{n,m} \le K a_{n,k})$ for each n and $0 \le k \le m \le n$.

Lemma 4.3.5: [7] Let A have monotone rows and satisfies the relation

$(19)(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1), \qquad r = [n/2]$

then for $0 < \alpha < 1$,

$(20) \sum_{k=0}^{n} a_{n,k}(k+1)^{-\alpha} = O(n^{-\alpha}).$

Proof: Let $r := [n/2]$,then

$$\sum_{k=0}^{n} a_{n,k}(k+1)^{-\alpha} = \sum_{k=0}^{r} a_{n,k}(k+1)^{-\alpha} + \sum_{k=r+1}^{n} a_{n,k}(k+1)^{-\alpha}$$

Case I:If $\{a_{n,k}\}$ is non-decreasing in k. Then, using (19), we have

$$\sum_{k=0}^{n} a_{n,k}(k+1)^{-\alpha} \le a_{n,r}\sum_{k=0}^{r}(k+1)^{-\alpha} + (r+1)^{-\alpha}\sum_{k=r+1}^{n} a_{n,k}$$

$$\le a_{n,r}\sum_{k=0}^{n}(k+1)^{-\alpha} + (r+1)^{-\alpha}s_n^A$$

$$= O((n+1)^{-1})O((n+1)^{1-\alpha}) + O(n)^{-\alpha} = O(n)^{-\alpha}$$

Case II:If $\{a_{n,k}\}$ is non-increasing in k. Then, using (28),

$$\sum_{k=0}^{n} a_{n,k}(k+1)^{-\alpha} \le a_{n,0}\sum_{k=0}^{r}(k+1)^{-\alpha} + O(n^{-\alpha}) = O(n^{-\alpha}). \blacksquare$$

With same notation previously stated, we define the matrix transformation $T_n^A(f)$ as follows

$$T_n^A(f) = \sum_{k=0}^{n} a_{n,k} s_k(f;x).$$

For a given positive sequence (p_n), if we consider the lower triangular matrix with entries $a_{n,k} = p_{n-k}/P_n$, $P_n = \sum_{k=0}^{n} p_k$.Then the Nörlund transform can be regarded as a matrix transform, so this transformation is more general than the Nörlund transformations, also we note that the row sum of this matrix is clearly 1 since

$$s_n^A = \sum_{k=0}^{n} a_{n,k} = \sum_{k=0}^{n} p_{n-k}/P_n = P_n/P_n = 1.$$

Theorem 4.3.6: [7]Let$f \in Lip(\alpha, p)$, and let A have monotone rows and satisfy

(21)$|s_n^A - 1| = O(n^{-\alpha})$.

If one of the conditions

(i) if $p > 1,$ $0 < \alpha < 1,$ and A also satisfies

$$(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1), \text{where } r := [n/2],$$

(ii) if $p > 1,$ $\alpha = 1,$

(iii) if $p = 1,$ $0 < \alpha < 1,$ and A also satisfies

(22)$(n+1)\max\{a_{n,0}, a_{n,n}\} = O(1)$,

holds. Then

(23)$\|f - T_n(f)\|_p = O(n^{-\alpha})$.

Proof: Case I:$p > 1,$ $0 < \alpha < 1$.

$$\begin{aligned} T_n(f) - f &= \sum_{k=0}^{n} a_{n,k} s_k(f) - s_n^A \cdot f(x) + (s_n^A - 1)\cdot f(x) \\ &= \sum_{k=0}^{n} a_{n,k} s_k(f) - \sum_{k=0}^{n} a_{n,k} \cdot f(x) + (s_n^A - 1)\cdot f(x) \\ &= \sum_{k=0}^{n} a_{n,k} \cdot \big(s_k(f) - f(x)\big) + (s_n^A - 1)\cdot f(x) \end{aligned}$$

Using (21) and Theorem 4.2.5 and Lemma 4.3.5,

$$\begin{aligned} \|T_n(f) - f\|_p &\le \sum_{k=0}^{n} a_{n,k} \|s_k(f) - f\|_p + |s_n^A - 1|\|f\|_p \\ &= \sum_{k=0}^{n} a_{n,k} O((k+1)^{-\alpha}) + O(n^{-\alpha}) = O(n^{-\alpha}) + O(n^{-\alpha}) = O(n^{-\alpha}). \end{aligned}$$

Case II: $p > 1, \alpha = 1$.

$$\|T_n(f) - f\|_p \le \|T_n(f) - s_n(f)\|_p + \|s_n(f) - f\|_p.$$

form theorem 4.2.5, when $\alpha = 1$,

$$\|s_n(f) - f\|_p = O(n^{-1})$$

Therefore, it remains to prove that

$$\|T_n(f) - s_n(f)\|_p = O(n^{-1})$$

Define

$$A_{n,k} = \sum_{i=k}^{n} a_{n,i}\,,$$

and using the fact that

$$A_{n,0} = \sum_{i=0}^{n} a_{n,i} = s_n^A,$$

then we may write

$$\begin{aligned} T_n(f) = \sum_{k=0}^{n} a_{n,k} s_k(f) &= \sum_{k=0}^{n} a_{n,k} \sum_{i=0}^{k} u_i(f) \\ &= a_{n,0}u_0 + a_{n,1}(u_0 + u_1) + \cdots + a_{n,n}(u_0 + \cdots + u_n) \\ &= u_0\big(a_{n,0} + \cdots a_{n,n}\big) + \cdots + u_n(a_{n,n}) \\ &= \sum_{k=0}^{n} A_{n,k} u_k(f;x). \end{aligned}$$

also,

$$\begin{aligned} s_n(f) = \sum_{k=0}^{n} u_k(f;x) &= \sum_{k=0}^{n} A_{n,0}u_k(f;x) + \sum_{k=0}^{n} \big(1 - A_{n,0}\big) u_k(f;x) \\ &= \sum_{k=0}^{n} A_{n,0}u_k(f;x) + (1 - s_n^A) \sum_{k=0}^{n} u_k(f;x) \\ &= \sum_{k=0}^{n} A_{n,0}u_k(f;x) + (1 - s_n^A) s_n(f;x). \end{aligned}$$

Now since $\|s_n\|_p \le \|f\|_p$, then

$\|T_n(f) - s_n(f)\|_p$

$$= \left\| \sum_{k=0}^{n} A_{nk}u_k(f;x) - \sum_{k=0}^{n} A_{n0}u_k(f;x) + (s_n^A - 1)s_n(f;x) \right\|_p$$

$$\le \left\| \sum_{k=0}^{n} A_{nk}u_k(f;x) - \sum_{k=0}^{n} A_{n0}u_k(f;x) \right\|_p + |1 - s_n^A| \|s_n(f;x)\|_p$$

$$\le \left\| \sum_{k=1}^{n} \big(A_{n,k} - A_{n,0}\big) u_k(f;x) \right\|_p + |1 - s_n^A| \|f\|_p$$

Define for each $1 \leq k \leq n$,

$$b_{n,k} := \frac{A_{n,k} - A_{n,0}}{k}.$$

Using summation by parts (Abel's transformation), by setting $ku_k = v_k$

$$\begin{aligned}
\sum_{k=1}^{n} (A_{n,k} - A_{n,0}) u_k(f;x) &= \sum_{k=1}^{n} \frac{A_{n,k} - A_{n,0}}{k} k u_k(f;x) = \sum_{k=1}^{n} b_{n,k} v_k \\
&= \sum_{k=1}^{n-1} (b_{n,k} - b_{n,k+1}) \sum_{j=0}^{k} v_j + b_{n,n} \sum_{k=0}^{n} v_k \\
&= \sum_{k=1}^{n-1} (b_{n,k} - b_{n,k+1}) \sum_{j=1}^{k} j u_j + b_{n,n} \sum_{k=1}^{n} k u_k \\
&= b_{n,n} \sum_{k=1}^{n} k u_k(f;x) + \sum_{k=1}^{n-1} \Delta_k b_{n,k} \sum_{j=0}^{k} j u_j(f;x).
\end{aligned}$$

Therefore

$$\|T_n(f) - s_n(f)\|_p \leq \left\| b_{n,n} \sum_{k=1}^{n} k u_k(f;x) \right\|_p + \left\| \sum_{k=1}^{n-1} \Delta_k b_{n,k} \sum_{j=1}^{k} j u_j(f;x) \right\|_p + O(n^{-1})$$

Now from Lemma 4.2.7 and 4.2.10,

$$\begin{aligned}
\left\| \sum_{j=1}^{n} j u_j(f;x) \right\|_p &= \|(n+1)(s_n(f) - \sigma_n(f))\|_p \\
&= (n+1) O(n^{-1}) \\
&= O(1).
\end{aligned}$$

Note that

$$\begin{aligned}
|b_{n,n}| &= \frac{1}{n} |A_{n,0} - A_{n,n}| = \frac{1}{n} \left| s_n^{(A)} - a_{n,n} \right| \\
&\leq \frac{1}{n} \left| s_n^{(A)} - 1 \right| + \frac{1}{n} |1 - a_{n,n}| \\
&= \frac{1}{n} O(n^{-1}) + k = (n)^{-1} O(1).
\end{aligned}$$

Thus

$$\left\| b_{n,n} \sum_{k=1}^{n} j u_j(f;x) \right\|_p = O(n^{-1}).$$

We may write

$$\begin{aligned}
\Delta_k b_{n,k} = b_{n,k} - b_{n,k+1} &= \frac{A_{n,k} - A_{n,0}}{k} - \frac{A_{n,k+1} - A_{n,0}}{k+1} \\
&= \frac{A_{n,k} - A_{n,0} + A_{n,k+1} - A_{n,k+1} + A_{n,0} - A_{n,0}}{k} - \frac{A_{n,k+1} - A_{n,0}}{k+1}
\end{aligned}$$

$$= \frac{A_{n,k} - A_{n,0} - A_{n,k+1} + A_{n,0}}{k} + \frac{A_{n,k+1} - A_{n,0}}{k} - \frac{A_{n,k+1} - A_{n,0}}{k+1}$$

$$= \frac{1}{k}\Delta_k(A_{n,k}) + \frac{A_{n,k+1} - A_{n,0}}{k} - \frac{A_{n,k+1} - A_{n,0}}{k+1}$$

$$= \frac{1}{k}\Delta_k(A_{n,k}) + \frac{A_{n,k+1} - A_{n,0}}{k(k+1)}$$

$$= \frac{1}{k(k+1)}\left[(k+1)\Delta_k(A_{n,k}) + A_{n,k+1} - A_{n,0}\right]$$

$$= \frac{1}{k(k+1)}\left[(k+1)(A_{n,k} - A_{n,k+1}) + (A_{n,k+1} - A_{n,0})\right]$$

$$= \frac{1}{k(k+1)}\left[k(A_{n,k} - A_{n,k+1}) + (A_{n,k} - A_{n,k+1}) - \sum_{r=0}^{k} a_{n,r}\right]$$

$$= \frac{1}{k(k+1)}\left[k\left(\sum_{r=k}^{n} a_{n,r} - \sum_{r=k+1}^{n} a_{n,r}\right) + \left(\sum_{r=k}^{n} a_{n,r} - \sum_{r=k+1}^{n} a_{n,r}\right) - \sum_{r=0}^{k} a_{n,r}\right]$$

$$= \frac{1}{k(k+1)}\left[k a_{n,k} + a_{n,k} - \sum_{r=0}^{k} a_{n,r}\right] = \frac{1}{k(k+1)}\left[(k+1)a_{n,k} - \sum_{r=0}^{k} a_{n,r}\right]$$

If $\{a_{n,k}\}$ is non increasing in k,then $\Delta_k b_{n,k} \leq 0$, and if $\{a_{n,k}\}$ non-decreasing in k implies that $\Delta_k b_{n,k} \geq 0$, so that

$$\sum_{k=1}^{n-1}|\Delta_k b_{n,k}| = |b_{n,1} - b_{n,n}| = \left|A_{n,1} - A_{n,0} - \frac{A_{n,n} - A_{n,0}}{n}\right| \leq |A_{n,1} - A_{n,0}| + \left|\frac{A_{n,n} - A_{n,0}}{n}\right|$$

$$= O(n^{-1}) + \frac{O(1)}{n} = O(n^{-1}).$$

and (23) is satisfied.

Case III: $p = 1, 0 < \alpha < 1$, From (21), using Abel's transformation, Lemma 4.2.6 , and the fact that $a_{n,n+1} = 0$,

$$\|T_n(f) - f\|_1 = \left\|\sum_{k=0}^{n} a_{n,k}(s_k(f) - f) + (1 - s_n^A)f\right\|_1$$

$$= \left\|\sum_{k=0}^{n-1} \Delta_k a_{n,k}\left(\sum_{i=0}^{k}(s_i(f) - f)\right) + a_{nn}\sum_{k=0}^{n}(s_k(f) - f)\right.$$

$$\left. + |(1 - s_n^A)|f\right\|_1$$

$$= \left\|\sum_{k=0}^{n} \Delta_k a_{n,k}\sum_{i=0}^{k}(s_i(f) - f) + (1 - s_n^A)f\right\|_1$$

$$\leq \left\| \sum_{k=0}^{n} \Delta_k a_{n,k} \sum_{i=0}^{k} (s_i(f) - f) \right\|_1 + |1 - s_n^A| \|f\|_1$$

$$\leq \sum_{k=0}^{n} (k+1) |\Delta_k a_{n,k}| \|\sigma_k(f) - (f)\|_1 + O(n^{-\alpha})$$

$$= \sum_{k=0}^{n} |\Delta_k a_{n,k}| O((k+1)^{1-\alpha}) + O(n^{-\alpha})$$

$$= O((n+1)^{1-\alpha}) \sum_{k=0}^{n} |\Delta_k a_{n,k}| + O(n^{-\alpha})$$

If $\{a_{n,k}\}$ is non-increasing in$k$, then

$$\sum_{k=0}^{n} |\Delta_k a_{n,k}| = a_{n,n} + \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1}) = a_{n,n} + a_{n,o} - a_{n,n} = a_{n,o}$$

If $a_{n,k}$ is non-decreasing in$k$, then

$$\sum_{k=0}^{n} |\Delta_k a_{n,k}| = a_{n,n} + a_{n,n} - a_{n,o} \leq 2a_{n,n}$$

Using (22),

$$\|T_n(f) - f\|_1 = O((n+1)^{-\alpha}) = O(n^{-\alpha}). \blacksquare$$

It easy to examine that all conditions of theorem 4.3.6 is more general than theorem 4.2.11, since if we consider the entries of the matrix A to be $a_{n,k} = {p_{n-k}}/{P_n}$, then A can be considered as a Nörlund matrix, that is, a matrix that defines a Nörlund means by it is rows, for this matrix $s_n^A = 1$, so (21) is satisfied.

Let us consider the case one of 4.2.11,that is, when $p > 1, 0 < \alpha < 1$, (1) holds, and (p_n) is non-decreasing then $a_{n,k}$ is non-increasing sequence in k so

$$(n+1) \max\{a_{n,0}, a_{n,r}\} = (n+1)a_{n,0} = (n+1)\frac{p_n}{P_n} = O(1)$$

Thus apart of condition I of theorem 4.3.6 is satisfied.

Now assume for the same conditions but for a non-increasing sequence (p_n) then $a_{n,k}$ is non-decreasing in k, and we may write

$$(n+1) \max\{a_{n,0}, a_{n,r}\} \leq \frac{(n+1)(n-r+1)p_{n-r}}{(n-r+1)P_{n-r}} = O(1)$$

for which the case one is satisfied.

Note that condition II of theorem 4.3.6 is always hold here whenever $p > 1, \alpha = 1$.

If we assume the last condition of 4.2.11, that is, for $p = 1, 0 < \alpha < 1$, condition (1) holds and p_n is non-decreasing sequence, then $a_{n,k}$ is non-increasing and

$$(n+1)\max\{a_{n,0}, a_{n,n}\} = (n+1)a_{n,0} = (n+1)\frac{p_n}{P_n} = O(1).$$

and that is condition III of 4.3.6, so by the previous argument we saw that theorem 4.3.6 is more general than theorem 4.2.11, also we can deduce it from this theorem when we restrict the matrix A to be Nörlund matrix.

Chapter Five

Trigonometric Approximation in the Weighted L^p Spaces

5.1 Trigonometrical approximation in the means

So far the theorems stated are giving the approximation of function in the non-weighted Lebesgue space, we discussed two ways, approximation in the means (Nörlund means) and the by matrix transformation, in this section an extension of these theorems is discussed, that is, the approximation will be for function in the weighted Lebesgue space, we note here that following theorems are just a generalization of the proceeding ones by giving conditions in weighted Lebesgue space.

The same theorems stated before, namely 4.2.11 and 4.2.12, are investigated here with more general class of functions, the weighted L^p space, again we develop these theorems by offering different manners of approximation, specially the matrix transformation method.

Note: In the weighted L^p space, we use the Muckenhoupt weights $\mathcal{A}_p$, this kind of weights plays a critical rule in many different aspects of mathematics.

Definition 5.1.1: Let $1 < p < \infty,\ w \in \mathcal{A}_p$, and let $f \in L^p_w$. Then the modulus of continuity is defined as

$$\omega_{p,w}(f;\delta) = \sup \|\Delta_h(f)\|_{p,w}\,, for\ \delta > 0.$$

and the supremum is taken over all h such that $|h| \le \delta$, where

$$\Delta_h(f)(x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|dt$$

Note: The Lipschitz class $Lip(\alpha, p, w)$ for $0 < \alpha \le 1$ is given by

$$Lip(\alpha, p, w) = \left\{f \in L^p_w : \omega_{p,w}(f;\delta) = O(\delta^\alpha), \delta > 0\right\}$$

We shall use the same notation as before, also we will do the approximation on the same means specially Nörlund and Riesz means.

Lemma 5.1.2: [2] Let $1 < p < \infty, \omega \in \mathcal{A}_p,\ 0 < \alpha \le 1$. Then the estimate

$$\|f - s_n(f)\|_{p,\omega} = O(n^{-\alpha})$$

holds for every $f \in Lip(\alpha, p, \omega)$ and $n = 1,2, \ldots$

Lemma 5.1.3: [2] Let $1 < p < \infty, \omega \in \mathcal{A}_p$. Then, for $f \in Lip(1, p, \omega)$ the estimate

$$\|s_n(f) - \sigma_n(f)\|_{p,\omega} = O(n^{-1}) \quad n = 1,2, \ldots$$

holds.

Theorem 5.1.4: [2] Let$1 < p < \infty,\ \omega \in \mathcal{A}_p, 0 < \alpha \leq 1,$ and let$(p_n)_0^\infty$ be a monotonic sequence of positive real numbers such that

$(1)(n+1)p_n = O(P_n)$

then, for $f \in Lip(\alpha, p, \omega)$ the estimate

$$\|f - N_n(f)\|_{p,\omega} = O(n^{-\alpha}) \quad n = 1,2, \dots$$

holds.

Proof: Let$0 < \alpha < 1$.Since

$$f\,(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m} f(x)$$

we have

$$f(x) - N_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\{f(x) - s_m(f)(x)\}$$

by Lemma 5.1.2, Lemma 4.2.8 (sec 4.2) and condition (1) we obtain

$$\begin{aligned}
\|f - N_n(f)\|_{p,\omega} &\leq \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}\|f - s_m(f)\|_{p,\omega} \\
&= \frac{1}{P_n}\sum_{m=1}^{n} p_{n-m}\|f - s_m(f)\|_{p,\omega} + \frac{p_n}{P_n}\|f - s_0(f)\|_{p,\omega} \\
&= \frac{1}{P_n}\sum_{m=1}^{n} p_{n-m}\ \ O(m^{-\alpha}) + \frac{p_n}{P_n}\|f - s_0(f)\|_{p,\omega} \\
&= \frac{1}{P_n} O(n^{-\alpha}P_n) + O\left(\frac{1}{n+1}\right) \\
&= O(n^{-\alpha})
\end{aligned}$$

Now let $\alpha = 1$, it is clear that (by Lemma 4.2.9)

$$N_n(f) = \frac{1}{P_n}\sum_{m=0}^{n} P_{n-m} u_m(f)\,(x)$$

by Abel transform,

$$s_n(f) - N_n(f) = \frac{1}{P_n}\sum_{m=1}^{n}(P_n - P_{n-m})u_m(f)$$

$$= \frac{1}{P_n}\sum_{m=1}^{n}\left(\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right)\left(\sum_{k=1}^{m} k u_k\,(f)\right) + \frac{1}{n+1}\sum_{k=1}^{n} k u_k\,(f)$$

and hence

$$\|s_n(f) - N_n(f)\|_{p,\omega}$$

$$\leq \frac{1}{P_n}\sum_{m=1}^{n} \Delta_m \left|\frac{P_n - P_{n-m}}{m}\right| \cdot \left\|\sum_{k=1}^{m} k u_k\,(f)\right\|_{p,\omega} + \frac{1}{n+1}\left\|\sum_{k=1}^{m} k u_k\,(f)\right\|_{p,\omega}$$

Since by lemma 4.2.10

$$s_n(f)(x) - \sigma_n(f)(x) = \frac{1}{n+1}\sum_{k=1}^{n} k\, u_k(f)(x),$$

Thus by Lemma 5.1.3 we get

$$\left\|\sum_{k=1}^{n} k u_k\,(f)\right\|_{p,\omega} = (n+1)\|s_n(f) - \sigma_n(f)\|_{p,\omega} = O(1)$$

Hence,

$$(2)\|s(f) - N_n(f)\|_{p,\omega} \le \frac{1}{P_n}\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| O(1) + O(n^{-1})$$

$$= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| + O(n^{-1})$$

By a simple computation, one can see that

$$\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} = \frac{1}{m(m+1)}\left(\sum_{k=n-m+1}^{n} p_k - mp_{n-m}\right)$$

which shows that

$$\left(\frac{P_n - P_{n-m}}{m}\right)_{m=1}^{n+1}$$

is non-increasing whenever $(p_n)$is non-decreasing and non-decreasing whenever $(p_n)$is non-increasing. This implies that

$$\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| = \left|p_n - \frac{P_n}{n+1}\right| = \frac{1}{n+1}O(P_n)$$

This and the inequality (2) yield

$$\|s_n(f) - N_n(f)\|_{p,\omega} = O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n}\left|\frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1}\right| + O(n^{-1})$$

$$= O\left(\frac{1}{P_n}\right)\frac{1}{n+1}O(P_n) + O(n^{-1})$$

$$= O\left(\frac{1}{n}\right) + O(n^{-1})$$

$$= O(n^{-1})$$

Combining the last estimate with that of Lemma 5.1.2 we obtain

$$\|f - N_n(f)\|_{p,\omega} \le \|f - s_n(f)\|_{p,\omega} + \|s_n(f) - N_n(f)\|_{p,\omega} = O(n^{-1})\blacksquare$$

Theorem 5.1.5: [2] Let$1 < p < \infty,\ \omega \in \mathcal{A}_p,\ 0 < \alpha \le 1,$and let$(p_n)$ be a sequence of positive real numbers satisfying the relation

(3) $\sum_{m=0}^{n-1}\left|\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right| = O\left(\frac{P_n}{n+1}\right)$

Then, for $f \in Lip(\alpha,p,\omega)$ the estimate

$$\|f-R_n(f)\|_{p,\omega} = O(n^{-\alpha}) \qquad n = 1,2,...$$

is satisfied.

Proof: Let $0 < \alpha < 1$,by definition of $R_n(f)(x)$.

$$f(x)-R_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_m\{f(x)-s_m(f)(x)\}\ .$$

from Lemma 5.1.2, we get

(4) $\|f-R_n(f)\|_{p,\omega} \le \frac{1}{P_n}\sum_{m=0}^{n} p_m\|f-s_m(f)\|_{p,\omega}$

$$= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} p_m\, m^{-\alpha} + \frac{p_0}{P_n}\|f-s_0(f)\|_{p,\omega}$$

$$= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} p_m\, m^{-\alpha}$$

by Abel transform,

$$\sum_{m=1}^{n} p_m\, m^{-\alpha} = \sum_{m=1}^{n-1} p_m\,\{m^{-\alpha} - (m+1)^{-\alpha}\} + n^{-\alpha}P_n$$

$$\le \sum_{m=1}^{n-1} m^{-\alpha}\frac{p_m}{m+1} + n^{-\alpha}P_n$$

$$= \sum_{m=1}^{n-1}\left(\frac{P_m}{m+1}-\frac{P_{m+1}}{m+2}\right)\left(\sum_{k=1}^{m} k^{-\alpha}\right) + \frac{P_n}{n+1}\sum_{m=1}^{n-1} m^{-\alpha} + n^{-\alpha}P_n$$

$$= O(n^{-\alpha}P_n)$$

by condition (3).This yield

$$\sum_{m=1}^{n-1} p_m m^{-\alpha} = O(n^{-\alpha}P_n)$$

and from this and (4) we get

$$\|f-R_n(f)\|_{p,\omega} = O(n^{-\alpha})$$

Let us consider the case $\alpha = 1$.By Abel transform,

$$R_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n-1}\{P_m(s_m(f)(x)-s_{m+1}(f)(x)) + P_n s_n(f)(x)\}$$

$$= \frac{1}{P_n}\sum_{m=0}^{n-1} P_m\big(-u_{m+1}(f)(x)\big) + s_n(f)(x),$$

Hence

$$R_n(f)(x) - s_n(f)(x) = -\frac{1}{P_n}\sum_{m=0}^{n-1} P_m\, u_{m+1}(f)(x)$$

Using Abel transform again yield

$$\sum_{m=0}^{n-1} P_m\, u_{m+1}(f)(x) = \sum_{m=0}^{n-1} \frac{P_m}{m+1}(m+1)u_{m+1}(f)(x)$$

$$= \sum_{m=0}^{n-1}\left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right)\left(\sum_{k=0}^{m}(k+1)u_{k+1}(f)(x)\right) + \frac{P_n}{n+1}\sum_{k=0}^{n-1}(k+1)u_{k+1}(f)(x)$$

Thus, by considering lemma 5.1.3 and (3) we obtain

$$\begin{aligned}\left\|\sum_{m=0}^{n-1} P_m\, u_{m+1}(f)\right\|_{p,\omega} &\le \sum_{m=0}^{n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right|\left\|\sum_{k=0}^{m}(k+1)\, u_{k+1}(f)\right\|_{p,\omega} \\ &\quad + \frac{P_n}{n+1}\left\|\sum_{k=0}^{n-1}(k+1)\, u_{k+1}(f)\right\|_{p,\omega} \\ &= \sum_{m=0}^{n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right|(m+2)\|s_{m+1}(f) - \sigma_{m+1}(f)\|_{p,\omega} \\ &\quad + P_n\|s_n(f) - \sigma_n(f)\|_{p,\omega} \\ &= O(1)\sum_{m=0}^{n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| + O\left(\frac{P_n}{n}\right)\end{aligned}$$

This gives

$$\begin{aligned}\|R_n(f) - s_n(f)\|_{p,\omega} &= \frac{1}{P_n}\left\|\sum_{m=0}^{n-1} P_m\, u_{m+1}(f)\right\|_{p,\omega} \\ &= \frac{1}{P_n} O\left(\frac{P_n}{n}\right) \\ &= O\left(\frac{1}{n}\right)\end{aligned}$$

combining this estimate with Lemma 5.1.2 yields

$$\|f - R_n(f)\|_{p,\omega} \le \|f - s_n(f)\|_{p,\omega} + \|R_n(f) - s_n(f)\|_{p,\omega} = O(n^{-1})\blacksquare$$

5.2 Approximation by matrix transformation

In the proceeding section we introduce the approximation by means, here we extend the method to the general case in which the matrix transformation is involved, with the same notations stated before we assume that A is a lower infinite triangular matrix and T_n^A is given by

$$T_n^A(f) = \sum_{k=0}^{n} a_{n,k} s_n(f)$$

Lemma 5.2.1: [1]Let $A = (a_{n,k})$ be an infinite lower triangular matrix and $0 < \alpha < 1$. If one of the conditions

1. A has almost monotone decreasing rows and
$$(n + 1)a_{n,0} = O(1),$$
2. A has almost monotone increasing rows,
$$(n + 1)a_{n,r} = O(1) \; where \; r := \left[\frac{n}{2}\right],$$

and
$$\left|s_n^{(A)} - 1\right| = O(n^{-\alpha}).$$

holds, then

$$(6) \sum_{k=1}^{n} k^{-\alpha} a_{n,k} = O(n^{-\alpha}).$$

Proof: Condition 1: since

$$\sum_{k=1}^{n} k^{-\alpha} = O(n^{1-\alpha}) \text{and} a_{n,k} \leq K a_{n,0} \text{for } k = 1, \ldots, n$$

we get

$$\sum_{k=1}^{n} k^{-\alpha} a_{n,k} \leq K a_{n,0} \sum_{k=1}^{n} k^{-\alpha}$$
$$= O\left(\frac{1}{n+1}\right) O(n^{1-\alpha})$$
$$= O(n^{-\alpha}).$$

Condition 2: Since

$$a_{n,k} \leq K a_{n,r} \text{for } k = 1, \ldots, r \text{ and} \left|s_n^{(A)} - 1\right| = O(n^{-\alpha}).$$

We have

$$\sum_{k=1}^{n} k^{-\alpha}\, a_{n,k} = \sum_{k=1}^{r} k^{-\alpha}\, a_{n,k} + \sum_{k=r+1}^{n} k^{-\alpha}\, a_{n,k}$$
$$\leq K a_{n,r} \sum_{k=1}^{r} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=r+1}^{n} a_{n,k}$$
$$\leq K a_{n,r} \sum_{k=1}^{n} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{n} a_{n,k}$$
$$= O\left(\frac{1}{n+1}\right) O(n^{1-\alpha}) + O(n^{-\alpha}) s_n^{(A)}$$
$$= O(n^{-\alpha}). \blacksquare$$

Theorem 5.2.2: [1]Let $1 < p < \infty$, $w \in \mathcal{A}_p$, $0 < \alpha < 1$, $f \in Lip(\alpha, p, w)$and $A = (a_{n,k})$ be a lower triangular regular matrix with

$$\left|s_n^{(A)} - 1\right| = O(n^{-\alpha}).$$

If one of the conditions

$i.$ Ahas almost monotone decreasing rows and

$$(n + 1)a_{n,0} = O(1),$$

$ii.$ Ahas almost monotone increasing rows and

$$(n + 1)a_{n,r} = O(1) \text{ where } r := [n/2],$$

holds, then

$$\left\|f - T_n^{(A)}(f)\right\|_{p,w} = O(n^{-\alpha}).$$

Proof: By definition of $T_n^{(A)}(f)$, we have

$$\begin{aligned} T_n^{(A)}(f)(x) - f(x) &= \sum_{k=0}^{n} a_{n,k} s_k(f)(x) - f(x) \\ &= \sum_{k=0}^{n} a_{n,k} s_k(f) - f(x) + s_n^{(A)} f(x) - s_n^{(A)} f(x) \\ &= \sum_{k=0}^{n} a_{n,k} s_k(f) - s_n^{(A)} f(x) + s_n^{(A)} f(x) - f(x) \\ &= \sum_{k=0}^{n} a_{n,k} s_k(f) - \sum_{k=0}^{n} a_{n,k} f(x) + \left(s_n^{(A)} - 1\right) f(x) \\ &= \sum_{k=0}^{n} a_{n,k}\big(s_k(f) - f(x)\big) + \left(s_n^{(A)}(f) - 1\right) f(x). \end{aligned}$$

Hence, by Lemma 5.1.2 and Lemma 5.2.1 we obtain

$$\begin{aligned} \left\|f - T_n^{(A)}(f)\right\|_{p,w} &\le \sum_{k=0}^{n} a_{n,k} \|s_k(f) - f\|_{p,w} + \left|s_n^{(A)}(f) - 1\right| \|f\|_{p,w} \\ &= \sum_{k=1}^{n} a_{n,k} \|s_k(f) - f\|_{p,w} + a_{n,0}\|s_0(f) - f\|_{p,w} + \left|s_n^{(A)}(f) - 1\right| \|f\|_{p,w} \\ &= \sum_{k=1}^{n} a_{n,k} O(k^{-\alpha}) + O\left(\frac{p_n}{P_n}\right) + O(n^{-\alpha}) \\ &= O(n^{-\alpha}) + O\left(\frac{1}{n+1}\right) + O(n^{-\alpha}) = O(n^{-\alpha}) + O\left(\frac{1}{n}\right) = O(n^{-\alpha}) \end{aligned}$$

Since $\left|s_n^{(A)} - 1\right| = O(n^{-\alpha})$. ■

Theorem 5.2.3: [1]Let $1 < p < \infty, w \in \mathcal{A}_p, f \in Lip(1, p, w)$ and $A = (a_{n,k})$be a lower triangular regular matrix with$\left|s_n^{(A)} - 1\right| = O(n^{-1})$. If one of the conditions

$$(i) \sum_{k=1}^{n-1} \left| a_{n,k-1} - a_{n,k} \right| = O(n^{-1}),$$

$$(ii) \sum_{k=1}^{n-1} (n-k) \left| a_{n,k-1} - a_{n,k} \right| = O(1),$$

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_{p,w} = O(n^{-1})$$

Proof: By Lemma 5.1.2

$$\begin{aligned} \left\| f - T_n^{(A)}(f) \right\|_{p,w} &= \left\| f - s_n(f) + s_n(f) - T_n^{(A)}(f) \right\|_{p,w} \\ &\le \left\| s_n(f) - T_n^{(A)}(f) \right\|_{p,w} + \| f - s_n(f) \|_{p,w} \\ &= \left\| s_n(f) - T_n^{(A)}(f) \right\|_{p,w} + O(n^{-1}). \end{aligned}$$

Thus, we have to show that

$$(7) \left\| s_n(f) - T_n^{(A)}(f) \right\|_{p,w} = O(n^{-1})$$

Set

$$A_{n,k} := \sum\nolimits_{m=k}^{n} a_{n,m}$$

Hence,

$$\begin{aligned} T_n^{(A)}(f) &= \sum_{k=0}^{n} a_{n,k} s_k(f)(x) = \sum_{k=0}^{n} a_{n,k} \left[\sum_{m=0}^{k} u_m(f)(x) \right] \\ &= a_{n,0} u_0 + a_{n,1}(u_0 + u_1) + \cdots + a_{n,n}(u_0 + u_1 + \cdots u_n) \\ &= u_0 \left(a_{n,0} + \cdots + a_{n,n} \right) + u_1 \left(a_{n,1} + \cdots + a_{n,n} \right) \ldots + u_n a_n \\ &= \sum_{k=0}^{n} \left[\sum_{m=k}^{n} a_{n,m} \right] u_k(f)(x) = \sum_{k=0}^{n} A_{n,k} u_k(f)(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} s_n(f)(x) &= \sum_{k=0}^{n} u_k(f)(x) \\ &= \sum_{k=0}^{n} u_k(f)(x) + A_{n,0} \sum_{k=0}^{n} u_k(f) - A_{n,0} \sum_{k=0}^{n} u_k(f) \\ &= A_{n,0} \sum_{k=0}^{n} u_k(f) + \left(1 - A_{n,0}\right) \sum_{k=0}^{n} u_k(f)(x) \\ &= \sum_{k=0}^{n} A_{n,0} u_k(f)(x) + \left(1 - s_n^{(A)}\right) s_n(f)(x). \end{aligned}$$

Thus,

$$T_n^{(A)}(f) - s_n(f)(x) = \sum_{k=0}^{n} A_{n,k} u_k(f)(x) - \sum_{k=0}^{n} A_{n,0} u_k(f)(x) - \left(1 - s_n^{(A)}\right) s_n(f)(x)$$
$$= \sum_{k=0}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f)(x) + \left(s_n^{(A)} - 1\right) s_n(f)(x)$$
$$= \sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f)(x) + \left(s_n^{(A)} - 1\right) s_n(f)(x).$$

By boundedness of the partial sums in the space L_w^p(see [9]) we get

(8) $$\left\| s_n(f)(x) - T_n^{(A)}(f) \right\|_{p,w} = \left\| \sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f)(x) + \left(s_n^{(A)} - 1\right) s_n(f)(x) \right\|_{p,w}$$
$$\leq \left\| \sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f)(x) \right\|_{p,w} + \left| s_n^{(A)} - 1 \right| \|f\|_{p,w}$$
$$= \left\| \sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f)(x) \right\|_{p,w} + O(n^{-1}).$$

Thus, the problem reduced to proving that

$$\left\| \sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f)(x) \right\|_{p,w} = O(n^{-1})$$

If we set

$$b_{n,k} := \frac{A_{n,k} - A_{n,0}}{k}, \quad k = 1, \dots, n,$$

Abel transform yields

$$\sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f) = \sum_{k=1}^{n} b_{n,k} k u_k(f)$$
$$= \sum_{k=1}^{n-1} \left(b_{n,k} - b_{n,k+1}\right) \left(\sum_{m=1}^{k} m u_m(f) \right) + b_{n,n} \sum_{m=1}^{n} m u_m(f)$$

Hence,

$$\left\| \sum_{k=1}^{n} \left(A_{n,k} - A_{n,0}\right) u_k(f) \right\|_{p,w}$$
$$\leq \left| b_{n,n} \right| \left\| \sum_{k=1}^{n} k u_k(f) \right\|_{p,w} + \sum_{k=1}^{n-1} \left| b_{n,k} - b_{n,k+1} \right| \left(\left\| \sum_{m=1}^{k} m u_m(f) \right\|_{p,w} \right).$$

we have by Lemma 4.2.10

$$\sum_{k=1}^{n} k u_k(f) = (n+1)[s_n - \sigma_n]$$

Therefore,considering Lemma 5.1.3,

$$\left\|\sum_{k=1}^{n} k u_m(f)\right\|_{p,w} = (n+1)\|s_n(f)-\sigma_n(f)\|_{p,w}$$
$$= (n+1)O(n^{-1})$$
$$= O(1).$$

This and the previous inequality yield

$$\left\|\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)u_k(f)(x)\right\|_{p,w} = O(1)\left|b_{n,n}\right| + O(1)\sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right|$$

Since $\left|s_n^{(A)}-1\right| = O(n^{-1})$,

$$\left|b_{n,n}\right| = \frac{\left|A_{n,n}-A_{n,0}\right|}{n} = \frac{\left|a_{n,n}-s_n^{(A)}\right|}{n} = \frac{1}{n}\left(s_n^{(A)}-a_{n,n}\right) \le \frac{1}{n}s_n^{(A)}$$
$$= \frac{1}{n}O(1) = O(n^{-1}).$$

Therefore, it is remained to prove that

$$(9)\ \sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right| = O(n^{-1}).$$

A simple calculation yields

$$b_{n,k}-b_{n,k+1} = \frac{A_{n,k}-A_{n,0}}{k} - \frac{A_{n,k+1}-A_{n,0}}{k+1}$$
$$= \frac{kA_{n,k}+A_{n,k}-kA_{n,0}-A_{n,0}-kA_{n,k+1}+kA_{n,0}}{k(k+1)}$$
$$= \frac{kA_{n,k}+\sum_{m=k}^{n}a_{n,m}-\sum_{k=0}^{n}a_{n,k}-kA_{n,k+1}}{k(k+1)}$$
$$= \frac{\sum_{m=k}^{n}a_{n,m}-\left(\sum_{m=k}^{n}a_{n,k}+\sum_{m=0}^{k}a_{n,k}-a_{n,k}\right)+ka_{n,k}}{k(k+1)}$$
$$= \frac{-\sum_{m=0}^{k}a_{n,k}+a_{n,k}+ka_{n,k}}{k(k+1)}$$
$$= \frac{1}{k(k+1)}\left\{(k+1)a_{n,k}-\sum_{m=0}^{k}a_{n,m}\right\}.$$

(i)Let $\sum_{k=0}^{n-1}\left|a_{n,k-1}-a_{n,k}\right| = O(n^{-1})$

Let's verify by induction that

$$(10)\ \left|\left(\sum_{m=0}^{k}a_{n,m}\right)-(k+1)a_{n,k}\right| \le \sum_{m=1}^{k} m\left|a_{n,m-1}-a_{n,m}\right| \text{ for } k=1,\dots,n.$$

If $k = 1$, then

$$\left|\left(\sum_{m=0}^{1} a_{n,m}\right) - (2)a_{n,1}\right| = \left|a_{n,0} - a_{n,1}\right|$$

thus (10) holds.

Now let us assume that (10) is true for $k = v$. For $k = v + 1$,

$$\left|\left(\sum_{m=0}^{v+1} a_{n,m}\right) - (v+2)a_{n,v+1}\right| = \left|\left(\sum_{m=0}^{v} a_{n,m}\right) - (v+1)a_{n,v+1}\right|$$

$$\leq \left|\left(\sum_{m=0}^{v} a_{n,m}\right) - (v+1)a_{n,v}\right| + \left|(v+1)a_{n,v} - (v+1)a_{n,v+1}\right|$$

$$\leq \left(\sum_{m=1}^{v} m\left|a_{n,m-1} - a_{n,m}\right|\right) + (v+1)\left|a_{n,v} - a_{n,v+1}\right|$$

$$= \sum_{m=1}^{v+1} m\left|a_{n,m-1} - a_{n,m}\right|,$$

and hence (10) holds for $k = 1, \dots, n$. Therefore,

$$\sum_{k=1}^{n-1}\left|b_{n,k} - b_{n,k+1}\right| = \sum_{k=1}^{n-1}\left|\frac{1}{k(k+1)}\left\{(k+1)a_{n,k} - \sum_{m=0}^{k} a_{n,m}\right\}\right|$$

$$= \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\left|\left(\sum_{m=0}^{k} a_{n,m}\right) - (k+1)a_{n,k}\right|$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k} m\left|a_{n,m-1} - a_{n,m}\right|$$

Set $m\left|a_{n,m-1} - a_{n,m}\right| = c_m$ for simplicity, by expanding we got

$$\sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k} c_m = \frac{1}{1\cdot 2}c_1 + \frac{1}{2\cdot 3}(c_1 + c_2) + \cdots + \frac{1}{n\cdot(n-1)}(c_1 + \cdots + c_{n-1})$$

$$= \left(\frac{1}{2} + .. + \frac{1}{n\cdot(n-1)}\right)c_1 + \cdots + \frac{1}{n\cdot(n-1)}c_{n-1}$$

$$= \sum_{m=1}^{n-1} c_m \sum_{k=m}^{n-1}\frac{1}{k(k+1)} = \sum_{m=1}^{n-1} m\left|a_{n,m-1} - a_{n,m}\right| \sum_{k=m}^{n-1}\frac{1}{k(k+1)}$$

$$\leq \sum_{m=1}^{n-1} m\left|a_{n,m-1} - a_{n,m}\right| \sum_{k=m}^{\infty}\frac{1}{k(k+1)} = \sum_{m=1}^{n-1}\left|a_{n,m-1} - a_{n,m}\right|$$

$$= O(n^{-1}).$$

so relation (9) holds, thus

$$\left\|s_n(f)(x) - T_n^{(A)}(f)\right\|_{p,w} = \left\|\sum_{k=1}^{n}\left(A_{n,k} - A_{n,0}\right)u_k(f)(x)\right\|_{p,w} + O(n^{-1})$$

$$= O(n^{-1}) + O(n^{-1}) = O(n^{-1})$$

Finally, we have

$$\left\|(f)-T_n^{(A)}(f)\right\|_{p,w} \le \left\|s_n(f)(x)-T_n^{(A)}(f)\right\|_{p,w}+O(n^{-1})$$
$$=O(n^{-1})+O(n^{-1})$$
$$=O(n^{-1}).$$

(ii)Let $\sum_{k=1}^{n-1}(n-k)\left|a_{n,k-1}-a_{n,k}\right|=O(1)$

By (9),

$$\sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right| \le \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|$$
$$\le \sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|$$
$$+\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|,$$

where$r :=\ [n/2]$. By Abel transform,

$$\sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \le \sum_{k=1}^{r}\left|a_{n,k-1}-a_{n,k}\right|$$
$$=\sum_{k=1}^{r}\frac{1}{n-k}(n-k)\left|a_{n,k-1}-a_{n,k}\right|$$
$$\le \frac{1}{n-r}\sum_{k=1}^{r}(n-k)\left|a_{n,k-1}-a_{n,k}\right|$$
$$=\frac{1}{n-r}O(1)=O(n^{-1}).$$

On the other hand

$$\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|$$
$$\le \sum_{k=r}^{n-1}\frac{1}{k(k+1)}\left\{\sum_{m=1}^{r}m\left|a_{n,m-1}-a_{n,m}\right|+\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right|\right\}$$
$$=\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{r}m\left|a_{n,m-1}-a_{n,m}\right|$$
$$+\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right| =: I_{n1}+I_{n2}.$$

Now since

$$\sum_{k=1}^{r}\left|a_{n,k-1}-a_{n,k}\right| = O(n^{-1}),$$

$$\begin{aligned}
I_{n1} &= \sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{r} m\left|a_{n,m-1}-a_{n,m}\right| \\
&= \sum_{k=r}^{n-1}\frac{1}{(k+1)}\sum_{m=1}^{r}\frac{m}{k}\left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \sum_{k=r}^{n-1}\frac{1}{(k+1)}\sum_{m=1}^{r}\left|a_{n,m-1}-a_{n,m}\right| \\
&= O(n^{-1})\sum_{k=r}^{n-1}\frac{1}{(k+1)} = O(n^{-1})(n-r)\frac{1}{r+1} = O(n^{-1}).
\end{aligned}$$

Let us also estimateI_{n2}.

$$\begin{aligned}
I_{n2} &= \sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=r}^{k} m\left|a_{n,m-1}-a_{n,m}\right| \\
&= \sum_{k=r}^{n-1}\frac{1}{(k+1)}\sum_{m=r}^{k}\frac{m}{k}\left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \sum_{k=r}^{n-1}\frac{1}{k+1}\sum_{m=r}^{k}\left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \frac{1}{r+1}\sum_{k=r}^{n-1}\left(\sum_{m=r}^{k}\left|a_{n,m-1}-a_{n,m}\right|\right) \\
&\leq \frac{2}{n}\sum_{k=r}^{n-1}\left(\sum_{m=r}^{k}\left|a_{n,m-1}-a_{n,m}\right|\right) \\
&= \frac{2}{n}\sum_{k=r}^{n-1}(n-k)\left|a_{n,k-1}-a_{n,k}\right| \\
&\leq \frac{2}{n}\sum_{k=1}^{n-1}(n-k)\left|a_{n,k-1}-a_{n,k}\right| \\
&= \frac{2}{n}O(1) = O(n^{-1})
\end{aligned}$$

Thus

$$\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k} m\left|a_{n,m-1}-a_{n,m}\right| = O(n^{-1}),$$

and hence

$$\sum_{k=1}^{n-1}\left|b_{n,k}-b_{n,k+1}\right| = O(n^{-1}).$$

Therefore,

$$\left\|s_n(f)(x)-T_n^{(A)}(f)\right\|_{p,w} = \left\|\sum_{k=1}^{n}\left(A_{n,k}-A_{n,0}\right)u_k(f)(x)\right\|_{p,w} + O(n^{-1})$$
$$= O(n^{-1}) + O(n^{-1}) = O(n^{-1})$$

Finally,

$$\left\|(f)-T_n^{(A)}(f)\right\|_{p,w} \leq \left\|s_n(f)(x)-T_n^{(A)}(f)\right\|_{p,w} + O(n^{-1})$$
$$= O(n^{-1}) + O(n^{-1})$$
$$= O(n^{-1}).$$

and the proof is complete■

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