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Oscillation Criteria of First and Second Order Neutral Difference Equations

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Dedication

To my parents,
To my husband Hisham and his parents,
To my brothers and my sisters and
To my daughters Nada and Tala.

Ruba Mohammed Ghaleb Al-Hamouri

Declaration:

I certify that this thesis submitted for the degree of Master is the result of my own research, except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ....

Ruba Mohammed Ghaleb Al-Hamouri

Date: 24 / 6 / 2006

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Abstract

This thesis is concerned with the oscillation criteria of solutions of certain types of neutral difference equations. We study the conditions that control the oscillation of solutions of the first and second order linear and nonlinear neutral difference equations with and without forcing terms, we classify these neutral difference equations depending on the coefficients, whether they are constant values, constant and variable values or all the coefficients are variables, also we study special cases when the neutral difference equations include maximum and sign functions.

The thesis presents main concepts and basic definitions of neutral difference equations and oscillations, also it contains several results in the oscillation theory of certain types of neutral difference equations, in addition to several examples given to illustrate the main theorems in this thesis.

In fact we were interested to study this topic in difference equations because difference equations have many applications in our real life and in the last few years there were many researches that considered studies of qualitative properties of different advanced types of difference equations like neutral difference equations, which arises in many applications in economics, delay reaction diffusion and electrical transmission lines in lossless transmission lines between circuits in high speed computers.

The thesis contains some extensions to theorems, including generalization, deduction and modifications on some conditions so as to use them to prove the oscillation of certain neutral difference equations.

ملخص

اهتمت هذه الرسالة بمعايير التذبذب لحلول فئة معينة من المعادلات الفرقية المتعادلة، حيث تعرضنا للشروط التي تحكم التذبذب لحلول الدرجة الأولى و الثانية الخطية و غير الخطية من المعادلات الفرقية المتعادلة سواء احتوت أو كانت خالية من حدود القوة ، قمنا بتصنيف هذه المعادلات الفرقية المتعادلة اعتمادا على المعاملات سواء أكانت قيما ثابتة أو قيما ثابتة و متغيرة أو كل المعاملات متغيرة. أيضا درسنا حالات خاصة عندما تشمل المعادلات الفرقية المتعادلة اقترانات القيمة العظمى و الاشارة.

تعرض الرسالة المفاهيم الرئيسية و التعريفات الأساسية للمعادلات الفرقية المتعادلة و التذبذب ، أيضا تحتوي على العديد من النتائج التي تتعلق بنظريات التذبذب لتلك الفئة المعينة من المعادلات الفرقية المتعادلة. بالاضافة لعدد من الأمثلة أدرجت لتوضيح النظريات الرئيسية في هذه الرسالة.

في الحقيقة توجهنا لدراسة هذا الموضوع في المعادلات الفرقية لأن لها العديد من التطبيقات في حياتنا العملية ، و في السنوات القليلة الماضية كان هناك العديد من الأبحاث التي اشتملت دراسات للصفات النوعية لأشكال مختلفة و متقدمة من المعادلات الفرقية ، مثل المعادلات الفرقية المتعادلة التي تظهر في كثير من التطبيقات ، في الاقتصاد و الانتشار الحراري في التفاعلات و في النقل الكهربائي في خطوط النقل غير الفاقدة للطاقة بين الدوائر في أجهزة الحاسوب عالية السرعة.

تحتوي الرسالة بعض الاضافات لنظريات تشمل التعميم ، الاستنتاج و التغير على بعض الشروط لاستعمالها في برهنة التذبذب لبعض المعادلات الفرقية المتعادلة.

Definitions

Through the research we used

- Δx_n to denote the forward difference operator $\Delta x_n = x_{n+1} - x_n$.
- NDE's to denote neutral difference equations.
- $X = l_\infty^N$ to denote the space of all real bounded sequences $x = \{x_n\}$ with sup norm $\|x\| = \sup\{|x_n| : n \geq N\}$.
- $\mathbf{R}^+$ to denote positive real numbers.
- $\mathbf{R}^-$ to denote negative real numbers.
- $\mathbf{R} = \mathbf{R}^+ \cup \{0\} \cup \mathbf{R}^-$.
- $\mathbf{Z}$ to denote the set of integers.
- $\mathbf{Z}^+$ to denote the set of positive integers.
- $\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}$.
- W.L.O.G to denote the sentence "without loss of generality".
- $\mathbf{N}$ to denote the set of natural numbers $= \{0,1,2,....\}$.
- $\mathbf{N}(n_0)$ denote the set $\{n_0, n_0 + 1, ...\}$.
- $\mathbf{N}(n_0, b-1) = \{n_0, n_0 + 1, ..., b-1\}$ where $n_0 < b - 1 < \infty$ and $n_0, b \in \mathbf{N}$.

  Any one of the last three sets will be denoted by $\overline{\mathbf{N}}$.

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Introduction

About 1960, the study of difference equations (DE's) appeared. Works at that time, did not exceed simple forms of linear DE's with constant coefficients and sometimes with variable coefficients. Mathematicians like Boole, Brand, Chorlton, Fort, Miller, Richardson, Spiegal and others, who introduced and presented many books that talk about finite differences and difference equations, so that one can not ignore that those works are quite important, for they are the foundation upon which the more advanced mathematical theory in DE's is built.

The importance of DE's arises in some mathematical models, for example, in probability theory, statistical problems, number theory, electrical networks, economics, psychology and sociology. These mathematical models describe realistic situations, when the variables under study take or assumed to take only a discrete set of values. Moreover, in view of the close relation between the finite difference operator Δ and the differential operator D, it was possible to approximate a differential equation by a difference equation, where the solution of the differential equation is an appropriate limit of the solution of the corresponding DE. So we can say that the subject of DE's is gradually forcing its way out of the difference calculus to become one of the most important instruments in the hands of mathematical physicists when concerned with discontinuous processes.

Since difference equations appeared as a discrete analog to differential equations, different types of DE's were treated and solved using techniques similar to those used in solving differential equations. for instance, the characteristic equation method was used to solve linear homogenous DE's with constant coefficients. While the method of undetermined coefficients was applied successfully to linear nonhomogenous DE's with constant coefficients under certain conditions, reduction of order was used for first order DE's with variable coefficients and also the variation of parameters was used successfully to solve DE's with variable coefficients, provided (in the last two methods) that the general solution of the corresponding homogenous DE is known. Here, we referred to [7] and [15], as they considered basic topics in finite differences and difference equations, also they contain methods that solve linear DE's with constant coefficients or with variable coefficients in addition to some applications of DE's.

About 1980 and after and as a result of the important and useful applications of DE's in many fields of science, mathematicians were encouraged to spend more efforts in working and developing many studies and researches that involve new and more complicated forms of DE's, like delay difference equations (DDE's), advanced difference equations (ADE's), neutral difference equations (NDE's), neutral difference equations of mixed type, partial difference equations (PDE's) nonlinear and higher order DE's, we referred to [1], [4], [6], [8], and [11]. Also new features of DE's were considered, such as the periodicity of solutions, asymptotic behavior and stability of solutions, oscillation and nonoscillation of solutions, boundary value problems, eigenvalue problems, etc. we referred to [1], [21], [22], [24], and [27]. In fact the valuable book of Agarwal "Difference Equations and Inequalities" [1] which is full of advanced topics so that one can find answers to many questions about qualitative properties of solutions of both difference systems and higher order DE's, solutions of boundary value problems of linear, nonlinear systems and higher order DE's.

As we will see later the study of each new form or new feature of DE's concerned many authors, for example in NDE's and oscillations of its solutions one can find many works for Agarwal, Zhang, Cheng, Grace, Parhi, Tripathy, Lalli, Thandapani and others [2-6], [8], [10], [12], [13], [14], [18-20], [24-30], these works were devoted to study the oscillation of NDE's of different orders of NDE's, they also considered the forced and unforced NDE's the linear and nonlinear NDE's, under several conditions on the arguments, coefficients, forcing terms and the nonlinear terms. The importance of NDE's comes from the fact that NDE's arise in many applications, for example, delay reaction diffusion, also in "cobweb" models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission in lossless transmission lines between circuits in high speed computers.

In this thesis we consider theorems that provide sufficient conditions for the oscillation of solutions of first and second order, linear and nonlinear NDE's, taking different forms depending on the coefficients and the forcing terms of those NDE's. the difference between the forms we considered mainly depends on the coefficients and forcing terms of NDE's. most works that concern the oscillation of NDE's was done in the last three decades. Again authors employed techniques similar to that used in differential equations to prove the oscillation of NDE's. All the articles were used in this thesis referred to one or more reference in differential equations especially in NDE's, most of the articles referred to [11] which includes a chapter that discusses the oscillation of delay difference equations (DDE's), as we will see later, the results in NDE's mainly depend on those of DDE's, of course with some improvements or needed extra conditions.

This thesis consists of four chapters:

Chapter one: contains basic definitions, notes, lemmas and theorems which are essential in the rest of our research.

Chapter two: studies the oscillation theorems for the first order linear NDE's of the form

$$\Delta\left(x_n + \delta c_n x_{n-m} - d_n x_{n+g}\right) + p_n x_{n-k} + q_n x_{n\pm l} = 0 \quad , \quad n \geq 0$$

where $\{c_n\}, \{d_n\}, \{p_n\}$ and $\{q_n\}$ are sequences of real numbers, m, g, k and l are integers and $\delta = \pm 1$.

First we discussed the proof of the oscillation of the first order linear NDE with constant coefficients by contradiction and by means of the characteristic equation. Sufficient and necessary conditions and comparison theorems we considered to the same form but this time when the coefficients are both constants and variables. When the coefficients are all variable theorems involved just sufficient conditions and comparison statements were studied. Finally, we introduced oscillation results for the first order NDE's with positive and negative variable coefficients depending on the treatment of an ordinary difference equation.

Chapter three: studies the oscillation theorems for the first order nonlinear NDE's of the form

$$\Delta\left(x_n + \delta_1 p_n x_{n-m}\right) + \delta_2 q_n G\left(x_{n-k}\right) = f_n \quad , \quad n \geq 0$$

where $\{p_n\}$, $\{q_n\}$ and $\{f\}_n$ are sequences of real numbers, $\delta_1 = \pm 1$, $\delta_2 = \pm 1$ and G is a real valued function.

We considered theorems for the special case of the unforced NDE with both constant and variable coefficients when $G(x_{n-k}) = x_{n-k}^{\alpha}$ where α is the quotient of odd positive integers, then we discussed the general case of $G(x_{n-k})$. Next we presented theorems for the same NDE but when the coefficients are all variables, the oscillation theorems considered several ranges of $\{p_n\}$, some of the proofs depend on a non-neutral difference inequality. After that we considered the forced form of NDE's with both constant and variable coefficients, the method used here was defining an oscillatory sequence of real numbers $\{h_n\}$ with $\Delta h_n = f_n$ in order to transform the NDE to an unforced form. Also the forced form of the NDE's with all coefficients are variables was discussed, two approaches to the proof were used, one by operators and the other by defining an oscillatory sequence $\{h_n\}$ with restricting conditions on $\{h_n\}$. Finally, we introduced some oscillation theorems for special cases when $G(x_{n-k}) = |x_{n-k}|^{\alpha} \operatorname{sgn} x_{n-k}$, α is a positive constant, when $G(x_{n-k}) = \prod_{i=1}^{n} |x_{n-k_i}|^{\alpha_i} \operatorname{sgn} x_{n-k_i}$ and $G(x_{n-k}) = \max_{s\in[n-k,n]} x_s$

Chapter four: studies the oscillation criteria for the second order NDE's of the form

$$\Delta(c_n \Delta(x_n + p_n x_{n-m})) + q_n G(x_{n-k}) = f_n \ , \quad n \geq 0$$

where $\{c_n\}$, $\{p_n\}$, $\{q_n\}$ and $\{f_n\}$ are sequences of real numbers , G is a real valued function, m and k are nonnegative integers. First we considered the case when $p_n = p$ (real number), $c_n = 1$ and $f_n = 0$, we got oscillation criteria under certain conditions involving coefficients and arguments, we depended on similar discussion applied to delay difference equations. Next we discussed the oscillation theorems for the forced form when $p_n = p$ constant, as in chapter three we defined a periodic oscillatory sequence $\{h_n\}$ with $\Delta^2 h_n = f_n$ in order to transform the forced NDE's to the unforced form. After that the second order NDE's with all coefficients are variables was studied considering various ranges of p_n with $f_n = 0$, a new approach was applied in [3] depending on dividing the set of all nontrivial solutions into four sets, one of them is the oscillatory solutions set and trying to find when this set is nonempty. Next we studied the forced form with variable coefficients, once again we need a periodic oscillatory sequence $\{h_n\}$ with $\Delta^2 h_n = f_n$ and $\lim_{n\to\infty} h_n = 0$ to achieve the oscillation. Finally, we considered the special case when $G(x_{n-k}) = x_{n-k}^{\beta}$, where β is the ratio of odd positive integers, which includes the sublinear, linear and superlinear NDE's depending on β.

Chapter One

Preliminaries

1.0 Introduction

This chapter mainly contains the basic definitions, results, notes, lemmas and theorems to be used later in this research. Section 1.1 introduces the definition of difference equations, the definition of NDE's and their classification. Section 1.2 defines the solutions of the NDE's and the oscillation of solutions. Section 1.3 contains basic definitions and notes on metric space, characteristic equation, Hölder inequality and Lipschitz condition. Section 1.4 includes lemmas and theorems we need in the rest of this research.

1.1 Difference equations and neutral difference equations (NDE's)

A difference equation in one independent variable $n \in \overline{\mathbf{N}}$ and one unknown x_n is a functional equation of the form

$$f(n, x_n, x_{n+1}, \ldots, x_{n+k}) = 0 \quad , \tag{1.1.1}$$

where f is a given function of n and the values of x_n at $n \in \overline{\mathbf{N}}$.

If the equation

$$E^m f(n) = (I + \Delta)^m f(n) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i f(n) \ , \ \Delta^0 = I$$

is substituted in equation (1.1.1) the later equation takes the form

$$g(n, x_n, \Delta x_n, \ldots, \Delta^k x_n) = 0$$

which is a difference equation.

A neutral difference equation (NDE) is a difference equation in which the highest difference of the unknown function appears both with and without delays, NDE's are classified into three classes, delay, advanced and mixed type.

If we consider the first order linear homogenous NDE

$$\Delta(x_n + p_n x_{n-m}) + \sum_{i=1}^{m} Q_{in} x_{n-k_i} = 0 \ , \ n \in \overline{\mathbf{N}} \tag{1.1.2}$$

where $\{p_n\}$ and $\{Q_{in}\}$ are sequences of real numbers.

Let $k = \max\{0, k_1, \ldots, k_m\}$

$l = \max\{1, -k_1, \ldots, -k_m\}$

Then equation (1.1.2) is a difference equation of order $(k + l)$.

If $k \geq 0$ and $l = 1$, then equation (1.1.2) is a delay NDE (i.e. it has nonnegative arguments) .

If $k = 0$ and $l \geq 2$, then equation (1.1.2) is an advanced NDE (i.e. it has nonpositive arguments).

If $k \geq 1$ and $l \geq 2$, then equation (1.1.2) is a NDE of mixed type (i.e. negative and positive arguments).

For the first order NDE

$$\Delta\left(x_n + p_n x_{n-m}\right) + \sum_{i=1}^{m} Q_{in} x_{n-k_i}^{\alpha} = f_n \tag{1.1.3}$$

If $\alpha = 1$, $f_n \neq 0$ then equation (1.1.3) is called linear forced NDE.

If $0 < \alpha < 1$, $f_n \neq 0$ then equation (1.1.3) is called sublinear forced NDE.

If $\alpha > 1$, $f_n \neq 0$ then equation (1.1.3) is called superlinear forced NDE.

Example 1.1.1:

(1) $\Delta\left(x_n + p_n x_{n-m}\right) + q_n x_{n-k} + c_n x_{n+l} = (-1)^n$

where c_n, p_n and q_n are sequences of real numbers.

(2) $\Delta^2\left(x_n + \frac{1}{2}x_{n-m}\right) + \frac{n^2}{n+1}x_{n-s}^{\frac{1}{3}} = 0$

(3) $\Delta^n\left(x_n + \sqrt{3}x_{n-m}\right) + (-1)^{3n} G\left(x_{n+k}\right) = f_n$

Equation (1) is a first order forced linear NDE of mixed type.
Equation (2) is a second order homogenous sublinear delay NDE.
Equation (3) is an *n-th* order forced nonlinear advanced NDE.

1.2 Solutions and oscillatory solutions of NDE's

By a solution of equation (1.1.2) we mean a nontrivial sequence $\{x_n\}$ which is defined for $n \geq -\mu$, where $\mu = \max\{m, k_i\text{'}s\}$ and satisfies equation (1.1.2) for $n \geq 0$.
If the initial condition $x_n = \phi_n$ for $n = -\mu, \ldots, -1, 0$ is given, then equation (1.1.2) has a unique solution satisfying the initial condition.

A solution $\{x_n\}$ of equation (1.1.2) is said to be oscillatory if for every $n_0 > 0$ there exists an $n \geq n_0$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory. Equation (1.1.2) is said to be oscillatory if all its solutions are oscillatory.

1.3 Basic definitions and results

Characteristic equation

Let k, r be positive integers and for each i, $i = 1, 2, \ldots, k$
Let p_i be an $r \times r$ matrix with real entries
Consider the equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0$$

then the associated characteristic equation is

$$\det\left(\lambda^k I + \lambda^{k-1} p_1 + \ldots + \lambda\, p_{k-1} + p_k\right) = 0$$

where I is an $r \times r$ identity matrix

Hölder inequality

Let $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. If $x \in l_p$ and $y \in l_q$ then (x_j, y_j), $j \in \mathbf{N}$ is in l_1 and

$$\sum_{j=1}^{\infty} \left| x_j y_j \right| \le \|x\|_p \|y\|_q$$

Lipschitz condition

Let $f(x,y)$ be a function defined on $\mathbf{R}^2$, we say that $f(x,y)$ satisfies Lipschitz condition in $\mathbf{R}^2$, if there exists a constant k, such that the function f satisfies the condition

$$|f(x,y_1) - f(x,y_2)| \le k|y_1 - y_2|$$

for every pair of points (x,y_1), and (x,y_2) in $\mathbf{R}^2$. The constant k is called Lipschitz constant.

Superlinear NDE:

We said that a nonlinear NDE of the form

$$\Delta(x_n + p_n x_{n-m}) + q_n G(x_{n-k}) = f_n \quad ,$$

where $\{p_n\}, \{q_n\}$ and $\{f_n\}$ are sequences of real numbers ,

is superlinear when G is nondecreasing and for $u \neq 0$

$$\sum_{i=n^*}^{\pm\infty} \frac{\Delta u_i}{G(u_{i+1})} < \infty \quad ,$$

where n^* is any positive integer.

The following notes are from analysis:

- A fixed point: a point $x \in M$, M is a subset of a Banach space is called a fixed point if $Tx = x$.

- A metric space is a set X together with a given distance.

- Banach space is a complete normed vector space. That is, a Banach space is a vector space X with a norm in which every Cauchy sequence converges to a vector in X .

- Let (X,d) be a vector space and let $T : X \to X$ we say that T is a contraction mapping on X if there exists a number $r \in [0,1]$ such that

  $$d(Tx,Ty) \le r d(x,y) \text{ for every } x, y \in X .$$

- We say that a Banach space X is partially ordered if X contains a cone K with nonempty interior. The ordering $\le$ in X is defined as follows:
  $x \le y$ if and only if $y - x \in K$.

1.4 Some basic lemmas and theorems

Lemma 1.4.1: Suppose that There exists a positive integer N such that

$$c_{N+im} \le 1 \text{ , for } i = 0,1,2,...$$

Let $p_n \geq 0$ and is not identically zero for all large n, then for any eventually positive solution $\{x_n\}$ of the difference inequality

$$\Delta(x_n - c_n x_{n-m}) + p_n x_{n-k} \leq 0 \quad , \tag{1.4.1}$$

the sequence $\{z_n\}$ defined by

$$z_n = x_n - c_n x_{n-m} \quad , \quad n \geq 0 \tag{1.4.2}$$

will satisfy $\Delta z_n \leq 0$ and $z_n > 0$ for all large n.

Proof: Since $\{x_n\}$ satisfies (1.4.1) then $\Delta z_n \leq 0$, we have that $\{z_n\}$ is a nonincreasing sequence, and we want to show that $z_n > 0$.

On the contrary suppose that $z_n < 0$ for $n \geq n_1$ and $x_n > 0$, so $z_n \leq z_{n_1} < 0$ for $n \geq n_1$. Choose n^* large enough so that $N + n^* m \geq n_1$, from (1.4.2) and for $j \geq 0$ we have

$$z_{N+n^*m+jm} = x_{N+n^*m+jm} - c_{N+n^*m+jm}\, x_{N+n^*m+(j-1)m}$$

$$\begin{aligned} x_{N+n^*m+jm} &= z_{N+n^*m+jm} + c_{N+n^*m+jm}\, x_{N+n^*m+(j-1)m} \\ &\leq z_{n_1} + x_{N+n^*m+(j-1)m} \\ &= z_{n_1} + z_{N+n^*m+(j-1)m} + c_{N+n^*m+(j-1)m}\, x_{N+n^*m+(j-2)m} \\ &\leq \ldots \leq (j+1)\, z_{n_1} + x_{n_1+n^*m} \quad , \quad j \geq 0 \end{aligned}$$

as $j \to \infty$

$$(j+1)\; z_{n_1} + x_{n_1+n^*m} \to -\infty \; ,$$

which is a contradiction, so $z_n > 0$. □

Lemma 1.4.2: Assume that $\alpha > 1$. Suppose further that there exists $\lambda > k^{-1} \ln \alpha$ such that

$$\liminf_{n\to\infty} q_n \exp(-e^{\lambda n}) > 0 \; ,$$

then every solution of the equation

$$\Delta x_n + q_n x_{n-k}^{\alpha} = 0 \quad , \quad n = 0,1,2,\ldots$$

is oscillatory.

Proof: See [17]. □

Lemma 1.4.3: If

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} q_i > \frac{k^{k+1}}{(k+1)^{k+1}} \quad , \qquad k \geq 1$$

then the inequality $\Delta y_n + q_n y_{n-k} \leq 0$ does not have a positive solution and hence the inequality $\Delta y_n + q_n y_{n-k} \geq 0$ does not have a negative solution.

Proof: See [8] page 184. □

Lemma 1.4.4: Let $\{f_n\}$, $\{g_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \geq n_0 > 0$ such that

$$f_n = g_n + p_n g_{n-m} \; , \quad n \geq n_0 + m \tag{1.4.3}$$

where $m \geq 0$ is an integer , suppose that there exist real numbers b_1, b_2, b_3 and b_4 such that p_n is in one of the following ranges:

(i) $-\infty < b_1 \leq p_n \leq 0$,

(ii) $0 \leq p_n \leq b_2 < 1$, or

(iii) $1 < b_3 \leq p_n \leq b_4 < \infty$.

If $g_n > 0$ for $n \geq n_0$, $\lim_{n\to\infty} \inf g_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists, then $L = 0$.

Proof: we may write (1.4.3) as

$$f_{n+m} - f_n = g_{n+m} + (p_{n+m} - 1)g_n - p_n g_{n-m} \ , \ n \geq n_0 + m \tag{1.4.4}$$

Since $\lim_{n\to\infty} \inf g_n = 0$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\lim_{k\to\infty} g_{n_k} = 0$.

Suppose that (i) holds. As the sequence $\{p_{n_k+m} - 1\}$ is bounded, we have $\lim_{k\to\infty}(p_{n_k+m} - 1)g_{n_k} = 0$ and hence (1.4.4) yields that

$$\lim_{k\to\infty}\left(g_{n_k+m} - p_{n_k} g_{n_k-m}\right) = 0$$

Since $g_{n_k+m} > 0$ for large k, we have $\lim_{k\to\infty} p_{n_k} g_{n_k-m} = 0$. From (1.4.3) it follows that

$$L = \lim_{k\to\infty} f_{n_k} = \lim_{k\to\infty}\left(g_{n_k} + p_{n_k} g_{n_k-m}\right) = 0$$

Next suppose that (ii) holds. Replacing n by $n_k - m$ in (1.4.4) and then taking limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty}\left((1 - p_{n_k})g_{n_k-m} + p_{n_k-m} g_{n_k-2m}\right) = 0$$

Since $1 - b_2 > 0$, we have

$$0 \leq (1 - b_2)\liminf_{k\to\infty} g_{n_k-m} \leq \liminf_{k\to\infty}\left((1 - p_{n_k})g_{n_k-m} + p_{n_k-m} g_{n_k-2m}\right) = 0$$

and

$$0 \leq (1 - b_2)\limsup_{k\to\infty} g_{n_k-m} \leq \limsup_{k\to\infty}\left((1 - p_{n_k})g_{n_k-m} + p_{n_k-m} g_{n_k-2m}\right) = 0$$

Hence $\lim_{k\to\infty} g_{n_k-m} = 0$

From (1.4.3) we get

$$L = \lim_{k\to\infty} f_{n_k} = \lim_{k\to\infty}\left(g_{n_k} + p_{n_k} g_{n_k-m}\right) = 0$$

Finally, let (iii) hold. Putting $n_k + m$ in place of n in (1.4.4) and letting $k \to \infty$, one obtains

$$\lim_{k\to\infty}\left(g_{n_k+2m} + (p_{n_k+2m} - 1)g_{n_k+m} - p_{n_k+m} g_{n_k}\right) = 0$$

As the sequence $\{p_{n_k+m}\}$ is bounded, we have

$$\lim_{k\to\infty}\left(g_{n_k+2m} + (p_{n_k+2m} - 1)g_{n_k+m}\right) = 0$$

Since $g_{n_k+2m} > 0$ for large k and $\{p_{n_k+2m} - 1\}$ is a positive bounded sequence, we conclude that $\lim_{k\to\infty} g_{n_k+m} = 0$. Thus from (1.4.3) we obtain

$$L = \lim_{k\to\infty} f_{n_k+m} = \lim_{k\to\infty}\left(g_{n_k+m} + p_{n_k+m} g_{n_k}\right) = 0$$

Hence the lemma is proved. □

Lemma 1.4.5: Let $1 \leq m \leq k-1$ and x_n be defined on $\mathbf{N}(a)$. Then

(i) $\liminf_{n\to\infty} \Delta^m x_n > 0$ implies $\lim_{n\to\infty} \Delta^i x_n = \infty$, $0 \le i \le m-1$

(ii) $\limsup_{n\to\infty} \Delta^m x_n < 0$ implies $\lim_{n\to\infty} \Delta^i x_n = -\infty$, $0 \le i \le m-1$

Proof:

(i) $\liminf_{n\to\infty} \Delta^m x_n > 0$ implies that there exists a large $n_1 \in \mathbf{N}(a)$ such that $\Delta^m x_n \ge c > 0$ for all $n \in \mathbf{N}(n_1)$, since

$$\Delta^{m-1} x_n = \Delta^{m-1} x_{n_1} + \sum_{l=n_1}^{n-1} \Delta^m x_l$$

It follows that $\Delta^{m-1} x_n \ge \Delta^{m-1} x_{n_1} + c(n-n_1)$ and hence $\lim_{n\to\infty} \Delta^{m-1} x_n = \infty$. The rest of the proof is by induction.

(ii) Similar to case (i) □

Theorem 1.4.1: (Discrete Kneser's Theorem)

Let x_n be defined on $\mathbf{N}(a)$, and $x_n > 0$ with $\Delta^k x_n$ of constant sign on $\mathbf{N}(a)$ and not identically zero. Then there exits an integer m, $0 \le m \le k$ with $k+m$ odd for $\Delta^k x_n \le 0$, or $k+m$ even for $\Delta^k x_n \ge 0$, and such that

$m \le k-1$ implies that $(-1)^{m+i} \Delta^i x_n > 0$ for all $n \in \mathbf{N}(a)$, $m \le i \le k-1$

$m \ge 1$ implies $\Delta^i x_n > 0$ for all large $n \in \mathbf{N}(a)$, $1 \le i \le m-1$

Proof: There are two cases to consider

<u>Case 1:</u> $\Delta^k x_n \le 0$ on $\mathbf{N}(a)$

Before that we prove that $\Delta^{k-1} x_n > 0$ on $\mathbf{N}(a)$. If not, then there exists some $n_1 > a$ in $\mathbf{N}(a)$ such that $\Delta^{k-1} x_{n_1} \le 0$. Since $\Delta^{k-1} x_n$ is decreasing and not identically constant on $\mathbf{N}(a)$, there exists $n_2 \in \mathbf{N}(n_1)$, such that $\Delta^{k-1} x_n \le \Delta^{k-1} x_{n_2} \le \Delta^{k-1} x_{n_1} \le 0$ for all $n \in \mathbf{N}(n_2)$. But from lemma 1.4.4 we find $\lim_{n\to\infty} x_n = -\infty$ which is a contradiction to $x_n > 0$.

Thus $\Delta^{k-1} x_n > 0$ on $\mathbf{N}(a)$ and there exists a smallest integer m, $0 \le m \le k-1$ with $k+m$ odd and

$$(-1)^{m+i} \Delta^i x_n > 0 \text{ on } \mathbf{N}(a), \; m \le i \le k-1 \tag{1.4.5}$$

next let $m > 1$ and

$$\Delta^{m-1} x_n < 0 \text{ on } \mathbf{N}(a), \tag{1.4.6}$$

then again from lemma 1.4.5 it follows that

$$\Delta^{m-2} x_n > 0 \text{ on } \mathbf{N}(a), \tag{1.4.7}$$

Inequalities (1.4.5) – (1.4.7) can be unified to

$$(-1)^{(m-2)+i} \Delta^i x_n > 0 \text{ on } \mathbf{N}(a), \; m-2 \le i \le k-1$$

which is a contradiction to the definition of m.

so, inequality (1.4.6) fails and

$$\Delta^{m-1} x_n \ge 0 \text{ on } \mathbf{N}(a) \; .$$

From inequality (1.4.5), $\Delta^{m-1} x_n$ is nondecreasing and hence

$$\lim_{m\to\infty} \Delta^{m-1} x_n > 0$$

We found from lemma 1.4.5 that

$$\lim_{n\to\infty} \Delta^i x_n = \infty \quad , \quad 1 \le i \le m-2$$

Thus

$$\Delta^i x_n > 0 \text{ for all large } n \in \mathbf{N}(a) \ , \ 1 \le i \le m-1$$

<u>Case2:</u> $\Delta^k x_n \ge 0$ on $\mathbf{N}(a)$

Let $n_3 \in \mathbf{N}(n_2)$ be such that $\Delta^{k-1} x_{n_3} \ge 0$, then since $\Delta^{k-1} x_n$ is nondecreasing and not identically constant, there exists some $n_4 \in \mathbf{N}(n_3)$ such that

$$\Delta^{k-1} x_n > 0 \text{ for all large } n \in \mathbf{N}(n_4)$$

Thus

$$\lim_{n\to\infty} \Delta^{k-1} x_n > 0$$

and from lemma 1.4.5

$$\lim_{n\to\infty} \Delta^i x_n = \infty \ , \quad 1 \le i \le k-2$$

and so

$$\Delta^i x_n > 0 \text{ for all large } n \in \mathbf{N}(a) \ , 1 \le i \le k-1$$

This proves the theorem for $m = k$.

In case $\Delta^{k-1} x_n < 0$ for all $n \in \mathbf{N}(a)$, we find from lemma 1.4.4 that $\Delta^{k-2} x_n > 0$ for all $n \in \mathbf{N}(a)$. And we continue as in the proof of case1. □

Corollary 1.4.1: Let x_n be defined on $\mathbf{N}(a)$, and $x_n > 0$ with $\Delta^k x_n \le 0$ on $\mathbf{N}(a)$ and not identically zero. Then there exists a large n in $\mathbf{N}(a)$ such that

$$x_n \ge \frac{1}{(k-1)!} \Delta^{k-1} x_{2^{k-m-1} n} (n - n_1)^{k-1} \quad , \quad n \in \mathbf{N}(n_1)$$

or

$$x_n \ge \frac{\left(\frac{n}{2}\right)^{k-1}}{(k-1)!} \Delta^{k-1} x_{2^{k-2} n} \quad , \quad n \in \mathbf{N}(2n_1)$$

Proof: See [1]. □

Theorem 1.4.2: (Knaster-Tarski fixed-point theorem)

Let X be a partially ordered Banach space with ordering $\le$. Let M be a subset of X with the following properties: the infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $T : M \to M$ be an increasing mapping, that is $x \le y$ implies $Tx \le Ty$. Then T has a fixed point in M.

Chapter Two

Oscillation Criteria of First-Order Linear Neutral Difference Equations

2.0 Introduction

In this chapter we will study the oscillation of the first-order linear NDE's of the form

$$\Delta\left(x_n + \delta c_n x_{n-m} - d_n x_{n+g}\right) + p_n x_{n-k} + q_n x_{n\pm l} = 0 \ , \qquad (2.0.1)$$

where $\{c_n\},\{d_n\},\{p_n\}$ and $\{q_n\}$ are sequences of real numbers, m, g, k and l are integers and $\delta = \pm 1$

Oscillation of NDE's with delays has received a great attention since 1980.

Many valuable works for Agarwal [1] , [2], Grace [8], Lalli [5], [14], Zhang [30], Cheng [26], [30], … etc deal with different topics in oscillation and nonoscillation, these works can be considered as good references in this field.

This chapter contains four sections. In the first section we studied the sufficient conditions for the oscillation of equation (2.0.1) when the coefficients are all constants. Section two introduces sufficient and necessary conditions for the oscillation of equation (2.0.1) taking $c_n = c$, $-1 < c \le 0$ and $d_n = q_n = 0$. Section three presents the oscillation of equation (2.0.1) with variable coefficients letting $d_n = q_n = 0$. Finally, section four concerns with the oscillation theorems of equation (2.0.1) with positive and negative variable coefficients.

2.1 Oscillation criteria of first-order linear NDE's with constant coefficients

Consider the NDE of the form

$$\Delta\left(x_n + \delta c x_{n-m} - d x_{n+g}\right) + p x_{n-k} + q x_{n+l} = 0 \ , \qquad (2.1.1)$$

with c, d, p and $q \in \mathbf{R}^+ \cup \{0\}$, m, g, k and $l \in \mathbf{Z}^+ \cup \{0\}$ and $\delta = 1$.

In fact equations of the form (2.1.1) are called NDE's of mixed type, it is clear here that the sequences of coefficients are all constants. This section shows two different approaches to achieve the oscillation of equation (2.1.1); one by establishing sufficient conditions for oscillation and the other by means of the characteristic equation.

Lemma 2.1.1: Assume that $p \in \mathbf{R}^+$ and $g \in \mathbf{Z}^+$, then

a. If

$$p > \frac{(g-1)^{g-1}}{g^g} \quad , \quad g > 1$$

then the difference inequality

$$\Delta y_n \ge p y_{n+g} \quad ,$$

has no eventually positive solution $\{y_n\}$ which satisfies $\Delta^j y_n \ge 0$ eventually, $j = 0,1$.

b. If

$$p > \frac{g^g}{(g+1)^{g+1}} \quad , \quad g > 1$$

then the difference inequality

$$(-1)\Delta y_n \geq p y_{n-g}$$

has no eventually positive solution $\{y_n\}$ which satisfies $(-1)^j \Delta^j y_n > 0$ eventually, $j = 0,1$.

Proof: See [8]. □

Lemma 2.1.2: Consider the linear difference equation

$$x_{n+s} + \sum_{j=1}^{s} q(j) x_{n+s-j} = 0 \quad , \tag{2.1.2}$$

for $n = 0,1,2,...$, where $s \in \mathbf{Z}^+ \cup \{0\}$ and $q(j) \in \mathbf{R}$, $j = 1,2,...,s$, then the following statements are equivalent:

(I$_1$) Every solution of (2.1.14) oscillates

(I$_2$) The associated characteristic equation

$$\lambda^s + \sum_{j=1}^{s} q(j) \lambda^{s-j} = 0 \quad ,$$

has no positive roots.

Proof: See [11]. □

Theorem 2.1.1: Let $d > 0$, if

$$\frac{p}{1+c} > \frac{(k-m)^{k-m}}{(1+k-m)^{1+k-m}} \quad , k-m \geq 1 \tag{2.1.3}$$

and

$$\frac{q}{d} > \frac{(l-g-1)^{l-g-1}}{(l-g)^{l-g}} \quad , \quad l-g > 1 \tag{2.1.4}$$

Then equation (2.1.1) is oscillatory.

<u>(I) The proof by contradiction:</u>

Suppose to contrary that $\{x_n\}$ is an eventually positive solution of equation (2.1.1) (i.e. $x_n > 0$ for $n \geq n_0$).

Let

$$y_n = x_n + c x_{n-m} - d x_{n+g} \quad , \tag{2.1.5}$$

then

$$\Delta y_n = -p x_{n-k} - q x_{n+l} \leq 0 \quad , \quad n \geq n_1 \geq n_0 \tag{2.1.6}$$

so either $y_n < 0$ or $y_n > 0$

Case 1: If $y_n < 0$, for $n \geq n_1$, set

$$0 < v_n = -y_n = d x_{n+g} - c x_{n-m} - x_n \leq d x_{n+g}$$

$$\frac{1}{d}v_{n-g} \le x_n \quad , \quad n \ge n_2 \tag{2.1.7}$$

Using (2.1.7) we have

$$-\Delta v_n = -px_{n-k} - qx_{n+l}$$

$$\Delta v_n \ge \frac{p}{d}v_{n-(g+k)} + \frac{q}{d}v_{n-g+l}$$

$$\Delta v_n \ge \frac{q}{d}v_{n-g+l} \tag{2.1.8}$$

Using lemma 2.1.1 (a) and inequality (2.1.4) then inequality (2.1.8) has no eventually positive solution

Case2: If $y_n > 0$, for $n \ge n_1$, let

$$w_n = y_n + cy_{n-m} - dy_{n+g} \quad , \tag{2.1.9}$$

then

$$\Delta w_n + py_{n-k} + qy_{n+l} = 0 \tag{2.1.10}$$

and

$$\Delta\left(w_n + cw_{n-m} - dw_{n+g}\right) + pw_{n-k} + qw_{n+l} = 0 \tag{2.1.11}$$

Clearly that $\Delta w_n < 0$ and $\Delta^2 w_n > 0$, for $n \ge N_1 \ge n_1$, there are two cases to consider: either

(i) $w_n < 0$ eventually , or (ii) $w_n > 0$ eventually

(i) If $w_n < 0$ for $n \ge N_1$, let

$$0 < W_n = -w_n = dy_{n+g} - cy_{n-m} - y_n \le dy_{n+g} \quad ,$$

then

$$\frac{1}{d}W_{n-g} \le y_n \tag{2.1.12}$$

Using inequality (2.1.12) in equation (2.1.10) to get

$$\Delta W_n \ge \frac{q}{d}W_{n+l-g} \tag{2.1.13}$$

Using condition (2.1.4) and lemma 2.1.1(a), inequality (2.1.13) has no eventually positive solution, which is a contradiction

(ii) If $w_n > 0$ for $n \ge N_1$

We have that $\{-\Delta w_n\}$ is decreasing together with equation (2.1.11) we have

$$0 = \Delta\left(w_n + cw_{n-m} - dw_{n+g}\right) + pw_{n-k} + qw_{n+l} \ge (1+c)\Delta w_{n-m} + pw_{n-k}$$

or

$$\Delta w_n + \frac{p}{1+c}w_{n-(k-m)} \le 0 \quad , \quad n \ge N_1 \tag{2.1.14}$$

so using condition (2.1.3) and lemma 2.1.1 (b), inequality (2.1.14) has no eventually positive solution, which is a contradiction.

So every solution of equation (2.1.1) is oscillatory. □

<u>(II) The proof by using characteristic equation:</u>

The associated characteristic equation with (2.1.1) is

$$F(\lambda) = (\lambda - 1)\left[1 + c\lambda^{-m} - d\lambda^{g}\right] + p\lambda^{-k} + q\lambda^{l} = 0 \quad , \tag{2.1.15}$$

we want to show that equation (2.1.15) has no positive roots, two cases are to be discussed:

Case (I): $\lambda > 1$

$$\frac{F(\lambda)\lambda^{-g}}{(\lambda-1)} = \frac{p\lambda^{-(k+g)} + q\lambda^{l-g}}{(\lambda-1)} + \lambda^{-g} + c\lambda^{-(m+g)} - d$$

$$\geq \frac{q\lambda^{l-g}}{(\lambda-1)} - d \quad ,$$

since the minimum of $f(x) = \dfrac{x^{\alpha}}{(x-1)^{\beta}}$, $\alpha > \beta$ and $x > 1$ occurs at $x = \dfrac{\alpha}{\alpha-\beta}$ we see that

$$\frac{F(\lambda)\lambda^{-g}}{(\lambda-1)} \geq q\frac{\left(\dfrac{l-g}{l-g-1}\right)^{l-g}}{\left(\dfrac{1}{l-g-1}\right)} - d > 0$$

Case (II): $0 < \lambda < 1$

We have

$$-\frac{F(\lambda)\lambda^{m}}{(\lambda-1)} = -\frac{\left(p\lambda^{-(k-m)} + q\lambda^{(m+g)}\right)}{(\lambda-1)} - \left(\lambda^{m} + c - d\lambda^{m+g}\right)$$

$$= \frac{p\lambda^{-(k-m)} + q\lambda^{(m+g)}}{(1-\lambda)} - \left(\lambda^{m} + c - d\lambda^{m+g}\right)$$

$$\geq \frac{p\lambda^{-(k-m)}}{(1-\lambda)} - \lambda^{m} - c$$

$$\geq \frac{p\lambda^{(m-k)}}{(1-\lambda)} - 1 - c$$

$$\geq \frac{p\left(\dfrac{k-m}{k-m+1}\right)^{-(k-m)}}{\left(\dfrac{1}{k-m+1}\right)} - 1 - c > 0 \ ,$$

since the minimum of the function $f(x) = \dfrac{x^{-\alpha}}{(1-x)^{\beta}}$ occurs at $x = \dfrac{\alpha}{\alpha+\beta}$, where α and β are positive.

From cases (I) and (II) we got that $F(\lambda) > 0$ on $(0,1)\cup(1,\infty)$ and $F(1) > 0 \Rightarrow F(\lambda) > 0$ for $\lambda \in \mathbf{R}^{+}$, which is a contradiction. □

Example 2.1.1: Consider the linear NDE

$$\Delta\left(x_n + 2x_{n-3} - 2x_{n+3}\right) + x_{n-4} + x_{n+4} = 0 \tag{2.1.16}$$

It is clear that conditions (2.1.3) and (2.1.4) of theorem 2.1.1 are satisfied, so every solution of equation (2.1.16) is oscillatory. Indeed, one of the solutions is $x_n = (-1)^n$, which is an oscillatory solution.

Remark 2.1.1: The results of this section are mainly referred to [2] and [8].

2.2 Oscillation criteria of first-order NDE's with both constant and variable coefficients

Consider the NDE of the form

$$\Delta\left(x_n + \delta c x_{n-m}\right) + p_n x_{n-k} = 0 \quad , \tag{2.2.1}$$

with $p_n > 0$, $-1 < c \le 0$ and $\delta = 1$.

The main results of this section are included in theorem 2.2.1 and theorem 2.2.3.

Theorem 2.2.1: Assume that $p_n > 0$ and $-1 < c \le 0$. Then every solution of equation (2.2.1) is oscillatory if and only if the following difference inequality

$$\Delta\left(x_n + c x_{n-m}\right) + p_n x_{n-k} \le 0 \quad , \tag{2.2.2}$$

has no eventually positive solution.

Proof: On the contrary, suppose W.L.O.G that $\{x_n\}$ is a nonoscillatory solution of inequality (2.2.2), say $x_n > 0$ eventually, we want to show that equation (2.2.1) has also an eventually positive solution.

Let

$$z_n = x_n + c x_{n-m} \quad , \tag{2.2.3}$$

then $z_n > 0$ and $\Delta z_n < 0$ eventually by lemma 1.4.1.

Define

$$w_n = \frac{-\Delta z_n}{z_n} > 0 \ , \quad n \ge N \tag{2.2.4}$$

we have $w_n < 1$ for $n \ge N$, using (2.2.2) we have

$$\Delta\left(x_n + c x_{n-m}\right) + p_n x_{n-k} \le 0$$

$$\Delta z_n + p_n x_{n-k} \le 0$$

$$\Delta z_n + p_n x_{n-k} + \left(c p_n x_{n-m-k} - c p_n x_{n-m-k}\right) \le 0$$

$$\Delta z_n + p_n\left(x_{n-k} + c x_{n-m-k}\right) - c p_n x_{n-m-k} \le 0$$

$$\Delta z_n + p_n z_{n-k} - \frac{c p_n}{p_{n-m}} p_{n-m} x_{n-m-k} \le 0 \ , \tag{2.2.5}$$

dividing (2.2.5) by z_n to get

$$\frac{\Delta z_n}{z_n} + \frac{p_n z_{n-k}}{z_n} - \frac{c\, p_n \;\, p_{n-m}}{p_{n-m} z_n} x_{n-m-k} \le 0$$

$$-w_n + p_n \prod_{i=n-k}^{n-1} (1 - w_i)^{-1} - \frac{c\, p_n}{p_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1 - w_i)^{-1} \le 0 \tag{2.2.6}$$

Taking in consideration that

$$\frac{z_{n-k}}{z_n} = \prod_{i=n-k}^{n-1} (1 - w_i)^{-1}$$

Equation (2.2.6) becomes

.

$$p_n \prod_{i=n-k}^{n-1} (1 - w_i)^{-1} - \frac{c\, p_n}{p_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1 - w_i)^{-1} \le w_n \tag{2.2.7}$$

Define

$$\lambda_n^{(0)} = \{0\}\ ,\quad n = N, N+1, \ldots, N+M-1\ ,\quad \text{where } M = \max\{m,k\}$$

$$\{\lambda_n^{(r)}\} = \left\{ p_n \prod_{j=n-k}^{n-1} (1-\lambda_j^{(r-1)})^{-1} - \frac{cp_n}{p_{n-m}}\ \lambda_{n-m}^{(r-1)} \prod_{j=n-m}^{n-1} (1-\lambda_j^{(r-1)})^{-1} \right\}\ , \text{ for } n \ge N+M \qquad (2.2.8)$$

We have that

$$\lambda_n^{(0)} \le \lambda_n^{(1)} \le \ldots \le \lambda_n^{(r)} \le \ldots \le w_n\ ,\quad r > 0\ ,\ n \ge N$$

and so

$$\lim_{r\to\infty} \lambda_n^{(r)} = \lambda_n \quad , \text{ for each fixed } n \ge N \qquad (2.2.9)$$

taking the limit in (2.2.8), to obtain

$$\lambda_n = 0\ ,\quad n = N, N+1, \ldots, N+M-1$$

$$\lambda_n = p_n \prod_{j=n-k}^{n-1} (1-\lambda_j)^{-1} - \frac{cp_n}{p_{n-m}}\ \lambda_{n-m} \prod_{j=n-m}^{n-1} (1-\lambda_j)^{-1}\ ,\quad n \ge N+M \qquad (2.2.10)$$

Define the recurrence relation

$$z_n = 1\ ,\quad n = N$$

$$z_{n+1} = z_n(1-\lambda_n)\ ,\quad n > N$$

so

$$z_n = \prod_{i=N}^{n-1} (1-\lambda_i) > 0\ ,\ n > N \qquad (2.2.11)$$

and

$$\lambda_n = \frac{-\Delta z_n}{z_n} > 0 \qquad (2.2.12)$$

Using (2.2.11) and (2.2.12) in (2.2.10) we get

$$-\frac{\Delta z_n}{z_n} = \frac{p_n\, z_{n-k}}{z_n} - \frac{cp_n}{p_{n-m}}\left(1-\frac{z_{n-m+1}}{z_{n-m}}\right)\frac{z_{n-m}}{z_n}\ ,\ \text{for } n \ge N+M$$

so

$$-\frac{\Delta z_n}{p_n} = z_{n-k} + \frac{c}{p_{n-m}}\ \Delta z_{n-m}\ ,\ \text{for } n \ge N+M \qquad (2.2.13)$$

since $-\dfrac{\Delta z_n}{p_n} > 0$, so define

$$y_n = -\frac{\Delta z_{n+k}}{p_{n+k}}\ ,\quad n \ge N+M-k \qquad (2.2.14)$$

and substitute y_n in (2.2.13) to get

$$y_{n-k} = z_{n-k} - cy_{n-m-k}\ ,\quad \text{for } n \ge N+M \qquad (2.2.15)$$

combining (2.2.15) with (2.2.14) , we get

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k} = 0\ ,\ n \ge N+M$$

which is a contradiction.

Suppose there exists a nonoscillatory solution $\{x_n\}$ for equation (2.2.1) , such that $x_n > 0$, then $\{x_n\}$ will be a solution of (2.2.2) which is a contradiction. □

Remarks 2.2.1 ([14])

(1) The oscillation of (2.2.1) implies the oscillation of equation

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} + f(n, x_{n-h_1}, \ldots, x_{n-h_l}) = 0 \quad ,$$

provided that the assumption of theorem 2.2.1 and $f(n, \eta_1, \ldots, \eta_l)\eta_1 \geq 0$ whenever $\eta_1 \eta_j > 0$, $j = 1,2,\ldots,l$ hold.

(2) The oscillation of (2.2.1) implies the oscillation of equation

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} + q_n x_{n+h} = 0 \quad ,$$

provided that $-1 < c \leq 0$, $p_n \geq 0$, $q_n \geq 0$, $k, h > 0$

The following theorem provides us with a comparison condition for the oscillation, when there is a relation between the sequences of coefficients.

Theorem 2.2.2: Assume that

$$\Delta(x_n + \bar{c}x_{n-m}) + q_n x_{n-k} = 0 \quad , \tag{2.2.16}$$

and $\bar{c}, c \in (-1,0]$, so that

$$q_n \geq p_n > 0 \tag{2.2.17}$$

and

$$\left(\frac{\bar{c}\, q_n}{q_{n-m}}\right) \leq \left(\frac{c\, p_n}{p_{n-m}}\right) \tag{2.2.18}$$

hold, then the oscillation of (2.2.1) implies the oscillation of (2.2.16)

Proof: Let $\{x_n\}$ be a positive solution of equation (2.2.16), $x_n > 0$ for $n \geq N$, proceeding as in the proof of theorem 2.2.1, there exists a sequence $\{w_n\}$, $w_n \in (0,1)$ such that

$$w_n = q_n \prod_{i=n-k}^{n-1} (1-w_i)^{-1} - \frac{\bar{c}\, q_n}{q_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1-w_i)^{-1} \quad ,$$

using conditions (2.2.17) and (2.2.18) it follows from the above equation that

$$w_n \geq p_n \prod_{i=n-k}^{n-1} (1-w_i)^{-1} - \frac{c\, p_n}{p_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1-w_i)^{-1}$$

Thus we get inequality (2.2.7), again proceeding as in proof of theorem 2.2.1 to get a contradiction. □

Theorem 2.2.3: If

$$q_i \geq p_i > 0 \quad , \tag{2.2.19}$$

and

$$\sum_{i=N}^{\infty} q_i = \infty \quad , \qquad -1 < \bar{c} \leq c < 0 \tag{2.2.20}$$

then the oscillation of (2.2.1) implies the oscillation of (2.2.16) .

Proof: Suppose that $\{x_n\}$ is a positive solution of equation (2.2.16), let

$$z_n = x_n + \bar{c}x_{n-m} \tag{2.2.21}$$

then

$$\Delta z_n + q_n x_{n-k} = 0$$

It follows that $z_n \to 0$ as $n \to \infty$, and

$$z_n = \sum_{n}^{\infty} q_i \, x_{i-k} \quad , \tag{2.2.22}$$

now from (2.2.21) and (2.2.22)

$$x_n = -\bar{c} x_{n-m} + \sum_{n}^{\infty} q_i \, x_{i-k} \quad ,$$

by (2.2.19)

$$\begin{aligned} x_n &\geq -c x_{n-m} + \sum_{n}^{\infty} p_i \, x_{i-k} \\ &\geq -c x_{n-m} \geq \ldots \geq (-c)^L x_{n-Lm} \\ &= (-c)^{\frac{(n-n_0)}{m}} x_{n_0} \\ &= \alpha (-c)^{\frac{n}{m}} \end{aligned}$$

where $\alpha = (-c)^{\frac{-n_0}{m}} x_{n_0}$ and $L = \dfrac{n-n_0}{m}$

Define

$$\left\{\lambda_n^{(0)}\right\} = \{x_n\} \quad , \quad n \geq n_0$$

$$\lambda_n^{(r)} = -c\lambda_{n-m}^{(r-1)} + \sum_{n}^{\infty} p_i \lambda_{i-k}^{(r-1)} \quad , \quad n \geq n_0 + \max\{m,k\}$$

$$\lambda_n^{(r)} = \lambda_n^{(0)} \; , \quad n_0 \leq n \leq n_0 + \max\{m,k\} \; , \; r = 1,2,3,\ldots \tag{2.2.23}$$

we have

$$\lambda_n^{(r)} \geq \alpha \, (-c)^{\frac{n}{m}} \quad ,$$

and

$$\lambda_n^{(r+1)} \leq \lambda_n^{(r)} \; , \; n \geq n_0 \, , \; r = 0,1,2,\ldots$$

hence, for each $n \geq n_0$ we have

$$\lambda_n^{(r)} \to \lambda_n^{(*)} \text{ as } r \to \infty$$

so

$$\lambda_n^{(*)} = -c\,\lambda_{n-m}^{*} + \sum_{i=n}^{\infty} p_i \; \lambda_{i-k}^{*}$$

hence,

$$\Delta(\lambda_n^{*} + c\,\lambda_{n-m}^{*}) + p_n \; \lambda_{n-k}^{*} = 0 \; , \; n \geq n_0 + M$$

which is a contradiction, so equation (2.2.1) is oscillatory. □

Remarks 2.2.2

(1) As a special case of theorem 2.2.3 , when $c = \bar{c} = 0$, then for the equations

$$\Delta y_n + p_n y_{n-k} = 0 \tag{2.2.24}$$

$$\Delta x_n + q_n x_{n-k} = 0 \tag{2.2.25}$$

with $q_n \geq p_n$, then

the oscillation of (2.2.24) implies that of (2.2.25) .

(2) One can extend the results in theorem 2.2.3 to the equation

$$\Delta(x_n + c\,x_{n-m}) + \sum_{j=1}^{l} p_{j\,n} \; x_{n-k_j} = 0 \tag{2.2.26}$$

Theorem 2.2.4: If

$$q_{ji} \geq p_{ji} > 0 \;, \quad j = 1,2,...,l \tag{2.2.27}$$

and

$$\sum_{i=N}^{\infty} q_{ji} = \infty \text{ , for } j = 1,2,...,l \tag{2.2.28}$$

with $-1 < \bar{c} \leq c < 0$, then the oscillation of equation (2.2.26) implies the oscillation of

$$\Delta(x_n + \bar{c}\; x_{n-m}) + \sum_{j=1}^{l} q_{jn}\; x_{n-k_j} = 0 \tag{2.2.29}$$

Proof: Suppose that $\{x_n\}$ is a positive solution of equation (2.2.29), let z_n as in (2.2.21), then

$$\Delta z_n + \sum_{j=1}^{l} q_{jn} x_{n-k_j} = 0 \tag{2.2.30}$$

We have $z_n \to 0$ as $n \to \infty$, let

$$z_n = \sum_{i=n}^{\infty} \left(\sum_{j=1}^{l} q_{ji} x_{i-k_j} \right) \tag{2.2.31}$$

so,

$$x_n = -\bar{c}\, x_{n-m} + \sum_{i=n}^{\infty} (\, q_{1i}\; x_{i-k_1} + q_{2i}\; x_{i-k_2} + ... + q_{li}\; x_{i-k_l}) \;,$$

by (2.2.27)

$$x_n \geq -\bar{c}\; x_{n-m} + \sum_{n}^{\infty} (\, p_{1i}\; x_{i-k_1} + p_{2i}\; x_{i-k_2} + ... + p_{li}\; x_{i-k_l})$$

$$\geq -\bar{c} x_{n-m} \geq ... \geq (-\bar{c}\,)^s\; x_{n-sm}$$

$$= (-c\,)^{\frac{(n-n_0)}{m}}\, x_{n_0} \quad , \quad n_0 = n - sm$$

Let $\alpha = (-c)^{\frac{-n_0}{m}}\; x_{n_0}$, then

$$x_n \geq \alpha\; (-c\,)^{\frac{n}{m}}$$

Define

$$\left\{\lambda_n^{(0)}\right\} = \left\{x_n\right\}$$

$$\lambda_n^{(r)} = -c\, \lambda_{n-m}^{(r-1)} + \sum_{n}^{\infty} \left(p_{1i}\; \lambda_{i-k_1}^{(r-1)} + ...\; p_{li}\; \lambda_{i-k_l}^{(r-1)}\right)$$

$$\lambda_n^{(r)} = \lambda_n^{(0)} \;,$$

we have

$$\lambda_n^{r+1} \leq \lambda_n^{r} \quad \text{(i.e. } \lambda_n^{(r)} \text{ is decreasing)}$$

Since $\lambda_n^{(r)} \geq \alpha\,(-c\,)^{\frac{n}{m}}$, $r = 0,1,2,...$

then for each $n \geq n_0$ we have

$$\lambda_n^{(r)} \to \lambda_n^{*} \quad \text{as } r \to \infty$$

so

$$\lambda_n^* = -c\,\lambda_{n-m}^* + \sum_n^{\infty} \left(p_{1i}\,\lambda_{i-k_1}^* + \ldots + p_{Li}\,\lambda_{i-k_l}^*\right)$$

Hence,

$$\Delta\left(\lambda_n^* + c\,\lambda_{n-m}^*\right) + p_{1n}\,\lambda_{n-k_1}^* + \ldots + p_{ln}\,\lambda_{n-k_l}^* = 0\ , \quad n \ge n_0 + M$$

This means that λ_n^* is a positive solution of equation (2.2.21), which is a contradiction. □

Example 2.2.1: Consider the NDE's

$$\Delta\left(x_n - \frac{1}{2}x_{n-1}\right) + 2^n x_{n-3} = 0\ , \qquad (2.2.32)$$

and

$$\Delta\left(x_n - \frac{1}{2}x_{n-1}\right) + 3^n x_{n-3} = 0\ , \qquad (2.2.33)$$

using theorem 2.2.2 we conclude that the oscillation of (2.2.32) implies the oscillation (2.2.33) since conditions (2.2.17) and (2.2.18) are satisfied.
Note also that example 2.2.1 satisfies the conditions of theorem 2.2.3.

Remark 2.2.3: The results of this section are mainly referred to [14].

2.3 Oscillation criteria of first-order linear NDE's with variable coefficients

Consider the first-order linear NDE of the form

$$\Delta(x_n + \delta\, c_n x_{n-m}) + p_n x_{n-k} = 0\ , \quad n \ge 0 \qquad (2.3.1)$$

where $\{c_n\}$ and $\{p_n\}$ are nonnegative sequences of real numbers, m and k are nonnegative integers and $\delta = -1$.

Equation (2.3.1) has been extensively investigated in literature. Authors like Tang, Yu and Ladas discussed the boundedness and the asymptotic behavior of the solutions of equation (2.3.1) with and without the condition $\sum_{n=0}^{\infty} p_n = \infty$, also Zhang, Cheng, Chen, Lalli and Yu as in [5] and [30] studied the oscillation property of equation (2.3.1) under several conditions.

In fact in this section we will consider some of those theorems in [5] and [30], but before that, we need the following assumptions:

(H_1) There exists a positive integer N such that

$$c_{N+im} \le 1\ , \text{ for } i = 0,1,2,\ldots$$

(H_2) For $n \ge 0$

$$\sum_{n=0}^{\infty} np_n \sum_{j=n}^{\infty} p_j = \infty$$

Lemma 2.3.1: Assume that (H_2) holds, and $c_n \ge 1$, $p_n \ge 0$, for $n = 0,1,2,\ldots$, then for any eventually positive solution $\{x_n\}$ of the inequality

$$\Delta(x_n - c_n x_{n-m}) + p_n x_{n-k} \le 0 \qquad (2.3.2)$$

the sequence $\{z_n\}$ defined by

$$z_n = x_n - c_n x_{n-m} \quad , \quad n \geq 0 \tag{2.3.3}$$

satisfies that $\Delta z_n \leq 0$ and $z_n < 0$.

Proof: see [5]. □

Theorem 2.3.1: Suppose that $c_n = 1$ and $p_n \geq 0$, for $n = 0,1,2,...$, and (H_2) holds then every solution of equation (2.3.1) is oscillatory.

Proof: From lemma 1.4.1, the sequence $\{z_n\}$ defined by (2.3.3) would be positive, and from lemma 2.3.1, the sequence $\{z_n\}$ would be negative for all large n. (i.e. positive and negative at the same time) , hence every solution of equation (2.3.1) is oscillatory. □

Theorem 2.3.2: Suppose that (H_1) and (H_2) hold, then every solution of equation (2.3.1) oscillates provided that for all large n.

$$p_{n-m} \leq c_{n-k} p_n \tag{2.3.4}$$

Proof: On the contrary, suppose that equation (2.3.1) has an eventually positive solution $\{x_n\}$, let z_n as in (2.3.3), then from (2.3.1)

$$\Delta z_n + p_n x_{n-k} = 0 \tag{2.3.5}$$

$$\begin{aligned} \Delta z_n &= -p_n x_{n-k} \\ &= -p_n \left(z_{n-k} + c_{n-k} x_{n-k-m} \right) \\ &= -p_n z_{n-k} - p_n c_{n-k} x_{n-k-m} \\ &\leq -p_n z_{n-k} - p_{n-m} x_{n-k-m} \quad , \qquad \text{by using (2.3.4)} \\ &= -p_n z_{n-k} + \Delta z_{n-m} \ , \quad n \geq n_0 + k + m, \qquad \text{by using (2.3.5)} \end{aligned}$$

i.e.

$$\Delta\left(z_n - z_{n-m}\right) + p_n z_{n-k} \leq 0 \ , \quad n \geq n_0 + k + m \tag{2.3.6}$$

using (H_2) and by lemma 1.4.1 and lemma 2.3.1 , then the sequence $\{z_n - z_{n-m}\}$ is eventually negative and positive at the same time, which is a contradiction, so every solution of equation (2.3.1) is oscillatory. □

Theorem 2.3.3: Suppose that (H_1) holds, $p_n \geq 0$ and there is some number $\alpha \in [0,1]$ such that

$$\alpha p_{n-m} \leq p_n c_{n-k} \ , \ \text{for all large } n \tag{2.3.7}$$

then every solution of equation (2.3.1) is oscillatory provided that the recurrence relation

$$\Delta w_n + \frac{p_n}{1-\alpha} w_{n-(k-m)} \leq 0 \quad , \quad n \geq 0 \tag{2.3.8}$$

has no eventually positive solution.

Proof: Suppose that (2.3.1) has an eventually positive solution $\{x_n\}$, define z_n as in (2.3.3) , then

$$\Delta z_n + p_n x_{n-k} = 0 \tag{2.3.9}$$

$$\Delta z_{n-m} + p_{n-m} x_{n-(m+k)} = 0$$

$$\alpha\ \Delta z_{n-m} = -\alpha\ p_{n-m} x_{n-(m+k)} \quad ,$$

using (2.3.7)

$$\alpha\ \Delta z_{n-m} \geq -c_n\ p_{n-m} x_{n-(m+k)} \quad ,$$

or

$$-\alpha\ \Delta z_{n-m} - c_n\ p_{n-k} x_{n-(m+k)} \leq 0 \quad , \tag{2.3.10}$$

adding (2.3.9) with (2.3.10) we get

$$\Delta z_n - \alpha\,\Delta z_{n-m} + p_n x_{n-k} - p_n c_{n-k} x_{n-(m+k)} \leq 0$$

using (2.3.3) we get

$$\Delta\left(z_n - \alpha\, z_{n-m}\right) + p_n z_{n-k} \leq 0$$

Define

$$w_n = z_n - \alpha\, z_{n-m} \ , \quad n \geq 0$$

satisfies $w_n > 0$ for all large n, we have

$$z_{n-m} \geq \frac{w_n}{1-\alpha} \quad ,$$

and

$$z_{n-k} \geq \frac{w_{n-(k-m)}}{1-\alpha} \quad , \quad n \geq 0$$

and hence $\{w_n\}$ is an eventually positive solution of (2.3.8), which is a contradiction. □

Remarks 2.3.1:

(1) In theorem 2.3.2, if we replace (H_1) by $c_n \geq 1$, for $n = 0,1,2,...$, and replace (2.3.4) by $c_{n-k}\, p_n \leq p_{n-m}$, then every solution of (2.3.1) is oscillatory.

(2) In theorem 2.3.3, if we replace (H_1) by $c_n \leq 0$, for $n = 0,1,2,...$, and replace (2.3.7) by $-c_{n-k}\, p_n \leq \alpha p_{n-m}$, we will have another criteria for the oscillation of equation (2.3.1) provided that

$$\Delta w_n + \frac{p_n}{1+\alpha} w_{n-(k-m)} \leq 0 \quad , \quad n \geq 0$$

has no eventually positive solution.

Remark 2.3.2: The results of this section are mainly referred to [5] and [30] .

2.4 Oscillation criteria of first-order linear NDE with positive and negative variable coefficients

Consider the NDE of the form

$$\Delta\left(x_n + \delta\, c_n x_{n-m}\right) + p_n x_{n-k} - q_n x_{n-l} = 0 \quad , \tag{2.4.1}$$

where m, k and l are integers such that $0 \leq l < k$, $m > 0$,$\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are nonnegative sequences of real numbers, the sequence $h_n = p_n - q_{n-k+l}$, $n \geq k-l$ is nonnegative and has a positive subsequence and $\delta = -1$.

Authors, since 1990 studied equation (2.4.1), they discussed several cases, when $q_n = 0$, $c_n = 0$ or when $c_n = c$ and $0 \leq c < 1$.

In [22] Tang, Yu and Peng investigated the oscillation of equation (2.4.1) depending on the treatment of an ordinary difference equation, while in [26] Tian and Cheng did some modifications on the results of [22], we will see later that the results in [22] are taken to be the special case of those in [26].

Before starting we have to consider the following definitions:

$$R_n(t) = c_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n-t+k-l-1} p_s \quad \text{, where } t = \{0,1,\ldots,k-l\} \tag{2.4.2}$$

$$M^*(t) = \begin{cases} m & , \ \textit{if } q_n = 0 \textit{ and } t = k-l \\ \max\{m,k\}, & \textit{otherwise} \end{cases} \tag{2.4.3}$$

$$m^*(t) = \begin{cases} m & , \ \textit{if } q_n = 0 \textit{ and } t = k-l \\ \min\{m,l+1\}, & \textit{otherwise} \end{cases} \tag{2.4.4}$$

Lemma 2.4.1: Suppose that $\{x_n\}$ is an eventually positive solution of the functional difference inequality

$$\Delta(x_n - c_n x_{n-m}) + p_n x_{n-k} - q_n x_{n-l} \le 0 \quad , \tag{2.4.5}$$

and there exists an integer $t \in \{0,1,\ldots,k-l\}$, such that $R_n(t) \le 1$, for all large n, then the sequence $\{z_n\}$ defined by

$$z_n = x_n - c_n x_{n-m} - \sum_{s=n-t}^{n-1} q_s x_{s-l} - \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k} \quad , \tag{2.4.6}$$

for all large n, will satisfy $z_n > 0$ and $\Delta z_n \le 0$ eventually.

Proof: Let $N_1 \in \mathbf{Z}^+$, such that for $n \ge N_1$, $x_n > 0$. By inequality (2.4.5) and equation (2.4.6), it is clear that $\Delta z_n \le 0$, which means that z_n is nonincreasing for $n \ge N_1$.

To prove that $z_n > 0$, suppose not that is $z_n < 0$ for all large n. Thus there exists a constant $\alpha > 0$ such that $z_n < -\alpha$ for $n \ge N_2 \ge N_1$, so from equation (2.4.6)

$$-\alpha > z_n = x_n - c_n x_{n-m} - \sum_{s=n-t}^{n-1} q_s x_{s-l} - \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k}$$

$$x_n \le -\alpha + c_n x_{n-m} + \sum_{s=n-t}^{n-1} q_s x_{s-l} + \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k} \tag{2.4.7}$$

The question appears here whether $\{x_n\}$ is bounded or not **?**

If $\{x_n\}$ is bounded: This means that $\limsup_{n\to\infty} x_n = \beta < \infty$, we can choose a sequence of integers $\{n_i^*\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} n_i^* = \infty$ and $\lim_{i\to\infty} x_{n_i} = \beta$.

Let $\{\lambda_i\}$ be a sequence of integers such that

$$n_i^* - M^*(t) \le \lambda_i \le n_i^* - m^*(t) \quad ,$$

and

$$x_\lambda = \max\{x_n : n_i^* - M^*(t) \le n \le n_i^* - m^*(t)\} \quad ,$$

with $\lim_{i\to\infty} \lambda_i = \infty$ and $\limsup_{n\to\infty} x_\lambda \le \beta$.

From equation (2.4.2) and inequality (2.4.7) we have

$$x_n \le -\alpha + c_n x_{n-m} + \sum_{s=n-t}^{n-1} q_s x_{s-l} + \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k}$$
$$\le -\alpha + x_\lambda$$

then

$$\limsup_{n\to\infty} x_n \le \limsup_{i\to\infty}(-\alpha + x_\lambda) < \beta$$
$$\beta \le -\alpha + \beta < \beta$$

which is a contradiction.

If $\{x_n\}$ is unbounded: This means that $\limsup_{n\to\infty} x_n = \infty$, so there exists a sequence of integers $\{n_i^*\}_{i=1}^{\infty}$ such that for $N_2 + M^*(t) \le n_i^*$, we have

$\lim_{i\to\infty} n_i^* = \infty$, $\lim_{i\to\infty} x_{n_i^*} = \infty$, and $x_n = \max\{x_n : N_2 \le n \le n_i^*, i = 1,2,...\}$.

From equation (2.4.2) and inequality (2.4.7), we get

$$x_n \le -\alpha + c_n x_{n-m} + \sum_{s=n-t}^{n-1} q_s x_{s-l} + \sum_{s=n}^{n+k-l-t-1} p_s x_{s-k}$$
$$\le -\alpha + x_n$$
$$< x_n$$

which is a contradiction. So z_n must be positive. □

Lemma 2.4.2: Suppose that the second-order difference inequality

$$\Delta^2 y_n + \frac{1}{M^*(t)} h_{n-t+k-l} y_n \le 0 \ , \qquad (2.4.8)$$

does not have any eventually positive solution, and there exists an integer $t \in \{0,1,...,k-l\}$ such that $R_n(t) \ge 1$, for all large n, then for any eventually positive solution $\{x_n\}$ of inequality (2.4.5), the sequence $\{z_n\}$ defined by equation (2.4.6) satisfies $z_n < 0$ and $\Delta z_n \le 0$ for all large n.

Proof: See [26]. □

Lemma 2.4.3: The difference inequality

$$\Delta^2 y_n + d_n y_n \le 0 \quad , \quad n \ge 0$$

has no eventually positive solutions if

$$\liminf_{n\to\infty} n \sum_{s=n}^{\infty} d_s > \frac{1}{4} \quad ,$$

where $\{d_n\}$ is a sequence of nonnegative real numbers.

Proof: See [22]. □

Theorem 2.4.1: Suppose there exist two integers $t, t_1 \in \{0,1,...,k-l\}$ such that

$$R_n(t_1) = c_n + \sum_{s=n-t_1}^{n-1} q_s + \sum_{s=n}^{n-t_1+k-l-1} p_s \le 1 \quad , \qquad (2.4.9)$$

$$R_n(t) = c_n + \sum_{s=n-t}^{n-1} q_s + \sum_{s=n}^{n-t+k-l-1} p_s \ge 1 \quad , \qquad (2.4.10)$$

for all large n. Further suppose that the functional inequality (2.4.8) does not have any eventually positive solution. Then every solution of equation (2.4.1) oscillates.

Proof: Using lemma 2.4.1, then we have that $z_n > 0$ and $\Delta z_n \le 0$ eventually. However, using lemma 2.4.2 gives that $z_n < 0$ eventually, so we obtain a contradiction, thus equation (2.4.1) can not have any eventually positive, nor eventually negative solution. □

Corollary 2.4.1: Suppose that (2.4.10) holds for $t = 0$, and all large n and

$$\liminf_{n\to\infty} n\sum_{s=n}^{\infty}\left(p_{s+k-l} - q_s\right) > \frac{\max\{m,k\}}{4} \quad , \tag{2.4.11}$$

holds. Further suppose $\left\{\dfrac{p_n}{p_n - q_{n-k+l}}\right\}$ is nondecreasing and there exist two nonnegative constants δ_1 and δ_2 such that $\delta_2 \ge p_n - q_{n-k+l}$ eventually and for all large n,

$$c_{n-k}\left(p_n - q_{n-k+l}\right) \le \delta_1\left(p_{n-m} - q_{n-k+l-m}\right) \quad , \tag{2.4.12}$$

$$p_n\left(p_{n+l} - q_{n-k+2l}\right) \le \delta_2\left(p_n - q_{n-k+l}\right) \quad , \tag{2.4.13}$$

and

$$\delta_1 + (k-l)\delta_2 = 1, \text{ and } h_n < \delta_2 \text{ eventually.}$$

Then every solution of equation (2.4.1) oscillates.

Proof: Suppose that equation (2.4.1) has an eventually positive solution $\{x_n\}$, and let z_n be defined by (2.4.6).
Applying lemma 2.4.2 and lemma 2.4.3 to get that $z_n < 0$ and $\Delta z_n \le 0$ eventually.
In view of inequalities (2.4.12) and (2.4.13) we have

$$\Delta z_n = -h_{n-t+k-l}x_{n-t-l} \quad ,$$

for simplicity let $h^* = h_{n-t+k-l}$, so

$$\begin{aligned}
\Delta z_n &= -h^* x_{n-t-l} \\
&= -h^*\left(z_{n-t-l} + c_{n-t-l}x_{n-t-l-m}\right) - h^*\left(\sum_{s=n-2t-l}^{n-t-l-1} q_s x_{s-l} + \sum_{s=n-t-l}^{n-2t+k-2l-1} p_s x_{s-k}\right) \\
&\ge -h^* z_{n-t-l} - \delta_1 h_{n-t+k-l-m}\, x_{n-t-l-m} - h^* \sum_{s=n-2t-l}^{n-t-l-1} \frac{q_s h_{s+k-l}}{h_{s+k-l}} x_{s-l} \\
&\qquad - h^* \sum_{s=n-t-l}^{n-2t+k-2l-1} \frac{p_s h_s}{h_s} x_{s-k} \\
&\ge -h^* z_{n-t-l} - \delta_1 \Delta z_{n-m} - \frac{q_{n-t-l}h^*}{h_{n-t+k-2l}} \sum_{s=n-2t-l}^{n-t-l-1}\left(-\Delta z_{s+t}\right) \\
&\qquad - \frac{p_{n-2t+k-2l}h^*}{h_{n-2t+k-2l}} \sum_{s=n-t-l}^{n-2t+k-2l-1}\left(-\Delta z_{s+t-k+l}\right) \\
&\ge -h^* z_{n-l} - \delta_1 \Delta z_{n-m} - \delta_2\left(z_{n-l} - z_{n-k}\right)
\end{aligned}$$

Thus

$$\Delta\left(z_n - \delta_1 z_{n-m}\right) - \left(\delta_2 + h_{n-t+k-l}\right)z_{n-l} + \delta_2 z_{n-k} \ge 0$$

Hence $\{-z_n\}$ is an eventually positive solution of inequality

$$\Delta(x_n - \delta_1 x_{n-m}) + \delta_2 x_{n-k} - (\delta_2 - h_{n+k-l})x_{n-l} \le 0$$

Comparing the last inequality with equation (2.4.1) and using lemmas 2.4.1, 2.4.2 and 2.4.3, we have that z_n is simultaneously positive and negative eventually , which is a contradiction. □

Remark 2.4.2: Similar results can be obtained for $(t = k - l)$, where $\left\{\dfrac{q_n}{p_{n+k-l} - q_n}\right\}$ is nondecreasing and for $\delta_1, \delta_2 \in \mathbf{Z}^+ \cup \{0\}$

$$q_{n-k}(p_n - q_{n-k+l}) \le \delta_2(p_{n-l} - q_{n-k}) \quad ,$$

holds together with (2.4.12) and $\delta_1 + (k-l)\delta_2 = 1$.

Theorem 2.4.2: Suppose there exists an integer $t \in \{0,1,...,k-l\}$ such that

$$c_n + \sum_{s=n-k+l}^{n-1} q_s = 1 \quad , \tag{2.4.14}$$

holds eventually, further suppose that there exists a constant $\alpha \in [0,1)$ such that

$$c_{n-k}(p_n - q_{n-k+l}) \ge \alpha(p_{n-m} - q_{n-k+l-m}) \quad , \tag{2.4.15}$$

then every solution of equation (2.4.1) is oscillatory. Provided that there exists a constant $\overline{\alpha} \in [0, \alpha)$ such that the following recurrence relation

$$\Delta u_n + \frac{\overline{\alpha}}{1-\alpha} h_{n-t+k-l}\, u_{n-t-l-m} \le 0 \quad , \quad n = 0,1,2,... \tag{2.4.16}$$

does not have an eventually positive solution.

Proof: Suppose to contrary that $\{x_n\}$ is an eventually positive solution of equation (2.4.1), from lemma 2.4.1 we have $z_n > 0$ and $\Delta z_n \le 0$ for all large n.

$$\Delta z_n \le -h_{n-t+k-l}\, x_{n-t-l}$$
$$= -h_{n-t+k-l}\,(z_{n-t-l} + c_{n-t-l}\, x_{n-t-l-m}) - h_{n-t+k-l}\left(\sum_{s=n-2t-l}^{n-t-l-1} q_s x_{s-L} + \sum_{s=n-t-l}^{n-2t+k-2l-1} p_s x_{s-k}\right),$$

hence,

$$\Delta z_n + h_{n-t+k-l} z_{n-t-l} + c_{n-t-l} h_{n-t+k-l} x_{n-t-l-m} \le 0 \quad , \tag{2.4.17}$$

for all large n, by inequality (2.4.15) and inequality (2.4.17) becomes

$$\Delta z_n + h_{n-t+k-l} z_{n-t-l} + \alpha\, h_{n-t+k-l-m} x_{n-t-l-m} \le 0 \quad , \tag{2.4.18}$$

Inequality (2.4.18) together with

$$-\alpha\, \Delta z_{n-m} - \alpha\, h_{n-t+k-m-l} x_{n-t-l-m} = 0 \; ,$$

we get

$$\Delta(z_n - \alpha\, z_{n-m}) - h_{n-t+k-l} z_{n-t-l} \le 0 \quad , \tag{2.4.19}$$

for all large n.

Let $u_n = z_n - \alpha\, z_{n-m}$, similar to the proof of lemma 2.4.1, we have $u_n > 0$ and $\Delta u_n \le 0$ for all large n. Hence, there exists an integer $N > 0$ such that $z_n > 0$ and $\Delta z_n \le 0$, and $u_n > 0$ and $\Delta u_n \le 0$, for $n \ge N$. Thus

$$z_n = u_n + \alpha\, z_{n-m}$$
$$= u_n + \alpha(u_{n-m} + \alpha\, z_{n-2m})$$
$$\vdots$$

$$= u_n + \alpha u_{n-m} + \ldots + \alpha^i u_{n-im} + \alpha z_{n-(i+1)m}$$
$$\geq \left(\alpha + \alpha^2 + \ldots + \alpha^i\right) u_{n-m}$$
$$= \frac{\alpha\left(1-\alpha^{i+1}\right)}{1-\alpha} u_{n-m}$$

for $n \geq (i+1)m + N$, for all large n, which is a contradiction. □

Example 2.4.1: Consider the NDE

$$\Delta\left(x_n - \frac{1}{2}x_{n-1}\right) + \left(\frac{1}{4} + (n+1)^{-2}\right)x_{n-3} - \left(\frac{1}{4} - (n+3)^{-2}\right)x_{n-1} = 0 \quad , \qquad (2.4.20)$$

using theorem 2.4.1 we have for $c_n = \frac{1}{2}$ and $t = 0,\ t_1 = 2$,

$$R_n(t_1) = R_n(2) = \frac{1}{2} + q_{n-2} + q_{n-1}$$
$$= 1 - (n+1)^{-2} - (n+2)^{-2} < 1$$

and

$$R_n(t) = R_n(0) = \frac{1}{2} + p_n + p_{n+1}$$
$$= 1 + (n+1)^{-2} + (n+2)^{-2} > 1 \text{ , for all large } n$$

but $h_n = p_n - q_{n-2} = 2(n+1)^{-2}$ and

$$\liminf_{n\to\infty}\left(n\sum_{s=n}^{\infty} h_{s+k-l}\right) = 2\liminf_{n\to\infty}\left(n\sum_{s=n+3}^{\infty} s^{-2}\right) = 2 > \frac{3}{4} = \frac{\max\{m,k\}}{4}$$

So we can conclude that every solution of equation (2.4.20) is oscillatory.

Remark 2.4.3: The results of this section are mainly referred to [22], and [26].

Chapter Three

Oscillation Criteria of First-Order Nonlinear Neutral Difference Equations

3.0 Introduction

This chapter mainly concerns with the oscillation of nonlinear NDE's of the form

$$\Delta(x_n + \delta_1\, p_n x_{n-m}) + \delta_2\, q_n G(x_{n-k}) = f_n \quad , \quad n \geq 0 \tag{3.0.1}$$

where $\{p_n\}$, $\{q_n\}$ and $\{f_n\}$ are sequences of real numbers, G is a real valued function and $\delta_1 = \pm 1, \delta_2 = \pm 1$.

We studied oscillation of equation (3.0.1) under several conditions depending on the form we consider.

Authors like Zhang [29], Lalli [12] studied nonlinear NDE's with both constant and variable coefficients. However, as in [23] and [20] authors considered the case when all the coefficients are variables, also homogenous and nonhomogenous (forced) cases are discussed as in [18] and [10].

In this chapter, the first section concerns with the oscillation of equation (3.0.1) when $p_n = p$, $f_n = 0$ and $G(x_{n-k}) = x_{n-k}^{\alpha}$, where α is the quotient of odd positive integers. In section two we discuss the oscillation of equation (3.0.1) with several conditions on $G(x_{n-k})$. Section three considered the forced form of (3.0.1) with both constant and variable coefficients. While in section four the coefficients are all variables. Finally, section five studies oscillation theorems when $G(x_{n-k}) = |x_{n-k}| \operatorname{sgn} x_{n-k}$ and when $G(x_{n-k}) = \max\limits_{s \in [n-k,n]} x_s$.

3.1 Oscillation criteria of first - order nonlinear NDE's with both constant and variable coefficients

Consider the first-order nonlinear NDE's of the form

$$\Delta(x_n + \delta_1\, p x_{n-m}) + \delta_2\, q_n G(x_{n-k}) = 0 \ , \quad n \geq 0 \tag{3.1.1}$$

where $\{q_n\}$ is a sequence of nonnegative real numbers, m and k are positive integers , $\delta_1 = -1, \delta_2 = +1$ and $q_n \geq 0$.

We start this chapter by taking the special case where $G(x_{n-k}) = x_{n-k}^{\alpha}$, where $\alpha \in (0,\infty)$ is a quotient of odd positive integers and $0 \leq p < 1$, the oscillation here is discussed for two cases , (i) $\alpha < 1$, (ii) $\alpha > 1$

Theorem 3.1.1: Assume that $0 \leq p < 1$ and $0 < \alpha < 1$. Then every solution of the equation

$$\Delta(x_n - p x_{n-m}) + q_n x_{n-k}^{\alpha} = 0 \ , \tag{3.1.2}$$

Oscillates if and only if $\sum\limits_{n=0}^{\infty} q_n = \infty$.

Proof: Assume that $\alpha \in (0,1)$ and $\{x_n\}$ is an eventually positive solution of equation (3.1.2). If we let

$$z_n = x_n - px_{n-m} \quad \text{, assuming that } 0 \le p < 1$$

Using lemma 1.4.1 we have that $z_n > 0$ eventually

Now,

$$\begin{aligned} x_{n-k} &= z_{n-k} + px_{n-m-k} \\ &\ge z_{n-k} + pz_{n-m-k} \\ &\ge (1+p)z_n \quad , \end{aligned}$$

so equation (3.1.2) becomes

$$\Delta z_n + q_n (1+p)^{\alpha} z_n^{\alpha} \le 0 \quad ,$$

multiplying the last inequality by $z_n^{-\alpha}$

$$z_n^{-\alpha} \Delta z_n + q_n (1+p)^{\alpha} \le 0 \quad . \tag{3.1.3}$$

Define

$$r(t) = z_L + (t-L)\Delta z_L \quad , \quad L \le t \le L+1 \text{ ,}$$

so $z_{L+1} \le r(t) \le z_L$, since $\Delta z_L \le 0$,

but

$$r'(t) = \Delta z_L$$

$$r^{\alpha}(t) = (z_L + (t-L)\Delta z_L)^{\alpha} \le z_L^{\alpha} \quad ,$$

so

$$\frac{r'(t)}{r^{\alpha}(t)} \le \frac{\Delta z_L}{z_L^{\alpha}} \quad , \tag{3.1.4}$$

using (3.1.3) , (3.1.4) and the condition $\sum_{n=0}^{\infty} q_n = \infty$, we get

$$\int_{r(N)}^{r(\infty)} \frac{dr}{r^{\alpha}} = -\infty \quad ,$$

which contradicts the fact that $\alpha \in (0,1)$. □

Note that to prove the other case we assume that $\alpha > 1$ there exists $\lambda > k^{-1} \ln \alpha$ such that $\liminf_{n \to \infty} q_n \exp(-e^{\lambda n}) > 0$. Then if $\{x_n\}$ is an eventually positive solution of equation (3.1.2). we have

$$\begin{aligned} x_{n-k} &= z_{n-k} + px_{n-m-k} \\ &= z_{n-k} + pz_{n-m-k} + p^2 x_{n-m-2k} \\ &\quad \vdots \\ &= z_{n-k} + pz_{n-m-k} + \ldots + p^L z_{n-m-Lk} + p^{L+1} x_{n-m-(L+1)k} \\ &\le p^L z_{n-m-Lk} \quad . \end{aligned}$$

Back to (3.1.2) it becomes

$$\Delta z_n + q_n p^{L\alpha} z_{n-m-Lk}^{\alpha} \le 0 \quad ,$$

for any $L \ge 0$ and where $p > 0$

$$\liminf_{n\to\infty} q_n \exp(-e^{\lambda n}) > 0 \quad \text{iff} \quad \liminf_{n\to\infty} p^{L\alpha} q_n \exp(-e^{\lambda n}) > 0 \ .$$

Then by lemma 1.4.2, we get a contradiction, so $\{x_n\}$ is an oscillatory solution of equation (3.1.2). For other direction of the proof see [10]. □

Remark 3.1.1: In [17] Lin obtained a bounded oscillation for the higher order form of equation (3.1.1) if and only if $\sum_{n=n_0}^{\infty} n^s q_n = \infty$ where s is the order of the NDE.

Theorem 3.1.2: Suppose that $p = -1$, $q_n \geq 0$ and $\sum_{n=N}^{\infty} q_n = \infty$ where N is a positive integer. Suppose further that for G is nondecreasing, $xG(x) > 0$ for $x \neq 0$,
and

$$\int_0^{\pm\varepsilon} \frac{du}{G(u)} < \infty \quad \text{for all } \varepsilon > 0 \ ,$$

then every solution of equation (3.1.1) is oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (3.1.1), and define z_n again as in theorem 3.1.1, from equation (3.1.1) we have

$$\Delta z_n = -q_n G(x_{n-k}) \leq 0, \quad n \geq 0$$

(i.e. $\Delta z_n \leq 0$ eventually). Since q_n is not identically zero, z_n can not be eventually identically zero. Thus either z_n is eventually negative or eventually positive.

If $z_n < 0$ eventually, then

$$z_n \leq z_N \quad \text{for} \quad n \geq N \ ,$$

now

$$\begin{aligned}
z_n &= x_n + px_{n-m} \ , \qquad p = -1 \\
x_n &= z_n + x_{n-m} \\
x_{N+mn} &= z_{N+mn} + x_{N+mn-m} \\
&\leq z_N + x_{N+(n-1)m} \\
&\leq z_N + z_{N+(n-1)m} + x_{N-(n-2)m} \\
&\ \ \vdots \\
&\leq nz_N + x_N
\end{aligned}$$

(i.e. $x_{N+mn} \leq nz_N + x_N$, $n \geq N$)

As $n \to \infty$ then $x_{N+mn} < 0$ eventually, which is a contradiction, so z_n must be eventually positive. ($z_n \geq 0$ eventually).

Now

$$\Delta z_n + q_n G(z_{n-k}) \leq 0 \ ,$$

dividing by $G(z_{n-k})$ we get

$$\frac{\Delta z_n}{G(z_{n-k})} + q_n \leq 0 \ , \quad n \geq N \tag{3.1.5}$$

but we have that

$$G(t) \leq G(z_{n-k}) \ , \quad t \in [z_{n+1}, z_{n-k}] \ ,$$

this implies

$$-\int_{z_{n+1}}^{z_n} \frac{dt}{G(t)} \le \frac{\Delta z_n}{G(z_{n-k})} \quad , \tag{3.1.6}$$

using (3.1.5) and (3.1.6) we get

$$-\int_{z_{n+1}}^{z_n} \frac{dt}{G(t)} + q_n \le 0$$

$$-\int_{z_{n+1}}^{z_N} \frac{dt}{G(t)} + \sum_{i=N}^{n} q_i \le 0 \quad .$$

Which is a contradiction, so $\{x_n\}$ is an oscillatory solution of equation (3.1.1). □

Remark 3.1.2: The authors in [21] studied the Linearized oscillation of the first-order nonlinear NDE of the form

$$\Delta(x_n - p x_{n-m}) + \sum_{i=1}^{L} q_i G_i(x_{n-k_i}) = 0 \, , \quad n \ge n_0 \tag{3.1.7}$$

where , $p \in (-1,0]$, $G_i \in C(\mathbf{R},\mathbf{R})$, $q_i \in (0,\infty)$, $m, k_i \in \{0,1,2,...\}$ and $i = 1,...,L$

The theorems show that the oscillation of equation (3.1.7) is related to that of the linear case of equation (3.1.7). i.e. the equation of the form

$$\Delta(x_n - p x_{n-m}) + \sum_{i=1}^{L} q_i x_{n-k_i} = 0 \tag{3.1.8}$$

Remark 3.1.3: The results included in this section all mainly referred to [12], [21] and [29].

3.2 Oscillation criteria of first-order nonlinear NDE's with variable coefficients

Consider the first-order nonlinear NDE's of the form

$$\Delta(x_n + \delta_1 p_n x_{n-m}) + \delta_2 q_n G(x_{n-k}) = 0 \quad , \quad n \ge 0 \tag{3.2.1}$$

$\{p_n\}$ and $\{q_n\}$ are sequences of real numbers, k and m are integers and $\delta_1 = \delta_2 = 1$.

In this section we discussed theorems which give sufficient conditions for the oscillation of all bounded solutions of equation (3.2.1) when $q_n \ge 0$, $m > 0$ and $k \ge 0$ in theorem 3.2.3 and when $p_n > 0$, $q_n > 0$, $m > 0$ and k is any integer in theorem 3.2.4.

Theorem 3.2.1: Suppose that $-1 < b_1 \le p_n \le 0$, where b_1 is a constant.
Let

$$\liminf_{|u| \to 0} \frac{G(u)}{u} > \gamma > 0 \, , \tag{3.2.2}$$

and

$$\liminf_{n \to \infty} \sum_{i=n-k}^{n-1} q_i > \frac{1}{\gamma} \frac{k^{k+1}}{(k+1)^{k+1}} \quad , \tag{3.2.3}$$

then every solution of the equation

$$\Delta(x_n + p_n x_{n-m}) + q_n G_n(x_{n-k}) = 0 \ , \tag{3.2.4}$$

where, $\{p_n\}$ and $\{q_n\}$ are sequences of real numbers with $q_n \geq 0$, $G \in C(\mathbf{R},\mathbf{R})$ is nondecreasing and $uG(u) > 0$ for $u \neq 0$ and $m > 0$, $k \geq 0$ are integers, also the value of γ is limited by inequality (3.2.3).

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (3.2.4) ($x_n > 0$ for $n \geq N > 0$), let $r = \max\{m,k\}$ and define

$$z_n = x_n + p_n x_{n-m} \ , \tag{3.2.5}$$

then $\Delta z_n \leq 0$ for $n \geq N + r$. Then either

$$z_n > 0 \quad \text{or} \quad z_n < 0 \quad \text{for} \quad n \geq N_1 > N + r$$

Let $z_n > 0$ for $n \geq N_1$. If $\lim\limits_{n\to\infty} z_n = \alpha$, then $0 \leq \alpha < \infty$. Suppose that $0 < \alpha < \infty$, then $z_n > \beta > 0$ for $n \geq N_2 > N_1$. As $z_n < x_n$ for $n \geq N_1$, then for $n \geq N_3 > N_2 + r$

$$G(\beta)\sum_{n=N_3}^{j-1} q_n < \sum_{n=N_3}^{j-1} q_n G(z_{n-k}) \leq \sum_{n=N_3}^{j-1} q_n G(x_{n-k}) = -\sum_{n=N_3}^{j-1} \Delta z_n < z_{N_3} \tag{3.2.6}$$

From (3.2.6) we get a contradiction. Hence $\lim\limits_{n\to\infty} z_n = 0$.

From inequality (3.2.2) we have for $n \geq N_4 > N_1$,

$$G(z_n) > \gamma z_n \ ,$$

So we get from equation (3.2.1) that for $n \geq N_4 + r$

$$\Delta z_n + \gamma q_n z_{n-k} < \Delta z_n + q_n G(z_{n-k}) \leq \Delta z_n + q_n G(x_{n-k}) = 0 \ , \tag{3.2.7}$$

inequality (3.2.7) with lemma 1.4.3 leads to a contradiction. Hence $z_n < 0$ for $n \geq N_1$

Thus $x_n < x_{n-m}$ for $n \geq N_1$ consequently, $\{z_n\}$ is bounded, since $\lim\limits_{n\to\infty} z_n$ exists, then $z_n < \alpha < 0$ for $n \geq N_2 > N_1$.

Hence for $n \geq N_3 > N_2 + r$,

$$z_n = x_n + p_n x_{n-m} > b_1 x_{n-m} \ ,$$

that is

$$x_{n-k} > \left(\frac{1}{b_1}\right) z_{n+m-k} > \left(\frac{\alpha}{b_1}\right) \ ,$$

from equation (3.2.1) we obtain

$$G\left(\frac{\alpha}{b_1}\right)\sum_{n=N_3}^{j-1} q_n < \sum_{n=N_3}^{j-1} q_n G(x_{n-k}) = -\sum_{n=N_3}^{j-1} \Delta z_n < -z_j$$

Thus $\sum\limits_{n=N_3}^{\infty} q_n < \infty$, which is a contradiction, that completes the proof. □

In fact we will make use of theorem 3.2.1 to determine the sufficient conditions for which the nonlinear NDE

$$\Delta(x_n + p_n x_{n-m}) + q_n |x_{n-k}|^{\alpha} \operatorname{sgn}(x_{n-k}) = 0 \ , \tag{3.2.8}$$

oscillates.

Remark 3.2.1: Suppose that $-1 < b_1 \le p_n \le 0$, where b_1 is a constant, and the relations (3.2.2) and (3.2.3) hold. Then every solution of equation (3.2.8) is oscillatory, provided that $0 < \alpha \le 1$.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (3.2.7) for $n \ge N > 0$, let z_n as in (3.2.5) and $r = \max\{m, k\}$
then $\Delta z_n \le 0$, for $n \ge N + r$, two possible choices for z_n arise:
If $z_n < 0$ for $n \ge N_1$, then $x_n < x_{n-m}$ for $n \ge N_1$, but $\lim_{n\to\infty} z_n$ exists, this means that $\{z_n\}$ is bounded , then $z_n < \alpha_1 < 0$, for $n \ge N_2 > N_1$.
Now

$$z_n = x_n + p_n x_{n-m} > b_1 x_{n-m} \ ,$$

so

$$z_{n+m-k} > b_1 x_{n-k} \ ,$$

or

$$x_{n-k} > \frac{1}{b_1} z_{n+m-k} > \frac{\alpha_1}{b_1}, \quad n \ge N_3 > N_2 + r \tag{3.2.9}$$

Using equation (3.2.8) and taking the summation for (3.2.9) we have for $0 < \alpha \le 1$

$$\left|\frac{\alpha_1}{b_1}\right|^{\alpha} \operatorname{sgn}\left(\frac{\alpha_1}{b_1}\right) \sum_{n=N_3}^{j-1} q_n < \sum_{n=N_3}^{j-1} q_n |x_{n-k}|^{\alpha} \operatorname{sgn}(x_{n-k})$$

$$= -\sum_{n=N_3}^{j-1} \Delta z_n < -z_j \ ,$$

let $j \to \infty$, then $\sum_{n=N_3}^{\infty} q_n < \infty$, which contradicts inequality (3.2.3).

If $z_n \ge 0$ for $n \ge N_1$, then,
if $\lim_{n\to\infty} z_n = \alpha_2$ then $0 \le \alpha_2 < \infty$
Suppose to contrary that $0 < \alpha_2 < \infty$, then $z_n > \beta > 0$ for $n \ge N_2 > N_1$, similarity using (3.2.8) and taking the summation from N_3 to $(j-1)$,we get for $0 < \alpha \le 1$

$$|\beta|^{\alpha} \operatorname{sgn}(\beta) \sum_{n=N_3}^{j-1} q_n < \sum_{n=N_3}^{j-1} q_n |z_{n-k}|^{\alpha} \operatorname{sgn}(z_{n-k})$$

$$\le \sum_{n=N_3}^{j-1} q_n |x_{n-k}|^{\alpha} \operatorname{sgn}(x_{n-k})$$

$$= -\sum_{n=N_3}^{j-1} \Delta z_n < z_{N_3} \ , \quad n \ge N_3 > N_2 + r$$

which also a contradiction, hence

$$\lim_{n\to\infty} z_n = 0$$

now, from (3.2.2) we have

$$|z_n|^{\alpha} \operatorname{sgn}(z_n) > \gamma z_n \ , \quad \text{for} \quad n \ge N_4 > N_1$$

so

$$\Delta z_n + \gamma q_n z_{n-k} < \Delta z_n + q_n |z_{n-k}|^{\alpha} \operatorname{sgn}(z_{n-k})$$

$$\le \Delta z_n + q_n |x_{n-k}|^{\alpha} \operatorname{sgn}(x_{n-k}) = 0, \quad 0 < \alpha \le 1$$

which contradicts lemma 1.4.3, so every solution of equation (3.2.8) is oscillatory. □

Remark 3.2.2: When $\alpha > 1$ we can apply theorem 3.2.2 fails to hold.

Remark 3.2.3: We will see later (in section 3.5) some oscillation theorems for equation (3.2.8) when $p_n = -1$, for $\alpha > 0$ which includes the sublinear, the linear and the super linear cases.

Remark 3.2.4: In theorem 3.2.1 if we replace the condition $-1 < b_1 \le p_n \le 0$ by $-\infty < a_1 \le p_n \le -1$ where b_1, a_1 are constants then the conclusion we get, that every solution of equation (3.2.4) oscillates or tends to $\pm\infty$ as $n \to \infty$.

Theorem 3.2.2: Let $0 \le p_n \le 1$ and there exist $\lambda, \mu > 0$ such that for $v, u > 0$

$$G(u+v) \le \lambda[G(u) + G(v)] \quad , \tag{3.2.10}$$

and for $v, u < 0$

$$G(u+v) \ge \mu[G(u) + G(v)] \quad , \tag{3.2.11}$$

and suppose that G satisfies the condition

$$\int_0^{\pm\varepsilon} \frac{du}{G(u)} < \infty \quad , \quad \text{for all } \varepsilon > 0 \tag{3.2.12}$$

If $m \le k$ and $\sum_{n=k}^{\infty} q_n^* = \infty$, then every solution of equation (3.2.4) oscillates, where $q_n^* = \min\{q_n, q_{n-m}\}$, $n \ge m$.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (3.2.4), $x_n > 0$ for $n \ge N$. Setting $r = \max\{m, k\}$ and z_n as in (3.2.5) we get that $z_n > 0$ and $\Delta z_n \le 0$ for $n \ge N + r$. Hence $\lim_{n\to\infty} z_n$ exists.

Using (3.2.10), there exists $\lambda > 0$ such that

$$G(x_{n-k} + x_{n-m-k}) \le \lambda[G(x_{n-k}) + G(x_{n-m-k})], \qquad n \ge N_1 > N + 2r$$

but $z_n \le x_n + x_{n-m}$ for $n \ge N_1$, then from equation (3.2.4) we get:

$$\begin{aligned}
0 &= \lambda\Delta z_n + \lambda q_n G(x_{n-k}) + \lambda\Delta z_{n-m} + \lambda q_{n-m} G(x_{n-m-k}) \\
&\ge \lambda\Delta z_n + \lambda\Delta z_{n-m} + \lambda q_n^*[G(x_{n-k}) + G(x_{n-m-k})] \\
&\ge \lambda\Delta z_n + \lambda\Delta z_{n-m} + q_n^* G(x_{n-k} + x_{n-m-k}) \\
&\ge \lambda\Delta z_n + \lambda\Delta z_{n-m} + q_n^* G(z_{n-k})
\end{aligned}$$

Dividing by $G(z_{n-k})$ we get

$$q_n^* + \frac{\lambda\Delta z_n}{G(z_{n-k})} + \frac{\lambda\Delta z_{n-m}}{G(z_{n-k})} \le 0 \;, \tag{3.2.13}$$

also

$$q_n^* + \frac{\lambda\Delta z_n}{G(z_n)} + \frac{\lambda\Delta z_{n-m}}{G(z_{n-k})} \le 0 \;, \quad n \ge N_1 \tag{3.2.14}$$

for $z_{n+1} < x < z_n$ and $z_{n+1-m} < y < z_{n-m}$, we have

$$\int_{z_{n+1}}^{z_n} \frac{dx}{G(x)} \geq \int_{z_{n+1}}^{z_n} \frac{dx}{G(z_n)} = -\frac{\Delta z_n}{G(z_n)} \quad ,$$

and

$$\int_{z_{n+1-m}}^{z_{n-m}} \frac{dy}{G(y)} \geq \int_{z_{n+1-m}}^{z_{n-m}} \frac{dy}{G(z_{n-m})} = -\frac{\Delta z_{n-m}}{G(z_{n-m})}$$

From equation (3.2.14) we get

$$\sum_{n=N_1}^{j-1} q_n^* \leq -\sum_{n=N_1}^{j-1} \int_{z_n}^{z_{n+1}} \frac{dx}{G(x)} - \sum_{n=N_1}^{j-1} \int_{z_{n-m}}^{z_{n+1-m}} \frac{dy}{G(y)}$$

$$= -\int_{z_{N_1}}^{z_j} \frac{dx}{G(x)} - \int_{z_{N_1-m}}^{z_{j-m}} \frac{dy}{G(y)}$$

Hence, $\sum_{n=N_1}^{\infty} q_n^* < \infty$, which is a contradiction. □

Remark 3.2.5: In theorem 3.2.3 if we take the sequence $\{p_n\}$ to be $-1 \leq p_n \leq a_2 < \infty$ where a_2 is a constant and conditions (3.2.10) and (3.2.12) hold then every solution of equation (3.2.4) oscillates provided that $\sum_{n=k}^{\infty} Q_n = \infty$, $Q_n = \min\{q_n, \frac{q_{n-m}}{G(a_2)}\}$, and for $u, v \in \mathbf{R}$ we have $G(uv) = G(u)G(v)$

The following theorem will prove the oscillatory behavior of equation (3.2.4) without the asymptotic condition on $\{q_n\}$, here the oscillation is related by a non-neutral difference inequality. But before that we need the following conditions:

(C1) $G \in C(\mathbf{R}, \mathbf{R})$ is nondecreasing and $uG(u) > 0$ for $u \neq 0$.

(C2) There exists a continuous function $\varphi : \mathbf{R} \to \mathbf{R}$ such that $\varphi(u)$ is nondecreasing in $u \in \mathbf{R}$, $u\varphi(u) > 0$ for $u \neq 0$ and

$$|\varphi(u+v)| \leq |G(u) + G(v)| \ , \quad \text{for } uv > 0$$

(C3) There exists a continuous function $w : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$|G(uv)| \leq w(u)\ |G(v)| \ , \quad \text{for } u > 0 \text{ and } v \in \mathbf{R}$$

Theorem 3.2.3: Suppose that $p_n > 0$ and $q_n > 0$ real sequences, $m > 0$, k is any integer, suppose further that (C1)-(C3) hold. Then every solution of equation (3.2.4) is oscillatory if there exists a positive real sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ for $n \geq n_0$, and the difference inequality

$$\{\Delta y_n + Q_n \varphi(y_{n+m-k})\} \operatorname{sgn} y_n \leq 0 \ ,$$

does not have any nonoscillatory solution, where

$$Q_n = \min\left\{\lambda_n q_n, \frac{[1-\lambda_{n-m}]q_{n-m}}{w(p_{n-k})}\right\}$$

Proof: Suppose that $\{x_n\}$ is an eventually positive solution of equation (3.2.4). letting $z_n = x_n + p_n x_{n-m}$. Then $z_n > 0$ and z_n is decreasing for all large $n \geq n_0$. Summing equation (3.2.4) from n to ∞, we have

$$z_n = \lim_{n\to\infty} z_n + \sum_{s=n}^{\infty} q_s G(x_{s-k}) \geq \sum_{s=n}^{\infty} q_s G(x_{s-k}) \ ,$$

for all large $n \geq n_0$, we see that

$$\begin{aligned} z_n &\geq \sum_{s=n}^{\infty} \lambda_s q_s G(x_{s-k}) + \sum_{s=n}^{\infty} (1-\lambda_s) q_s G(x_{s-k}) \\ &= \sum_{s=n}^{\infty} \lambda_s q_s G(x_{s-k}) + \sum_{s=n+m}^{\infty} (1-\lambda_{s-m}) q_{s-m} G(x_{s-k-m}) \\ &\geq \sum_{s=n+m}^{\infty} Q_s G(x_{s-k}) + \sum_{s=n+m}^{\infty} Q_s w(p_{s-k}) G(x_{s-k-m}) \\ &\geq \sum_{s=n+m}^{\infty} Q_s G(x_{s-k}) + \sum_{s=n+m}^{\infty} Q_s G(p_{s-k} x_{s-k-m}) \\ &= \sum_{s=n+m}^{\infty} Q_s \left[G(x_{s-k}) + G(p_{s-k} x_{s-k-m}) \right] \\ &\geq \sum_{s=n+m}^{\infty} Q_s \varphi(x_{s-k} + p_{s-k} x_{s-k-m}) \\ &= \sum_{s=n+m}^{\infty} Q_s \varphi(z_{s-k}) \ , \ \text{for all } n \geq n_0 \ , \end{aligned}$$

now let

$$y_n = \sum_{s=n}^{\infty} Q_s \varphi(z_{s-k}) > 0 \ ,$$

then $z_n \geq y_{n+m}$ eventually. But

$$\Delta y_n = -Q_n \varphi(z_{n-k}) \leq -Q_n \varphi(y_{n-k+m}) \, , \ \text{for all} \quad n \geq n_0$$

this means that the last inequality has an eventually positive solution, which is a contradiction, so every solution of equation (3.2.4) is oscillatory. □

Corollary 3.2.1: Every solution of the equation

$$\Delta(x_n + p_n x_{n-m}) + q_n x_{n-k}^{\alpha} = 0 \ , \tag{3.2.15}$$

where α is ratio of odd positive integers , $\{p_n\}$ is a positive real sequence and $m > 0$, $\{q_n\}$ is a positive real sequence and k is any integer,
is oscillatory if the difference inequality

$$\Delta y_n + Q_n y_{n-k+m}^{\alpha} \leq 0 \ ,$$

does not have any eventually positive solution, where

$$Q_n = \min\{1, 2^{1-\alpha}\} \min\left\{ \frac{q_n}{1 + p_{n-k+m}^{\alpha}}, \frac{q_{n-m}}{1 + p_{n-k}^{\alpha}} \right\}$$

Proof: Suppose that $\{x_n\}$ is an eventually positive solution of equation (3.2.15). Let $z_n = x_n + p_n x_{n-m}$, it is clear that $z_n > 0$ and z_n is decreasing for all large $n \geq n_0$. Summing equation (3.2.15) from n to ∞, to get

$$z_n = \lim_{n\to\infty} z_n + \sum_{s=n}^{\infty} q_s x_{s-k}^{\alpha}$$

$$\geq \sum_{s=n}^{\infty} q_s x_{s-k}^{\alpha} \ , \text{ for all large } n \geq n_0 .$$

But we have

$$z_n \geq \sum_{s=n}^{\infty} \frac{q_s}{1+p_{s-k+m}^{\alpha}} x_{s-k}^{\alpha} + \sum_{s=n}^{\infty} \frac{p_{s-k+m}^{\alpha}}{1+p_{s-k+m}^{\alpha}} q_s x_{s-k}^{\alpha}$$

$$= \sum_{s=n}^{\infty} \frac{q_s}{1+p_{s-k+m}^{\alpha}} x_{s-k}^{\alpha} + \sum_{s=n+m}^{\infty} \frac{p_{s-k}^{\alpha}}{1+p_{s-k}^{\alpha}} q_{s-m} x_{s-m-k}^{\alpha}$$

$$\geq \sum_{s=n+m}^{\infty} Q_s x_{s-k}^{\alpha} + \sum_{s=n+m}^{\infty} Q_s p_{s-k}^{\alpha} x_{s-m-k}^{\alpha}$$

$$\geq \sum_{s=n+m}^{\infty} Q_s x_{s-k}^{\alpha} + \sum_{s=n+m}^{\infty} Q_s \left(p_{s-k} x_{s-m-k}\right)^{\alpha}$$

$$= \sum_{s=n+m}^{\infty} Q_s \left(x_{s-k}^{\alpha} + \left(p_{s-k} x_{s-m-k}\right)^{\alpha}\right)$$

$$\geq \sum_{s=n+m}^{\infty} Q_s \varphi\left(x_{s-k} + p_{s-k} x_{s-m-k}\right)$$

$$= \sum_{s=n+m}^{\infty} Q_s \varphi(z_{s-k}) \ , \text{ for all } n \geq n_0 \ ,$$

let

$$y_n = \sum_{s=n}^{\infty} Q_s \varphi(z_{s-k}) > 0 \ ,$$

hence, $z_n \geq y_{n+m}$ eventually. We see that

$$\Delta y_n = -Q_n \varphi(z_{n-k}) \leq -Q_n \varphi(y_{n-k+m}) ,$$

i.e. $\Delta y_n + Q_n y_{n-k+m}^{\alpha} \leq 0$, for all $n \geq n_0$

so y_n is an eventually positive solution of the above inequality which is a contradiction. □

Note: in the last corollary it is clear that

$$\lambda_n = \frac{1}{1+p_{n-k+m}^{\alpha}} \ , \ w(u) = u^{\alpha}$$

$$\varphi(u) = u^{\alpha}$$

Example 3.2.1: Consider the NDE

$$\Delta\left(x_n + \frac{1}{8}\left((-1)^{n+1} - 1\right)x_{n-1} \right) + \frac{5}{3} x_{n-2}^{\frac{1}{3}} = 0 \ , \quad n \geq 0 \qquad (3.2.16)$$

using theorem 3.2.1, we have

$$\sum_{i=n-2}^{n-1} q_i = \frac{10}{3} > \frac{1}{\gamma}\left(\frac{8}{27}\right) \text{ if and only if } \gamma > \frac{8}{90} .$$

As a consequence, every solution of the equation (3.2.16) is oscillatory.

Example 3.2.2: Consider the NDE

$$\Delta\left(x_n + \frac{1}{3}\left(1 + (-1)^n\right)x_{n-1} \right) + \frac{4}{3} x_{n-2}^{\frac{1}{3}} = 0 \ , \quad n \geq 0 \qquad (3.2.17)$$

Applying theorem (3.2.3), it is clear that the sequence $\{p_n\}$ satisfies $0 \le p_n \le 1$ since $0 \le \frac{1}{3}\left(1+(-1)^n\right) \le 1$.

To check conditions (3.2.10) and (3.2.11)

$$(u+v)^{\frac{1}{3}} \le \lambda\left(u^{\frac{1}{3}} + v^{\frac{1}{3}}\right) \text{ for } u,v > 0 \text{ and } \lambda > 0$$

$$(u+v)^{\frac{1}{3}} \ge \mu\left(u^{\frac{1}{3}} + v^{\frac{1}{3}}\right) \text{ for } u,v < 0 \text{ and } \mu > 0$$

also

$$\int_0^{\pm\sigma} \frac{du}{u^{\frac{1}{3}}} = \frac{3}{2}\left[u^{\frac{2}{3}}\right]_0^{\pm\sigma} < \infty \text{ , for all } \sigma > 0$$

and finally,

$$\sum_{n=k}^{\infty} q_n^* = \sum_{n=-2}^{\infty} \frac{4}{3} = \infty \quad \text{is satisfied.}$$

Then every solution of equation (3.2.17) is oscillatory. In particular $x_n = (-1)^{3n}$ is an oscillatory solution of equation (3.2.17).

Remark 3.2.6: The results of this section are mainly referred to [20] and [23].

3.3 Oscillation criteria of first-order forced nonlinear NDE's with both constant and variable coefficients

Consider the first-order forced nonlinear NDE of the form

$$\Delta(x_n + \delta_1\, p x_{n-m}) + \delta_2\, q_n G(x_{k_n}) = f_n \ , \qquad (3.3.1)$$

where, $\delta_1 = +1$, $\delta_2 = -1$, p is nonnegative real number, $\{k_n\}$ is a sequence of nonnegative integers with $\lim_{n\to\infty} k_n = \infty$, $\{f_n\}$ and $\{q_n\}$ are also sequences of real numbers with $q_n \ge 0$ eventually, the function G is a real valued function satisfying $xG(x) > 0$ for $x \ne 0$ and $m \in \mathbf{N}(1)$.

In this section we discuss theorems that establish sufficient conditions of the oscillation of equation (3.3.1) under certain assumptions on the forcing term.

Theorem 3.3.1: Suppose that there exists a sequence $\{h_n\}$ of real numbers such that

$$\Delta h_n = f_n \quad \text{and } \{h_n\} \text{ is oscillatory} \ , \qquad (3.3.2)$$

also $\{h_n\}$ satisfies

$$\limsup_{n\to\infty} h_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} h_n = -\infty \ , \qquad (3.3.3)$$

then every bounded solution of equation (3.3.1) is oscillatory.

Proof: Suppose to contrary that $\{x_n\}$ is an eventually positive bounded solution of (3.3.1), so there exists a positive integer n_0 such that $x_n > 0$, $x_{n-m} > 0$ and $x_{k_n} > 0$ for $n \ge n_0$.

Define

$$y_n = x_n + p x_{n-m} \quad \text{and} \quad z_n = y_n - h_n \quad , \tag{3.3.4}$$

equation (3.3.1) becomes

$$\Delta y_n - q_n G\left(x_{k_n}\right) = f_n ,$$

hence

$$\Delta z_n = q_n G\left(x_{k_n}\right) \geq 0 \text{ for } n \geq n_0 \quad ,$$

also $z_n > 0$ for $n \geq n_0$ because if not, then $z_n \leq 0$ for $n \geq n_1 \geq n_0$ and $y_n - h_n \leq 0 \Rightarrow$ $0 < y_n \leq h_n$ for $n \geq n_1$, which is a contradiction because $\{h_n\}$ is oscillatory.

Thus $z_n > 0$ for $n \geq n_0$ holds and

$$\Delta\left(y_n - h_n\right) \geq 0 \quad \text{and} \quad y_n - h_n \geq 0 \text{ for } n \geq n_0 \quad ,$$

then

$$\lim_{n \to \infty}\left(y_n - h_n\right) = \beta \text{ , such that } \beta \in [0, \infty)$$

but

$$\limsup_{n \to \infty} h_n = \infty \quad \text{and} \quad \liminf_{n \to \infty} h_n = -\infty$$

So there exists a sequence $\{n_L\}$ such that $\lim_{L \to \infty} h_{n_L} = \infty$, but we have $\lim_{L \to \infty}\left(y_{n_L} - h_{n_L}\right) = \beta$, then $\{y_{n_L}\}$ is unbounded, which is a contradiction, then every bounded solution of (3.3.1) is oscillatory. □

Theorem 3.3.2: Assume that $q_n \leq 0$ eventually, then every solution of equation (3.3.1) is oscillatory provided that the conditions (3.3.2) and (3.3.3) hold.

Proof: Suppose that $\{x_n\}$ is an eventually positive solution of equation (3.3.1), also we choose n_0 such that $x_n > 0$, $x_{n-m} > 0$ and $x_{k_n} > 0$ for $n \geq n_0 > 0$.

With y_n and z_n as in theorem (3.3.1) we have

$$\Delta z_n = q_n G\left(x_{k_n}\right) \leq 0 \text{ , for } n \geq n_0$$

again with $z_n > 0$ for $n \geq n_0$, so

$$\Delta\left(y_n - h_n\right) \leq 0 \quad \text{and} \quad y_n - h_n > 0 \text{ for } n \geq n_0 \quad ,$$

then $\lim_{n \to \infty}\left(y_n - h_n\right) = \beta^*$, but from (3.3.3) it follows that there exists a sequence $\{n_L\}$ such that

$$\lim_{L \to \infty} h_{n_L} = -\infty \quad ,$$

but we have

$$\lim_{L \to \infty}\left(y_{n_L} - h_{n_L}\right) = \beta^* \quad ,$$

then the sequence $\{y_{n_L}\}$ can not be positive, which is a contradiction, then every solution of (3.3.1) must be oscillatory. □

Theorem 3.3.3: Suppose that equation (3.3.1) is superlinear. Suppose further that the sequence $\{h_n\}$ is periodic of period ρ and satisfies (3.3.2) and (3.3.3). If

$$\sum_{n \in A} q_n = \infty$$

where $A = \{n \in N : k_n > n + 1\}$, then equation (3.3.1) is oscillatory provided that $0 \leq p < 1$

Proof: Suppose that $\{x_n\}$ is an eventually positive solution of equation (3.3.1), again we choose a positive integer n_0 such that x_n , x_{n-m} and x_{k_n} are all positive for $n \ge n_0 > 0$.
We define y_n and z_n as in theorem (3.3.1) ,

$$\Delta y_n - q_n G(x_{k_n}) = f_n \tag{3.3.5}$$

$$\Delta z_n = q_n G(x_{k_n}) \ge 0 \tag{3.3.6}$$

with $z_n > 0$ for $n \ge n_0$, now

$$x_n + p x_{n-m} = z_n + h_n$$

$$\begin{aligned} x_n &= z_n + h_n - p x_{n-m} \\ &= z_n + h_n - p(z_{n-m} + h_{n-m} - p x_{n-2m}) \\ &= z_n + h_n - p z_{n-m} - p h_{n-m} + p^2 (z_{n-2m} + h_{n-2m} - p x_{n-3m}) \end{aligned}$$

We proceed so that we can choose a sufficient large integer n_1 such that

$$x_n \ge (1-p)(z_n + h_n) \quad \text{for } n \ge n_1 \quad ,$$

also there exists an integer $n_2 \ge n_1$ such that

$$x_n \ge (1-p)(z_n + h_{n_2}) = \xi_n \quad \text{for } n \ge n_2 \quad , \tag{3.3.7}$$

but we have

$$(1-p)(z_{n+1} + h_{n_2}) = \xi_{n+1} \tag{3.3.8}$$

$$(1-p)(z_n + h_{n_2}) = \xi_n \quad . \tag{3.3.9}$$

Subtracting (3.3.9) from (3.3.8) to get

$$\Delta z_n = \frac{1}{1-p} \Delta \xi_n \quad \text{for} \quad n \ge n_2$$

and

$$\xi_n = (1-p)(z_n + h_{n_2})$$

$$\xi_n \ge (1-p)(z_{n_2} + h_{n_2}) > 0 \quad \text{for} \quad n \ge n_2 \quad .$$

It follows, from (3.3.6) and (3.3.7) that

$$\Delta \xi_n \ge (1-p) q_n G(\xi_{k_n}) \quad \text{for } n \ge n_2 \quad , \tag{3.3.10}$$

dividing (3.3.10) by $G(\xi_{n+1})$ and summing from n_2 to n, we get

$$\sum_{i=n_2}^{n} \frac{\Delta \xi_i}{G(\xi_{i+1})} \ge (1-p) \sum_{i \in D} q_i \quad ,$$

where $D = A \cap [n_2, n]$, taking in consideration that $\xi_{n+1} \ge \xi_{k_n}$ on D
and hence

$$\sum_{i \in D} q_i < \sum_{i=n_2}^{n} \frac{\Delta \xi_i}{G(\xi_{i+1})} < \infty$$

Which is a contradiction, and hence every solution of equation (3.3.1) is oscillatory. □

The following theorem is for the case when

$$\frac{G(x)}{x} \ge M \quad \text{for } x \ne 0 \ , \ M \text{ is a positive constant ,} \tag{3.3.11}$$

Theorem 3.3.4: there exists an integer $l \geq 2$ such that $k_n \geq n+l$, $n \in \mathbf{N}(1)$, also $\{h_n\}$ is periodic of period ρ and satisfies condition (3.3.2), then equation (3.3.1) is oscillatory provided that $0 \leq p < 1$, and

$$\lim_{n\to\infty} \inf \sum_{i=n-l+1}^{n-1} q_i > \gamma^* \left(\frac{l}{l+1}\right)^{l+1}$$

where, $\gamma^* = \dfrac{1}{M(1-p)}$.

Proof: Suppose that equation (3.3.1) has an eventually positive solution, then we proceed as in the proof of theorem 3.3.3 to have

$$\Delta z_n = q_n G(x_{k_n}) \;, \quad \text{using condition (3.3.11) for } n \geq n_0$$
$$\geq M q_n x_{k_n} \;,$$

also

$$x_n \geq (1-p)(z_n + h_{n_2}) = \xi_n \quad \text{for } n \geq n_2 \geq n_1$$
$$\Delta \xi_n = (1-p)(\Delta z_n)$$
$$\frac{\Delta \xi_n}{(1-p)} \geq q_n G(x_{k_n})$$
$$\geq M q_n x_{k_n}$$
$$\geq M q_n \xi_{k_n} \geq M q_n \xi_{n+l}$$
$$\Delta \xi_n \geq M(1-p) q_n \xi_{n+l} \;, \quad \text{for} \quad n \geq N^* \text{ (sufficiently large)}$$

in view of lemma 1.4.3 the last inequality has no eventually positive solution which is a contradiction, so every solution of equation (3.3.1) is oscillatory. □

Theorem 3.3.5: Suppose that $\{h_n\}$ is periodic of period ρ and satisfies condition (3.3.2), suppose that $G'(x) \geq 0$ for $x \neq 0$. If

$$\sum_{n=n_0} q_n = \infty \quad , \quad n_0 \geq 0$$

Then every bounded solution of equation (3.3.1) is oscillatory provided that $0 \leq p < 1$.

Proof: Suppose that $\{x_n\}$ is an eventually bounded positive solution, and we proceed as in the proof of theorem 3.3.3 until we reach to

$$\Delta \xi_n > (1-p) q_n G(\xi_{k_n}) \;, \quad n \geq n_2$$

but we know that $\{\xi_n\}$ is bounded and increasing sequence, so there exists $c_1, c_2 > 0$ such that

$$0 < c_1 \leq \xi_n \leq c_2 \;, \qquad \text{for } n \geq n_3 \geq n_2$$

then

$$\Delta \xi_n \geq (1-p) q_n G(c_1) \;, \quad \text{for} \quad n \geq n_3$$

summing both sides of the above inequality, we get

$$\sum_{i=n_3}^{n} \Delta \xi_i \geq (1-p) G(c_1) \sum_{i \in D} q_i \quad ,$$

then

$$\sum_{i\in D} q_i < \sum_{i=n_3}^{n} \Delta\xi_i = \xi_{n+1} - \xi_{n_3} \quad ,$$

where $D = A \bigcap [n_3, n]$, $A = \{n \in N : k_n > n+1\}$

which is a contradiction, so every bounded solution of equation (3.3.1) is oscillatory. □

Example 3.3.1: Consider the NDE

$$\Delta(x_n + x_{n-2}) + (1+2n)|x_{n-1}|^{5n} \operatorname{sgn} x_{n-1} = (-1)^{n+1}(2n+3) \ , \quad n \in N \tag{3.3.12}$$

it is clear that $h_n = (-1)^n (n+1)$, and $\Delta h_n = (-1)^{n+1}(2n+3) = F_n$, also $\{h_n\}$ is oscillatory and satisfies that

$$\lim_{n\to\infty} \sup(-1)^n (n+1) = \infty \, , \quad \text{and} \quad \lim_{n\to\infty} \inf (-1)^n (n+1) = -\infty \ ,$$

then by theorem 3.3.1 every bounded solution of equation (3.3.12) is oscillatory.

Example 3.3.2: Consider the NDE

$$\Delta\left(x_n + \frac{1}{2} x_{n-2} \right) - \frac{9(2n+7)}{n+9} |x_{n-1}|^3 = 4(-1)^{n+2} \ , \quad n \in N \tag{3.3.13}$$

taking $h_n = 2(-1)^{n+1}$ an oscillatory periodic sequence of period 2, applying theorem 3.3.3 to equation (3.3.13), it is clear that

$$\sum_{n\in D} \frac{9(2n+7)}{n+9} = \infty \ ,$$

so every solution of equation (3.3.13) is oscillatory.

Remark 3.3.1: Theorem 3.3.1 and theorem 3.3.2 are also true for the equation

$$\Delta(x_n + p x_{n-m}) + q_n G(x_{k_n}) = f_n \tag{3.3.14}$$

(i.e. $\delta_2 = +1$), but the same thing is not true with the remaining theorems in this section.

Remark 3.3.2: The results of this section are mainly referred to [10].

3.4 Oscillation criteria of first-order forced nonlinear NDE's with variable coefficients

Consider the first-order NDE of the form

$$\Delta(x_n + \delta_1 p_n x_{n-m}) + \delta_2 q_n G(x_{n-k}) = f_n \ , \qquad n \geq 0 \tag{3.4.1}$$

where $\{p_n\}$, $\{q_n\}$ and $\{f_n\}$ are sequences of real numbers with $q_n \geq 0$, $G \in C(\mathbf{R}, \mathbf{R})$ is nondecreasing and $xG(x) > 0$ for $x \neq 0$, $m > 0$ and $k \geq 0$ are integers and $\delta_1 = \delta_2 = 1$.

In fact this section takes its importance from its main equation (equation (3.4.1)), since we can extract many special cases from this equation, for instance, the linear homogenous case, the linear forced case and the nonlinear homogenous case. However, sometimes the results of this section are applicable to some special cases of equation (3.4.1) with some modifications.

Two approaches are used here to prove the oscillation of equation (3.4.1) by operators and by sufficient conditions while the characteristic equation here does not work.

Theorem 3.4.1: Let $-1 < b_1 \le p_n \le 0$, $G(x)$ is nondecreasing and

$$\left|\sum_{n=0}^{\infty} f_n\right| < \infty \quad , \tag{3.4.2}$$

every solution of equation (3.4.1) oscillates or tends to zero as $n \to \infty$ if and only if

$$\sum_{n=0}^{\infty} q_n = \infty \tag{3.4.3}$$

Proof: Suppose that $\{x_n\}$ is an eventually negative solution of equation (3.4.1), then $x_n < 0$ for $n \ge N_1$, we want to show that $\lim_{n\to\infty} x_n = 0$. Let

$$z_n = x_n + p_n x_{n-m} \quad , \tag{3.4.4}$$

and

$$w_n = z_n - \sum_{i=0}^{n-1} f_i \quad , \quad \text{for} \quad n \ge N_1 + m \quad , \tag{3.4.5}$$

we get

$$\Delta w_n = -q_n G(x_{n-k}) \ge 0 \;, \; \text{for} \;\; n \ge N_1 + m + k \quad , \tag{3.4.6}$$

either $w_n > 0$ or $w_n < 0$, let $w_n > 0$ for $n \ge N_2$, where $N_2 > N_1 + m + k$.

Claim: $\{x_n\}$ is bounded. If not, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$x_{n_j} \to -\infty \;\; \text{as} \;\; n_j \to \infty \;\; \text{and} \;\; j \to \infty$$

and

$$x_{n_j} = \min\{x_n : N_2 \le n \le n_j\} \quad ,$$

We may choose n_j sufficiently large so that $n_j - m > N_2$, and hence

$$\begin{aligned} w_{n_j} &= x_{n_j} + p_{n_j} x_{n_j - m} - \sum_{i=0}^{n_j - 1} f_i \\ &\le \left(1 + p_{n_j}\right) x_{n_j} - \sum_{i=0}^{n_j-1} f_i \le \left(1 + b_1\right) x_{n_j} - \sum_{i=0}^{n_j-1} f_i < 0 \end{aligned} \tag{3.4.7}$$

Thus $w_{n_j} < 0$ for large n_j, which is a contradiction, so $\{x_n\}$ is bounded, as a consequence $\{w_n\}$ is bounded and so $\lim_{n\to\infty} w_n$ exists.

If

$$\lim_{n\to\infty} \sup x_n = \alpha \,, \quad -\infty < \alpha < 0 \quad ,$$

then there exists $\beta < 0$ such that

$$x_n < \beta \;, \; \text{for} \;\; n \ge N_3 > N_2$$

but by (3.4.6) we get

$$\sum_{n=N_3+k}^{r-1} q_n G\left(x_{n-k}\right) = -\sum_{n=N_3+k}^{r-1} \Delta w_n = -w_r + w_{N_3+k} \ge -w_r \quad ,$$

hence

$$\sum_{n=N_3+k}^{r-1} q_n G\left(x_{n-k}\right) > -w_r \;, \quad \text{as} \quad r \to \infty$$

$$\sum_{n=N_3+k}^{\infty} q_n G(x_{n-k}) > -\infty \quad . \tag{3.4.8}$$

However, using equation (3.4.3) we get

$$\sum_{n=N_3+k}^{\infty} q_n G(x_{n-k}) < G(\beta) \sum_{n=N_3+k}^{\infty} q_n = -\infty \tag{3.4.9}$$

We get a contradiction, hence $\limsup_{n\to\infty} x_n = 0$.

As $\lim_{n\to\infty} z_n$ exists, using lemma 1.4.4 we get that $\lim_{n\to\infty} z_n = 0$.

Similarly, when $w_n < 0$ we have $\limsup_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} z_n = 0$.

We know that $z_n \le x_n + b_1 x_{n-m}$ for $n \ge N_2$, we infer that

$$\begin{aligned} 0 = \liminf_{n\to\infty} z_n &\le \liminf_{n\to\infty} [x_n + b_1 x_{n-m}] \\ &\le \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} (b_1 x_{n-m}) \\ &= \liminf_{n\to\infty} x_n + b_1 \liminf_{n\to\infty} (x_{n-m}) \\ &= (1+b_1) \liminf_{n\to\infty} x_n \end{aligned}$$

Then $\liminf_{n\to\infty} x_n = 0$. Hence $\lim_{n\to\infty} x_n = 0$, so the sufficiency part of the theorem is proved. Now for the necessity part we assume that

$$\sum_{n=0}^{\infty} q_n < \infty \quad .$$

We want to show that equation (3.4.1) has a positive solution $\{x_n\}$ such that $\liminf_{n\to\infty} x_n > 0$, since we have that $\lim_{n\to\infty} \sum_{i=0}^{n} f_i = 0$, we choose an integer $N > 0$, such that

$$\left| \sum_{n=N}^{\infty} f_n \right| < \frac{1+b_1}{10} \quad , \text{ and} \tag{3.4.10}$$

$$G(1) \sum_{n=N}^{\infty} q_n < \frac{1+b_1}{5} \quad , \tag{3.4.11}$$

let $\mathrm{X} = l_\infty^N$ is a Banach space , let $\mathrm{K} = \{\, x \in \mathrm{X} : x_n \ge 0 \quad \textit{for} \quad n \ge N \,\}$, and $W = \left\{ x \in \mathrm{X} : \frac{1+b_1}{10} \le x_n \le 1,\ n \ge N \right\}$, with X is partially ordered Banach space.

If $x^0 = \{x_n^0\}$, where $x_n^0 = \frac{(1+b_1)}{10}$ for $n \ge N$, then $x^0 = \inf W$ and $x^0 \in W$. Let W^* be a nonempty subset of W, the supremum of W^* is the sequence $x^* = \{x_n^* : n \ge N\}$ and $x_n^* = \sup\{x_n : x = \{x_i : i \ge N\} \in W^*\}$, clearly that $x^* \in W$.

Now, for $x \in W$, we define

$$(Tx)_n = \begin{cases} (Tx)_{N+r} & , N \le n \le N+r \\ -p_n x_{n-m} + \dfrac{1+b_1}{5} + \displaystyle\sum_{i=n}^{\infty} [q_i G(x_{i-k}) - f_i] & , n \ge N+r \end{cases} ,$$

where $r = \max\{m, k\}$.

Using (3.4.10) and (3.4.11), we get

$$(Tx)_n \le -b_1 + \frac{2(1+b_1)}{5} + \frac{1+b_1}{10} < 1 \quad ,$$

and

$$(Tx)_n \ge \frac{1+b_1}{5} - \frac{1+b_1}{10} = \frac{1+b_1}{10}$$

Thus $T:W \to W$ for $x, y \in W$, $x \le y$ implies $T_x \le T_y$. Hence T has a fixed point in W by Knaster-Tarski fixed point theorem (theorem 1.4.2).

If $x = \{x_n\} \in W$ is the fixed point of T, then

$$x_n = \begin{cases} x_{N+r} & , N \le n \le N+r \\ -p_n x_{n-m} + \dfrac{1+b_1}{5} + \displaystyle\sum_{i=n}^{\infty}[q_i G(x_{i-k}) - f_i] & , n \ge N+r \end{cases}$$

Hence, x_n is a positive solution of equation (3.4.1) with $\liminf\limits_{n\to\infty} x_n \ge \dfrac{1+b_1}{10} > 0$. □

Theorem 3.4.2: Let $0 \le p_n \le b_2 < 1$ and let $G(x)$ is nondecreasing and f_n satisfies inequality (3.4.2):

i. If equation (3.4.3) holds, then every solution of equation (3.4.1) oscillates or tends to zero as $n \to \infty$.
ii. Suppose that G satisfies the Lipschitz condition on intervals of the form $[a,b]$, $0 < a < b < \infty$, If every solution of equation (3.4.1) oscillates or tends to zero as $n \to \infty$, then equation (3.4.3) holds.

Proof:

***The proof of part (i)*:** is the same as the proof of the sufficiency condition of theorem (3.4.1)

The proof of part (ii): Suppose that $\displaystyle\sum_{n=0}^{\infty} q_n < \infty$,

choosing $N > 0$ sufficiently large such that

$$\left|\sum_{n=N}^{\infty} f_n\right| < \frac{1-b_2}{10} \quad \text{and} \quad L\sum_{n=N}^{\infty} q_n < \frac{1-b_2}{5} \quad ,$$

where, $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\dfrac{(1-b_2)}{10}, 1\right]$,

let $\mathrm{X} = l_\infty^N$ and

$$S = \left\{x \in \mathrm{X} : \frac{1-b_2}{10} \le x_n \le 1, n \ge N\right\}$$

Clearly that S is a complete metric space, where the metric is induced by the norm on X.

For $x \in S$ define

$$(Tx)_n = \begin{cases} (Tx)_{N+r} & , N \le n \le N+r \\ -p_n x_{n-m} + \dfrac{1+4b_2}{5} + \displaystyle\sum_{i=n}^{\infty}[q_i G(x_{i-k}) - f_i] & , n \ge N+r \end{cases} \quad ,$$

so

$$(Tx)_n < \frac{1+4b_2}{5} + L\sum_{i=N}^{\infty} q_i + \frac{1-b_2}{10}$$

$$< \frac{1+4b_2}{5} + \frac{1-b_2}{5} + \frac{1-b_2}{10} < 1 \ , \quad \text{for} \quad n \geq N$$

and

$$(Tx)_n > -b_2 + \frac{1+4b_2}{5} - \frac{1-b_2}{10} = \frac{1-b_2}{10} \ .$$

This implies that $T : S \to S$. Further, for $u, v \in S$, and $n \geq N + r$,

$$\left|(Tu)_n - (Tv)_n\right| \leq b_2\|u-v\| + \left(\frac{1-b_2}{5}\right)\|u-v\|$$

$$= \mu\|u-v\|$$

which implies that

$$\|Tu - Tv\| \leq \mu\|u-v\| \quad , \text{for } \ \mu > 0$$

where $\mu = \frac{1}{5}(1+4b_2) < 1$.

Thus T is a contraction, hence it has a unique fixed point $x = \{x_n\}$ in S, clearly that $\{x_n\}$ is a positive solution of equation (3.4.1) with $\liminf\limits_{n\to\infty} x_n > 0$. □

Corollary 3.4.1: Let $0 \leq p_n < b_2 < 1$ and let $G(x)$ is nondecreasing and (3.4.2) hold. Suppose that G satisfies the Lipschitz condition on interval of the form $[a,b]$, $0 < a < b < \infty$, Then every solution of equation (3.4.1) oscillates or tends to zero as $n \to \infty$ if and only if equation (3.4.3) holds.

Proof: The proof is an immediate consequence of theorem 3.4.2 . □

Theorem 3.4.3:

i. If $1 < b_3 \leq p_n \leq b_4 < \infty$, $G(x)$ is nondecreasing, and equations (3.4.2), (3.4.3) hold, then every solution of equation (3.4.1) oscillates or tends to zero as $n \to \infty$.

ii. If $1 < b_3 \leq p_n \leq b_4 \leq \frac{1}{2}b_3^2$, equation (3.4.2) holds, G is nondecreasing and satisfies Lipschitz condition on interval of the form $[a,b]$, $0 < a < b < \infty$ and every solution of equation (3.4.1) oscillates or tends to zero as $n \to \infty$, then equation (3.4.3) holds.

Proof: The proof is similar to that of theorem 3.4.2, for details see [16]. □

Remark 3.4.1: for the case when $-\infty < b_5 \leq p_n \leq b_6 < -1$, the authors in [18] obtained sufficient conditions for oscillation and asymptotic behavior of equation (3.4.3).

In the previous theorems the conditions guarantee the oscillation for equation (3.4.1) when $p_n \neq \pm 1$. The question here is whether equation (3.4.1) is oscillatory when $p_n = \pm 1$? The answer is yes, but with different sufficient conditions when $m > 0$. For that we need the following conditions:

(C1) There exists a sequence $\{h_n\}$ that changes sign with $\liminf\limits_{n\to\infty} h_n = \alpha$ and $\limsup\limits_{n\to\infty} h_n = \beta$

where $-\infty < \alpha < 0$, $0 < \beta < \infty$ and $\Delta h_n = f_n$

(C$_2$) $\sum_{n=k}^{\infty} q_n G\left(h_{n-k}^{+}\right) = \infty$ and $\sum_{n=k}^{\infty} q_n G\left(h_{n+m-k}^{-}\right) = \infty$

where $h_n^{+} = \max\{h_n, 0\}$ and $h_n^{-} = \max\{-h_n, 0\}$

(C$_3$) $\sum_{n=k}^{\infty} q_n G\left(-h_{n+m-k}^{+}\right) = -\infty$ and $\sum_{n=k}^{\infty} q_n G\left(-h_{n-k}^{-}\right) = -\infty$

(C$_4$) For $u > 0$ and $v > 0$, there exists $\lambda > 0$ such that
$G(u+v) \le \lambda\left(G(u) + G(v)\right)$

(C$_5$) For $u < 0$ and $v < 0$, there exists $\mu > 0$ such that
$G(u+v) \ge \mu\left(G(u) + G(v)\right)$

(C$_6$) $\sum_{n=r}^{\infty} q_n^{*} G\left(h_{n-k}^{+}\right) = \infty$ and $\sum_{n=r}^{\infty} q_n^{*} G\left(-h_{n-k}^{-}\right) = -\infty$,

where $q_n^{*} = \min\{q_n, q_{n-m}\}$, $n \ge m$ and $r = \max\{k, m\}$

Theorem 3.4.4: Let $-1 \le p_n \le 0$. Let conditions (C$_1$)-(C$_3$) hold. Then all solutions of equation (3.4.1) oscillate.

Proof: Suppose on the contrary that x_n is an eventually positive solution of equation (3.4.1), ($x_n > 0$ for $n \ge N_1$).
Setting

$$z_n = x_n + p_n x_{n-m} \quad \text{and} \quad w_n = z_n - h_n \tag{3.4.12}$$

Then

$$\Delta w_n = -q_n G(x_{n-k}) \le 0, \quad \text{for } n \ge N_1 + r, \tag{3.4.13}$$

then either $w_n > 0$ or $w_n < 0$, for $n \ge N_1 + r$. Let $w_n > 0$, $n \ge N_2$, then $x_n \ge h_n^{+}$ and equation (3.4.13) yields

$$\sum_{n=N_3}^{j-1} q_n G\left(h_{n-k}^{+}\right) \le \sum_{n=N_3}^{j-1} q_n G\left(x_{n-k}\right) < w_{N_3},$$

where $j > N_3 + 1$ and $N_3 > N_2 + r$, which contradicts (C$_2$), so $w_n < 0$ for $n \ge N_2$. Then $x_n \ge h_{n+m}^{-}$ and $\lim_{n\to\infty} w_n = l$, $-\infty \le l < 0$

Suppose that $l = -\infty$. Let $\lambda > \beta > 0$. For $0 < \varepsilon < \lambda - \beta$, there exists $N_3 > N_2$ such that

$$h_n < \beta + \varepsilon, \quad n \ge N_3$$

Further, we have

$$w_n < -\lambda, \quad n \ge N_4 > N_3$$

that is

$$\begin{aligned} x_n &\le -p_n x_{n-m} + h_n - \lambda \\ &\le x_{n-m} + \beta + \varepsilon - \lambda \end{aligned}$$

So for $n \ge N_4 + jm$

$$x_n \le x_{n-jm} + j(\beta+\varepsilon-\lambda)$$

In particular

$$x_{N_4+jm} \le x_{N_4} + j(\beta+\varepsilon-\lambda) < 0 \quad \text{for large } j,$$

which is a contradiction. So $-\infty < l < 0$, now from equation (3.4.13) we get

$$\sum_{n=N_3}^{j-1} q_n G\left(h_{n+m-k}^{-}\right) \le \sum_{n=N_3}^{j-1} q_n G(x_{n-k}) \le -w_j \quad ,$$

which is also a contradiction, so $\{x_n\}$ is an oscillatory solution of equation (3.4.1). □

Theorem 3.4.5: Let $0 \le p_n \le 1$. If conditions (C_1), (C_4), (C_5) and (C_6) hold then all solutions of equation (3.4.1) oscillate.

Proof: Let $\{x_n\}$ satisfies $x_n > 0$ for $n \ge N_1$. Setting z_n and w_n as in equation (3.4.12) to get equation (3.4.13). Using (C_1) we have $w_n > 0$, for $n \ge N_2 \ge N_1$
hence

$$x_n + x_{n-m} \ge h_n^{+} \ , \quad \text{for} \quad n \ge N_2$$

Using (C_4) we have

$$\begin{aligned}\lambda(f_n + f_{n-m}) &= \lambda\Delta z_n + \lambda\Delta z_{n-m} + \lambda q_n G(x_{n-k}) + \lambda q_{n-m} G(x_{n-k-m}) \\ &\ge \lambda\Delta(z_n + z_{n-m}) + q_n^{*} G(x_{n-k} + x_{n-k-m}) \\ &\ge \lambda\Delta(z_n + z_{n-m}) + q_n^{*} G\left(h_{n-k}^{+}\right)\end{aligned}$$

That is

$$q_n^{*} G\left(h_{n-k}^{+}\right) \le \lambda\Delta(h_n + h_{n-m}) - \lambda\Delta(z_n + z_{n-m}) \quad ,$$

taking the summation from N_2 to $j-1$ to the last inequality to get

$$\begin{aligned}\sum_{n=N_2}^{j-1} q_n^{*} G\left(h_{n-k}^{+}\right) &\le \lambda \sum_{n=N_2}^{j-1} \Delta(h_n + h_{n-m}) - \lambda \sum_{n=N_2}^{j-1} \Delta(z_n + z_{n-m}) \\ &< \lambda(h_j + h_{j-m}) - \lambda(h_{N_2} + h_{N_2-m}) + \lambda(z_{N_2} + z_{N_2-m})\end{aligned}$$

Consequently

$$\sum_{n=N_2}^{\infty} q_n^{*} G\left(h_{n-k}^{+}\right) \le 2\lambda \limsup_{j\to\infty} h_j - \lambda(h_{N_2} + h_{N_2-m}) + \lambda(z_{N_2} + z_{N_2-m}) < \infty$$

Which contradicts (C_6). So $\{x_n\}$ must be an oscillatory solution of equation (3.4.1). □

Example 3.4.1: Consider the NDE

$$\Delta(x_n + \frac{1}{4} x_{n-m}) + \frac{3^n}{2^{n+1}}\left(2^{n+1} - 1\right) x_{n-k}^{\frac{1}{3}} = \frac{3(-1)^n}{2^{n+1}} \quad , \quad n \ge 0 \qquad (3.4.14)$$

equation (3.4.14) satisfies the conditions of theorem 3.4.2(i) ,where $G(x) = x^{\frac{1}{3}}$ is nondecreasing and $f_n = \dfrac{3(-1)^n}{2^{n+1}}$ satisfies inequality (3.4.2) , also

$$\left|\sum_{n=0}^{\infty} q_n\right| = \left|\sum_{n=0}^{\infty} \frac{3^n}{2^{n+1}}\left(2^{n+1} - 1\right)\right| = \infty \quad ,$$

so every solution of equation (3.4.15) oscillates or tends to zero as $n \to \infty$.

Example 3.4.2: Consider the NDE

$$\Delta\left(x_n+\left(\frac{(-1)^n}{5}-\frac{1}{4}\right)x_{n-1}\right)+e^n x_{n-2}^3=(-1)^{n+1} \quad , \quad n\ge 0 \tag{3.4.15}$$

it is clear that equation (3.4.15) satisfies the conditions of theorem 3.4.4 since

$$-1\le-\frac{9}{20}\le p_n=\left(\frac{(-1)^n}{5}-\frac{1}{4}\right)<0 \ ,$$

also

$$h_n=\frac{1}{2}(-1)^n \ , \qquad h_n^+=\begin{cases}\frac{1}{2},\ n\ even\\ 0,\ n\ odd\end{cases} \ , \qquad \text{and} \qquad h_n^-=\begin{cases}0,\ n\ even\\ \frac{1}{2},\ n\ odd\end{cases} \ ,$$

so

$$\sum_{n=k}^{\infty}q_nG\left(h_{n-k}^+\right)=\sum_{n=2}^{\infty}e^n\left(h_{n-2}^+\right)^3=\sum_{i=0}^{\infty}e^i\left(h_{2i}^+\right)^3=\infty \ ,$$

and

$$\sum_{n=k}^{\infty}q_nG\left(h_{n-k+m}^-\right)=\sum_{n=2}^{\infty}e^n\left(h_{n-1}^-\right)^3=\sum_{i=0}^{\infty}e^i\left(h_{2i+1}^-\right)^3=\infty \ ,$$

also

$$\sum_{n=2}^{\infty}e^nG\left(-h_{n-1}^+\right)=\sum_{n=2}^{\infty}e^nG\left(-h_{n-2}^-\right)=-\infty \ ,$$

hence, all solutions of equation (3.4.15) oscillate.

Remark 3.4.2: The results of this section are mainly referred to [18] and [20].

3.5 Oscillation criteria of certain nonlinear NDE's

In this section we introduce some oscillation theorems for nonlinear NDE's with $G(x)$ replaced by special functions, such as the maximum function and the product function multiplied by the sign function, the theorems in [9] and [16] discuss the special case when $p_n\equiv 1$ and $f_n=0$. All the theorems of this section considered special cases of the form

$$\Delta(x_n+\delta_1 p_n x_{n-m})+\delta_2 q_n G(x_{n-k})=0 \ ,$$

with $\delta_1=-1$ and $\delta_2=1$.

Consider the nonlinear NDE of the form

$$\Delta(x_n-p_n x_{n-m})+q_n\left|x_{n-k}\right|^{\alpha}\operatorname{sgn} x_{n-k}=0 \ , \tag{3.5.1}$$

where $\{q_n\}$ is a sequence of nonnegative real numbers, α is a positive constant and m and k are positive integers and $\delta_1=-1$, $\delta_2=1$.

Theorem 3.5.1: If

$$\sum_{n=n_0}^{\infty}n^{\alpha}q_n\left(\sum_{j=n}^{\infty}q_j\right)^{\alpha}=\infty \quad , \quad n\ge n_0\ge 0$$

Then equation (3.5.1) is oscillatory, provided that

$$\sum_{j=n}^{\infty} q_j < \infty$$

$p_n = 1$ and $\alpha > 0$

Proof: Let $\{x_n\}$ be a nonoscillatory solution of equation (3.5.1), W.L.O.G let $x_{n-r} > 0$ for $n \geq n_1 \geq n_0$, where $r = \max\{m, k\}$,

let

$$z_n = x_n - x_{n-m}$$

then

$$\Delta z_n = -q_n x_{n-k}^{\alpha} \leq 0 \quad \text{for } n \geq n_1$$

Two cases to be considered here: either $z_n > 0$ or $z_n < 0$

Case1: $z_n < 0$ and $\Delta z_n < 0$ eventually:

z_n is nonincreasing, so there exists a constant $c_1 > 0$ such that

$$z_n < -c_1 \text{ , for } n \geq N \geq n_1$$

thus

$$x_N = z_N + x_{N-m} < -c_1 + x_{N-m} \quad ,$$

or

$$x_{N+m} = z_{N+m} + x_N < -c_1 + x_N < -2c_1 + x_{N-m} \ ,$$

hence, for any integer $h > 1$

$$x_{N+hm} < -(h+1)c_1 + x_{N-m} \to -\infty \ , \quad \text{as} \quad h \to \infty$$

which is a contradiction.

Case 2: $z_n > 0$ and $\Delta z_n < 0$ eventually:

We have $x_n > x_{n-m}$, for $n \geq n_1$, so there exists a constant $c_2 > 0$ such that

$$x_{n-k} \geq c_2 \ , \text{ for } n \geq N_1 \geq n_1 + k$$

then

$$\Delta z_n \leq -(c_2)^{\alpha} q_n \ , \text{ for } n \geq N_1$$

hence

$$z_s - z_n \leq -(c_2)^{\alpha} \sum_{j=n}^{s-1} q_j \quad , \quad n \geq N_1$$

now, letting $s \to \infty$, we have

$$z_n \geq (c_2)^{\alpha} \sum_{j=n}^{\infty} q_j \ , \quad n \geq N_1$$

and hence

$$\Delta z_n \geq (c_2)^{\alpha} q_n \quad ,$$

but since $c_2 > 0$ and α is a positive constant, then $\Delta z_n \geq 0$, which is a contradiction , so equation (3.5.1) is oscillatory. □

Theorem 3.5.2: If

$$\sum_{j=n}^{\infty} q_j = \infty \ , \quad n \geq n_0 \geq 0$$

then equation (3.5.1) with $p_n = 1$ is oscillatory.

Proof: Suppose that $\{x_n\}$ is an eventually positive solution of equation (3.5.1), and let z_n be defined as in theorem 3.5.1, the case $z_n > 0$ and $\Delta z_n > 0$ is as that in theorem 3.5.1.

* If $z_n > 0$ and $\Delta z_n < 0$,

so

$$x_n > x_{n-m}, \quad n \ge n_1$$

and there exists a constant number $c_2 > 0$ such that

$$x_{n-k} \ge c_2 \ , \text{for } \ n \ge n_2 \ge n_1$$

then

$$\Delta z_n = -q_n x_{n-k}^{\alpha} \le 0 \ ,$$

so

$$\Delta z_n \le -(c_2)^{\alpha} q_n$$

Taking the summation to the last inequality from n_2 to $h-1 \ge n_2$, to get

$$0 < z_h \le z_{n_2} - c_2^{\alpha} \sum_{n=n_2}^{h-1} q_n \to -\infty \ , \text{as } \ h \to \infty$$

which is a contradiction, so equation (3.5.1) is oscillatory. □

Now we will discuss the oscillation for solutions of a more general equation. Several criteria for the oscillation are to be discussed for the first-order nonlinear NDE of the form

$$\Delta(x_n - p_n x_{n-m}) + q_n \prod_{i=1}^{h} \left| x_{n-k_i} \right|^{\alpha_i} \operatorname{sgn} x_{n-k_i} = 0 \ , \quad n \ge 0 \tag{3.5.2}$$

where m is a positive integer, k_i's are nonnegative integers, $p_n \ge 0$, $q_n \ge 0$ such that q_n is not identically zero for all large n and $\alpha_i > 0$ with $\sum_{i=1}^{h} \alpha_i = 1$.

Lemma 3.5.1: Suppose there is an integer $N > 0$ such that $p_{N+im} \le 1$ for $i \ge 0$ holds. Suppose further that

(H1) $p_n + q_n \min\{k_1, k_2, ..., k_h\} > 0$ or

(H2) $\min\{k_1, k_2, ..., k_h\} > 0$

and q_n doesn't vanish identically over sets of consecutive integers of the form $\{a, a+1, ..., a+k_h\} \min\{k_1, ..., k_h\}$. Then every solution of equation (3.5.2) is oscillatory if and only if

$$\Delta(x_n - p_n x_{n-m}) + q_n \prod_{i=1}^{h} \left| x_{n-k_i} \right|^{\alpha_i} \operatorname{sgn} x_{n-k_i} \le 0 \ , \quad n = 0,1,2,... \tag{3.5.3}$$

has no eventually positive solution.

Proof: See [16]. □

Lemma 3.5.2: Suppose $p_n \ge 1$, $q_n \ge 0$, for $n \ge 0$ and

$$\sum_{n=0}^{\infty} q_n \prod_{j=0}^{n-1} \left(1 + \frac{j q_j}{m} \right) = \infty \tag{3.5.4}$$

Then for every eventually positive solution $\{x_n\}$ of inequality (3.5.3), the sequence z_n defined by $z_n = x_n - p_n x_{n-m}$ will satisfy $z_n < 0$ and $\Delta z_n \le 0$ for all large n.

Proof: It is clear that $\Delta z_n \le 0$, and is not identically zero for all large n. Suppose to the contrary that $x_n > 0$, $\Delta z_n \le 0$ and $z_n > 0$ for $n \ge T$, then

$$x_n > p_n x_{n-m} \ge x_{n-m} > 0 \ , \qquad n \ge T + m$$

thus

$$x_n \ge \min\{x_{T-m}, x_{T-m+1}, ..., x_{T-1}\} = M > 0, \quad n \ge T + 2m = T_1 \tag{3.5.5}$$

let

$$N(n) = \left[\frac{n - T_1}{m}\right] \ ,$$

where $\left[\frac{n - T_1}{m}\right]$ is the integer part of $\left(\frac{n - T_1}{m}\right)$, then

$$\begin{aligned} x_n &\ge z_n + x_{n-m} \\ &\ge z_n + z_{n-m} + ... + z_{n-(N(n)-1)m} + x_{n-N(n)m} \ , \ \ n \ge T_1 \end{aligned} \tag{3.5.6}$$

since $\{z_n\}$ is nonincreasing then

$$x_n \ge N(n) z_n + M \ , \ \ n \ge T_1 \tag{3.5.7}$$

substituting (3.5.7) into (3.5.3) to get

$$\Delta z_n + q_n \prod_{i=1}^{h} \left[N(n-k_i) z_{n-k_i} + M\right]^{\alpha_i} \le 0 \ , \quad n \ge T_1 + m = T_2 \tag{3.5.8}$$

by Hölder inequality we have

$$\prod_{i=1}^{h} \left[N(n-k_i) z_{n-k_i} + M\right]^{\alpha_i} \ge \prod_{i=1}^{h} \left[N(n-k_i)\right]^{\alpha_i} \prod_{i=1}^{h} \left[z_{n-k_i}\right]^{\alpha_i} + M$$

also

$$\Delta z_n + q_n \prod_{i=1}^{h} \left[N(n-k_i)\right]^{\alpha_i} z_n + q_n M \le 0 \ , \qquad n \ge T_2$$

then

$$\Delta\left[z_n \prod_{j=T_2}^{n-1} \left[1 + q_j \prod_{i=1}^{h} \left(N(i-k_i)\right)^{\alpha_i}\right]\right] + M q_n \prod_{j=T_2}^{n-1} \left[1 + q_j \prod_{i=1}^{h} \left(N(i-k_i)\right)^{\alpha_i}\right] \le 0 \ , \ \ n \ge T_2 \tag{3.5.9}$$

Taking the summation to inequality (3.5.9) from T_2 to $n \ge T_2$, to get

$$z_{n+1} \prod_{j=T_2}^{n} \left[1 + q_j \prod_{i=1}^{h} \left(N(i-k_i)\right)^{\alpha_i}\right] - z_{T_2} + M \sum_{s=T_2}^{n} q_s \prod_{j=T_2}^{s-1} \left[1 + q_j \prod_{i=1}^{h} \left(N(i-k_i)\right)^{\alpha_i}\right] \le 0 \ , \ n \ge T_2 \tag{3.5.10}$$

we assume that

$$\sum_{n=0}^{\infty} q_n < \infty$$

We have that

$$\prod_{i=1}^{h} \frac{\left[N(n-k_i)\right]^{\alpha_i}}{n} \to \frac{1}{m} \ \ , \ \text{as} \ \ n \to \infty$$

also $\sum_{n=T_2}^{\infty} q_n\left[\frac{n}{m}-\prod_{i=1}^{h}\left(N(n-k_i)\right)^{\alpha_i}\right]$ is absolutely convergent,

then, $\lim_{n\to\infty}\frac{\prod_{j=T_2}^{n-1}\left[1+q_j\prod_{i=1}^{h}\left(N(i-k_i)\right)^{\alpha_i}\right]}{\prod_{j=T_2}^{n-1}\left[1+q_j\left(\frac{i}{m}\right)\right]}$ exists.

By condition (3.5.4), we get

$$\sum_{n=T_2}^{\infty} q_n\prod_{j=T_2}^{n-1}\left[1+q_j\left(\frac{i}{m}\right)\right]=\infty \ ,$$

letting $n\to\infty$ in inequality (3.5.10), to get a contradiction, hence $z_n<0$. □

Theorem 3.5.3: Suppose that $p_n=1$, $q_n\geq 0$ for $n\geq 0$ and equation (3.5.4) holds. Then every solution of equation (3.5.2) is oscillatory.

Proof: Using lemma 1.4.1 and lemma 3.5.2 then z_n is simultaneously positive and negative for large n, which is a contradiction, so equation (3.5.2) is oscillatory. □

Remark 3.5.1: Theorem 3.5.3 is an extension and improvement to theorem 2.3.1.

Theorem 3.5.4: Suppose there is an integer $N>0$ such that $p_{N+im}\leq 1$, $i\geq 0$ holds suppose also that $p_n, q_n\geq 0$ for $n\geq 0$ and equation (3.5.4) holds. Suppose further that

$$q_n\prod_{i=1}^{n} p_{n-k_i}^{\alpha_i}\geq q_{n-m} \ , \text{ for all large } n.$$

Then every solution of equation (3.5.2) is oscillatory.

Proof: Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of equation (3.5.2). Then by lemma 1.4.1, $z_n>0$ for all large n, but $z_n=x_n-p_nx_{n-m}$, so

$$\begin{aligned}\Delta z_n &= -q_n\prod_{i=1}^{h} x_{n-k_i}^{\alpha_i}\\ &= -q_n\prod_{i=1}^{h}\left(z_{n-k_i}+p_{n-k_i}x_{n-m-k_i}\right)^{\alpha_i}\end{aligned}$$

By Hölder inequality we have

$$\Delta z_n\leq -q_n\left[\prod_{i=1}^{h} z_{n-k_i}^{\alpha_i}+\prod_{i=1}^{h} p_{n-k_i}^{\alpha_i}\prod_{i=1}^{h} x_{n-m-k_i}^{\alpha_i}\right] \text{, for all large } n \tag{3.5.11}$$

but we have

$$\Delta z_{n-m}+q_{n-m}\prod_{i=1}^{m} x_{n-m-k_i}^{\alpha_i}=0 \quad , \tag{3.5.12}$$

subtracting equation (3.3.10) from inequality (3.5.9) to get

$$\Delta z_n-\Delta z_{n-m}+q_n\prod_{i=1}^{h} z_{n-k_i}^{\alpha_i}\leq\left[q_{n-m}-q_n\prod_{i=1}^{h} p_{n-k_i}^{\alpha_i}\right]\prod_{i=1}^{h} x_{n-m-k_i}^{\alpha_i} \ ,$$

by the hypothesis of this theorem

$$\Delta(z_n - z_{n-m}) + q_n \prod_{i=1}^{h} z_{n-k_i}^{\alpha_i} \le 0 \quad ,$$

which is a contradiction to theorem 3.5.3, so every solution of equation (3.5.2) is oscillatory. □

Remark 3.5.2: Theorem 3.5.4 is an extension and improvement to theorem 2.3.2.

In a similar manner used in the proof of theorem 3.5.4 and using theorem 2.3.3 and theorem 4 in [30] we deduced the following result.

Theorem 3.5.5: Suppose that $p_n, q_n \ge 0$ for $n \ge 0$ such that $p_{N+im} \le 1$ for some $N > 0,\ i \ge 0$ and equation (3.5.4) hold, suppose further that there is some number $r \in (0,1)$ such that

$$-q_n \prod_{i=1}^{h} p_{n-k_i}^{\alpha_i} \le r\, q_{n-m} \ , \quad \text{for all large } n \tag{3.5.13}$$

Then equation (3.5.2) is oscillatory provided that the following difference inequality

$$\Delta w_n + \frac{q_n}{1+r} w_{n-(m-k)} \le 0 \ , \ \ k = \min\{k_1, \ldots, k_h\} \ , \ n \ge 0$$

has no eventually positive solution.

Proof: Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of equation (3.5.2), by lemma 1.4.1 then $z_n = x_n - p_n x_{n-m}$ satisfies $z_n > 0$ and $\Delta z_n \le 0$ for all large n.

From equation (3.5.2) we get

$$0 = r\Delta z_{n-m} + r q_{n-m} \prod_{i=1}^{h} \left| x_{n-m-k_i} \right|^{\alpha_i} \ge r\Delta z_{n-m} - q_n \prod_{i=1}^{h} p_{n-k_l}^{\alpha_i} \left| x_{n-m-k_i}^{\alpha_i} \right| \ , \quad \text{for large } n, \tag{3.5.14}$$

but we have

$$\Delta z_n + q_n \prod_{i=1}^{h} \left| x_{n-k_i}^{\alpha_i} \right| = 0 \ , \tag{3.5.15}$$

adding inequality (3.5.14) to equation (3.5.15) to get

$$\Delta(z_n + r z_{n-m}) + q_n \left(\prod_{i=1}^{h} \left| x_{n-k_i}^{\alpha_i} \right| - \prod_{i=1}^{h} p_{n-k_i}^{\alpha_i} \left| x_{n-m-k_i}^{\alpha_i} \right| \right) \le 0 \ , \tag{3.5.16}$$

however, by Hölder inequality we have

$$z_n \le \prod_{i=1}^{h} z_n^{\alpha_i} = \prod_{i=1}^{h} \left| x_n - p_n x_{n-m} \right|^{\alpha_i} \le \left| \prod_{i=1}^{h} x_n^{\alpha_i} - \prod_{i=1}^{h} p_n^{\alpha_i} x_{n-m}^{\alpha_i} \right| \quad ,$$

so inequality (3.5.16) becomes

$$\Delta(z_n + r z_{n-m}) + q_n z_{n-k} \le 0 \ , \quad \text{for all large } n$$

where $k = \min\{k_1, \ldots, k_h\}$,

so the sequence w_n defined by

$$w_n = z_n + r\, z_{n-m} > 0 \ , \quad n \ge 0$$

and

$$w_n \le (1+r) z_{n-m}$$

thus

$$z_{n-k} = z_{(n+m-k)-m} \geq \frac{w_{n-(k-m)}}{1+r} \quad ,$$

for all large n, so $\{w_n\}$ is an eventually positive solution of inequality

$$\Delta w_n + \frac{q_n}{1+r} w_{n-(k-m)} \leq 0 \quad ,$$

which is a contradiction. □

the following theorem concerns with another special case of the nonlinear NDE of the form

$$\Delta(x_n - p_n x_{n-m}) + q_n \max_{s\in[n-k,n]} x_s = 0 \quad , \quad n \geq 0 \qquad (3.5.17)$$

where m and k are positive integers , $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences and $\{q_n\}$ has a positive subsequence.

Theorem 3.5.6: Assume that there exists a nonnegative integer $N \geq 0$ such that $p_{N+im} \leq 1$ for $i = 0,1,2,...$ Suppose further that there exists some positive integer T such that the difference inequality

$$\Delta z_n + q_n \min_{s\in[n-k,n]} \sum_{i=0}^{T-1} \prod_{j=0}^{i} p_{s+jm} z_{n-m} \leq 0 \ ,$$

has no eventually positive solution. Then all solution of equation (3.5.17) are oscillatory.

Proof: Let x_n be an eventually positive solution of (3.5.17) then $\Delta z_n \leq 0$ and $z_n = x_n - p_n x_{n-m} > 0$ eventually. Thus

$$\begin{aligned} x_n &= z_n + p_n x_{n-m} \\ &= z_n + p_n z_{n-m} + p_n p_{n-m} x_{n-2m} \\ &\ \vdots \\ &= z_n + p_n z_{n-m} + ... + \prod_{j=0}^{T-1} p_{n-jm} z_{n-(j+1)m} \\ &\geq \sum_{i=0}^{T-1} \prod_{j=0}^{i} p_{n-jm} z_{n-m} \end{aligned}$$

Hence,

$$\begin{aligned} \max_{s\in[n-k,n]} x_s &\geq \max_{s\in[n-k,n]} \sum_{i=0}^{T-1} \prod_{j=0}^{i} p_{s-jm} z_{s-m} \\ &\geq \max_{s\in[n-k,n]} \sum_{i=0}^{T-1} \prod_{j=0}^{i} p_{s-jm} z_{n-m} \end{aligned}$$

Substituting the last inequality into (3.5.17), we have

$$\Delta z_n + q_n \max_{s\in[n-k,n]} \sum_{i=0}^{T-1} \prod_{j=0}^{i} p_{s-jm} z_{n-m} \leq 0 \quad ,$$

which is a contradiction, that complete the proof. □

Example 3.5.1: Consider the NDE

$$\Delta(x_n - x_{n-3}) + (e^5 - 1)(1 - e^{-1})^2 e^{-n} x_{n-2} = 0 \qquad (3.5.18)$$

It is clear

$$\sum_{j=n}^{\infty}\frac{(e^5-1)(1-e^{-1})^2}{e^j}=\infty$$

So every solution of equation (3.5.18) is oscillatory by theorem 3.5.2.

Example 3.5.2: Consider the NDE

$$\Delta(x_n-x_{n-1})+n^{-\frac{5}{3}}|x_{n-2}|^2\operatorname{sgn} x_{n-2}=0 \tag{3.5.19}$$

$$\sum_{j=n}^{\infty}q_j=j^{-\frac{5}{3}}\geq\frac{1}{\left(\frac{5}{3}-1\right)n^{\left(\frac{5}{3}-1\right)}}$$

so by theorem 3.5.1 equation (3.5.19) is oscillatory.

Example 3.5.3: Consider the NDE

$$\Delta\left(x_n-\frac{n}{n+1}x_{n-2}\right)+n^{\frac{7}{4}}x_{n-1}=0 \tag{3.5.20}$$

It is clear that $p_n, q_n\geq 0$, and that there exists $N>0$ such that $p_{N+im}\leq 1$, and

$$\sum_{n=0}^{\infty}n^{\frac{7}{4}}\prod_{j=0}^{n-1}\left(1+\frac{j(j)^{\frac{7}{4}}}{2}\right)=+\infty$$

Then by theorem 3.5.5 every solution of equation (3.5.20) is oscillatory.

Remark 3.5.3: the results of this section are mainly referred to [6], [9] and [16].

Chapter Four

Oscillation Criteria of Second-Order Neutral Difference Equations

4.0 Introduction

In this chapter we discuss the oscillatory behavior of the second-order NDE's of the form

$$\Delta(c_n\Delta(x_n + p_n x_{n-m})) + q_n G(x_{n-k}) = f_n \quad , \tag{4.0.1}$$

where $\{c_n\},\{p_n\},\{q_n\}$ and $\{f_n\}$ are sequences of real numbers , $G \in C(\mathbf{R},\mathbf{R})$ with $uG(u) > 0$ for $u \neq 0$ and m and k are nonnegative integers.

In fact the problem of oscillation for second-order NDE's has received a great deal of attention during the last few years, most of the researches to the best we know do not separate the linear and nonlinear second-order NDE's , they studied the nonlinear case as a general case, so they did not use the expression "nonlinear" in the titles of the researches. However, they used this expression when the brackets containing the difference operator is of degree more than one.

This chapter mainly includes five sections. The first section deals with the unforced form of equation (4.0.1) when $c_n = 1,\ p_n = p$. Section two concerns with the forced form of the equation studied in section one. Section three studies the unforced form of equation (4.0.1) with the coefficients are all variables. Section four discusses the oscillation theorems for equation (4.0.1) and finally section five studies the nonlinear form

$$\Delta(c_n[\Delta(x_n + p_n x_{n-m})]) + q_n x_{n-k}^{\beta} = 0 \ , \quad n > 0$$

where β is the ratio of odd positive integers.

4.1 Oscillation theorems for second-order NDE's with both constant and variable coefficients

Consider the second-order NDE of the form

$$\Delta^2(x_n + p x_{n-m}) + q_n G(x_{n-k}) = 0 \ , \quad n \geq 0 \tag{4.1.1}$$

where p is a positive real number, m and k are nonnegative integers with $k \geq m$, $\{q_n\}$ is a sequence of nonnegative real numbers and $G \in C(\mathbf{R},\mathbf{R})$ is a nonincreasing function.

Theorem 4.1.1: Suppose that for a nonnegative constant M the condition

$$xG(x) > 0 \text{ and } G(x)\operatorname{sgn} x \geq M|x|^{\gamma} \text{ for } x \neq 0 \quad \text{, for } \gamma = 1 \tag{4.1.2}$$

suppose further that there exists a positive sequence $\{\rho_n\}$ such that

$$\sum_{n=1}^{\infty}\left[M\rho_n q_n - \frac{(\Delta\rho_n)^2}{4\rho_n}\right] = \infty \tag{4.1.3}$$

Then every solution of equation (4.1.1) is oscillatory.

Proof: Suppose to contrary that $\{x_n\}$ is a positive solution of equation (4.1.1), there exists a positive integer n_0 such that

$$x_{n-k} > 0 \ , \quad \text{for } n \geq n_0$$

let

$$z_n = x_n + px_{n-m} \quad ,$$

then $z_n > 0$ and

$$\Delta^2 z_n = -q_n G(x_{n-k}) \le 0 \ , \ \text{for } n \ge n_0 \tag{4.1.4}$$

hence $\{\Delta z_n\}$ is an eventually nonincreasing sequence, either $\Delta z_n > 0$ or $\Delta z_n \le 0$ for $n \ge n_0$.

Suppose that $\Delta z_n \le 0$, then there exists an integer $n_1 \ge n_0$ such that $\Delta z_n \le 0$ for all $n \ge n_1$. using inequality (4.1.4) we get

$$\Delta z_n \le \Delta z_{n_1} \le 0 \ , \text{for } n \ge n_1$$

but q_n is not eventually identically zero, so there exists an integer $n_2 \ge n_1$ such that

$$\Delta^2 z_n < 0 \quad , \text{for } n \ge n_2$$

so

$$\Delta z_n \le \Delta z_{n_2+1} < \Delta z_{n_2} \le \Delta z_{n_1} \le 0 \quad , \text{for} \quad n \ge n_2 + 1 \tag{4.1.5}$$

Taking the summation to the first two terms of inequality (4.1.5) we have

$$\sum_{n_2+1}^{n} \Delta z_n \le \sum_{n_2+1}^{n} \Delta z_{n_2+1} \quad , \quad n \ge n_2 + 1$$

so

$$z_{n+1} - z_{n_2+1} \le (n - n_2)\Delta z_{n_2+1} \to -\infty \ \text{ as } n \to \infty$$

which contradicts the fact that $z_n > 0$ for $n \ge n_0$, so $\Delta z_n > 0$ and so we have

$$z_{n-k} > 0 \ , \ \Delta z_n > 0 \ , \ \Delta^2 z_n \le 0 \ \text{ for } n \ge n_0 \tag{4.1.6}$$

Set

$$w_n = \rho_n \frac{\Delta z_n}{z_{n-k}} \quad , \quad n \ge n_0$$

then

$$\Delta w_n = \rho_{n+1} \frac{\Delta z_{n+1}}{z_{n+1-k}} - \rho_n \frac{\Delta z_n}{z_{n-k}} \quad , \tag{4.1.7}$$

adding and subtracting the terms $\left(\rho_n \dfrac{\Delta z_{n+1}}{z_{n+1-k}}\right)$ and $\left(\rho_n \dfrac{\Delta z_{n+1}}{z_{n-k}}\right)$ to equation (4.1.7) we get

$$\Delta w_n = \Delta\rho_n \frac{\Delta z_{n+1}}{z_{n+1-k}} + \frac{\rho_n}{z_{n-k}}\left[\Delta^2 z_n - \frac{\Delta z_{n+1}}{z_{n+1-k}} \Delta z_{n-k}\right]$$

using equation (4.1.1), the last equation becomes

$$\Delta w_n = -\rho_n \frac{q_n G(x_{n-k})}{z_{n-k}} + \Delta\rho_n \frac{\Delta z_{n+1}}{z_{n+1-k}} - \rho_n \left(\frac{\Delta z_{n+1}}{z_{n+1-k}}\right)\left(\frac{\Delta z_{n-k}}{z_{n-k}}\right)$$

then

$$\Delta w_n \le -\rho_n \frac{q_n G(z_{n-k})}{z_{n-k}} + \Delta\rho_n \frac{\Delta z_{n+1}}{z_{n+1-k}} - \rho_n \left(\frac{\Delta z_{n+1}}{z_{n+1-k}}\right)\left(\frac{\Delta z_{n-k}}{z_{n-k}}\right) \tag{4.1.8}$$

since $\Delta z_n > 0$ and nonincreasing , there exists an integer $n_1 \ge n_0$ such that $\Delta z_{n-k} \ge \Delta z_n$ for $n \ge n_1$. Thus

$$\Delta w_n \le -M\rho_n q_n + \Delta\rho_n \frac{\Delta z_{n+1}}{z_{n+1-k}} - \rho_n \left(\frac{\Delta z_{n+1}}{z_{n+1-k}}\right)\left(\frac{\Delta z_n}{z_{n-k}}\right) \tag{4.1.9}$$

Now we have

$$\Delta z_n \le \Delta z_{n_2+1} \quad \text{, for } n \ge n_2+1$$

so

$$z_n - z_{n_1} = \sum_{i=n_1}^{n-1} \Delta z_i \ge (n-n_1)\Delta z_{n-1} \quad ,$$

and so

$$z_n \ge \left(\frac{n}{2}\right)\Delta z_n \quad \text{, for } n \ge n_2 \ge 2n_1+1 \tag{4.1.10}$$

There exists an integer $n_3 \ge n_2 + k$ such that

$$z_{n-k} \ge \frac{n-k}{2}\Delta z_{n-k} \ge \frac{n-k}{2}\Delta z_n \quad , \quad n \ge n_3 \tag{4.1.11}$$

and

$$0 < \frac{\Delta z_n}{z_{n-k}} \le \frac{2}{n-k} \to 0 \quad \text{as } n \to \infty .$$

Thus letting $h_n = \dfrac{\Delta z_n}{z_{n-k}}$, $h_n > 0$ and nonincreasing (i.e. $h_{n+1} \le h_n$, $n \ge n_3$) substituting the value of h_n in inequality (4.1.9) it becomes

$$\begin{aligned}
\Delta w_n &\le -M\rho_n q_n + \Delta\rho_n h_{n+1} - \rho_n h_{n+1}^2 \quad , \quad n \ge n_3 \\
&\le -M\rho_n q_n + \frac{(\Delta\rho_n)^2}{4\rho_n} - \left(\sqrt{\rho_n}\, h_{n+1} - \frac{\Delta\rho_n}{2\sqrt{\rho_n}}\right)^2 \\
&\le -M\rho_n q_n + \frac{(\Delta\rho_n)^2}{4\rho_n} \quad , \quad n \ge n_3
\end{aligned}$$

summing both sides of the above inequality from n_3 to $m \ge n_3$, we have

$$w_{m+1} - w_{n_3} \le -\sum_{n=n_3}^{m}\left(M\rho_n q_n - \frac{(\Delta\rho_n)^2}{4\rho_n}\right) \to -\infty \quad \text{as } m \to \infty$$

using equation (4.1.3).This contradicts the fact that $w_m > 0$ for $m \ge n_3$, so every solution of equation (4.1.1) is oscillatory and the proof is complete. □

Theorem 4.1.2: Suppose that condition (4.1.2) holds with $\gamma = 1$ and $k \ge 1$. If

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} (i-k) q_i > \frac{2}{M}\left(\frac{k}{k+1}\right)^{k+1} \tag{4.1.12}$$

Then every solution of equation (4.1.1) is oscillatory.

Proof: Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of equation (4.1.1). Proceeding as in the proof of theorem 4.1.1, we have that the inequalities in (4.1.6) holds for $n \ge n_0 > 0$. From equation (4.1.1) and condition (4.1.2) we have

$$\begin{aligned}
\Delta^2 z_n &= -q_n\left(\frac{G(x_{n-k})}{z_{n-k}}\right) z_{n-k} \\
&\le -q_n\left(\frac{G(z_{n-k})}{z_{n-k}}\right) z_{n-k}
\end{aligned}$$

$$\leq -Mq_n z_{n-k} \ , \text{ for } n \geq n_0 \tag{4.1.13}$$

using inequality (4.1.11) we get

$$\Delta^2 z_n \leq -Mq_n\left(\frac{n-k}{2}\right)(\Delta z_{n-k}) \ , \ n \geq n_1 \geq n_0$$

where n_1 is sufficiently large.

Set

$$0 < \Delta z_n = \xi_n \ , \ n \geq n_1$$

$$\Delta\xi_n + Mq_n\left(\frac{n-k}{2}\right)(\xi_{n-k}) \leq 0 \ , \text{ for } n \geq n_1 \tag{4.1.14}$$

by lemma 1.4.3 inequality (4.1.14) has no eventually positive solution, which is a contradiction. □

Remark 4.1.1: The oscillation of all solutions of equation (4.1.1) with condition (4.1.2) was proved in [13] for

(i) $0 < \gamma < 1$, provided that $\sum_{n=1}^{\infty} n^{\gamma} q_n = \infty$

(ii) $\gamma > 1$, provided that there exists a sequence $\{\rho_n\}$ such that $\rho_n > 0$, $\Delta\rho_n > 0$, $\Delta^2\rho_n \leq 0$ and $\sum_{n=1}^{\infty} \rho_n q_n = \infty$

Theorem 4.1.3: Suppose that G is monotone nondecreasing with $xG(x) > 0$ for $x \neq 0$, if inequalities in (4.1.6) hold and

$$\sum_{n=1}^{\infty} q_n = \infty \ ,$$

Then every solution of equation (4.1.1) is oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.1.1). There exists a sufficiently large positive integer n_0 such that the inequalities in (4.1.6) are satisfied for $n \geq n_0$, so we can find an integer $n_1 \geq n_0$ and a constant $c > 0$ such that

$$x_{n-k} \geq c \ , \text{ for } n \geq n_1$$

then equation (4.1.12) becomes

$$\Delta^2 z_n + q_n G(c) \leq 0 \ , \text{ for } n \geq n_1$$

and then

$$\sum_{n=n_1}^{L} \Delta^2 z_n \leq -G(c) \sum_{n=n_1}^{L} q_n \ , \ L \geq n_1$$

hence

$$\Delta z_{L+1} - \Delta z_{n_1} \leq -G(c) \sum_{n=n_1}^{L} q_n \ ,$$

as $L \to \infty$ then $\lim_{L\to\infty} \Delta z_{L+1} = -\infty$, which is a contradiction. □

Remark 4.1.2: The result in theorems 4.1.1 - 4.1.3 can be found in [13] but for the delay difference equations, we make some modifications on the conditions, and the proof so as to apply these theorems on the neutral delay difference equation (4.1.1).

Example 4.1.1: Consider the NDE

$$\Delta^2(x_n + px_{n-m}) + \frac{m}{n^2}x_{n-k} = 0 \quad , \tag{4.1.15}$$

with $\rho_n = n \in \mathbf{N} - \{0\}$, $k \in \mathbf{N}$

It is clear that $xG(x) = x^2 > 0$ and

$$G(x)\operatorname{sgn} x = x\operatorname{sgn} x \ge M|x| \quad , \quad x \ne 0$$

taking $M = 1$, we have

$$\sum_{n=1}^{\infty}\left(M\rho_n q_n - \frac{(\Delta\rho_n)^2}{4\rho_n}\right) = \sum_{n=1}^{\infty}\left(n\left(\frac{m}{n^2}\right) - \frac{1}{4n}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{m}{n} - \frac{1}{4n}\right) = \infty \quad \text{iff } m > \frac{1}{4}$$

Hence using theorem 4.1.1 we have that every solution of equation (4.1.15) is oscillatory provided that $m > \frac{1}{4}$.

Example 4.1.2: Consider the NDE

$$\Delta^2\left(x_n + \frac{5}{3}x_{n-1}\right) + \left(\frac{4n^2 + 8n + 6}{(n-2)^4}\right)(x_{n-2})^4 \operatorname{sgn} x_{n-2} = 0 \quad , \quad n > 2 \tag{4.1.16}$$

taking $\rho_n = n$

condition (4.1.2) is already satisfied for $M = 1$. Now

$$\sum_{n=3}^{\infty}\rho_n q_n = \sum_{n=3}^{\infty} n\left(\frac{4n^2 + 8n + 6}{(n-2)^4}\right) = \infty$$

so using remark 4.1.1(ii) equation (4.1.16) is oscillatory.

Example 4.1.3: Consider the NDE

$$\Delta^2\left(x_n + \frac{1}{3}x_{n-1}\right) + (n-2)^{\frac{3}{2}}\left(\frac{1}{(n+2)^3} + \frac{2}{(n+1)^3} + \frac{1}{n^3}\right)|x_{n-2}|^{\frac{1}{2}} = 0 \quad , \quad n > 2 \tag{4.1.17}$$

clearly that condition (4.1.2) is satisfied with $0 < \gamma < 1$. Now

$$\sum_{n=3}^{\infty} n^{\frac{1}{2}}\left[(n-2)^{\frac{3}{2}}\left(\frac{1}{(n+2)^3} + \frac{2}{(n+1)^3} + \frac{1}{n^3}\right)\right] = \infty$$

Since $(n-2)^{\frac{3}{2}}\left(\frac{1}{(n+2)^3} + \frac{2}{(n+1)^3} + \frac{1}{n^3}\right) \ge \frac{4(n-2)^{\frac{3}{2}}}{(n+2)^3} \quad , \quad n > 2$

Then every solution of equation (4.1.17) is oscillatory by remark 4.1.1(i).

Remark 4.1.3: The results of this section are mainly referred to [13].

4.2 Oscillation criteria of second-order forced NDE's with both constant and variable coefficients

Consider the second-order NDE of the form

$$\Delta^2(x_n + px_{n-m}) + q_n G(x_{n-k}) = f_n \quad , \quad n \ge 0 \tag{4.2.1}$$

where p is a positive real number, $\{q_n\}$ and $\{f_n\}$ are sequences of real numbers with $q_n \geq 0$ for all $n \in \mathbf{N}$ and $G \in C(\mathbf{R},\mathbf{R})$ with $uG(u) > 0$ for $u \neq 0$, m,k are nonnegative integers and $r = \max\{m,k\}$.

Authors in [25] established sufficient conditions for the oscillation of equation (4.2.1) for the special case when p is identically one and other cases. However, more general case was studied by Zafer in [28] when $0 \leq p_n < 1$ with different sufficient conditions. Also in [10] Grace and Lalli treated equation (4.2.1) but for the case when p is nonnegative real number.

We need the following conditions:

(C_1) G is nondecreasing and there exists $K > 0$ such that

$$|G(uv)| \geq K|G(u)||G(v)| \text{ , for all } u,v \in \mathbf{R}$$

and

$$\int_0^{\pm c} \frac{ds}{G(s)} < \infty \text{ , for all } c > 0$$

(C_2) There exists a real sequence $\{h_n\}$ such that

$$\Delta^2 h_n = f_n \text{ with } h_n \text{ is } m \text{ periodic}$$

(C_3) There exists $\gamma > 0$ such that $\dfrac{G(u)}{u} \geq \gamma > 0$ for $u \neq 0$.

(C_4) $\displaystyle\sum_{n=n_0}^{\infty} q_n = \infty$

Theorem 4.2.1: If $p_n = p = 1$ and the conditions (C_1)-(C_4) hold then all the solutions of equation (4.2.1) are oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.2.1) with $x_n > 0$, $x_{n-m} > 0$ and $x_{n-k} > 0$ for all $n \geq n_1 \geq n_0$. Since $\{h_n\}$ is periodic, there is a real number w such that the sequence $\{h_n - w\}$ is oscillatory. For $n \geq n_1$ let $z_n = x_n + x_{n-m} - (h_n - w)$. Then

$$\Delta^2 z_n = -q_n G(x_{n-k}) \leq 0 \quad , \tag{4.2.2}$$

and so $\{z_n\}$ is monotonic, so either $z_n < 0$ eventually or $z_n > 0$ eventually.

If $z_n < 0$ eventually then $0 < x_n < h_n - w$ for large n, which is impossible. Thus $z_n > 0$ for $n \geq n_2$ for some $n_2 \geq n_1$, by lemma 1.4.1, we have $\Delta z_n > 0$ for $n \geq n_2$. Taking the summation to equation (4.2.2) and applying (C_3), we obtain

$$\begin{aligned} \Delta z_{n_2} &= \sum_{s=n_2}^{n-1} q_s G(x_{s-k}) + \Delta z_n \\ &> \sum_{s=n_2}^{n-1} q_s G(x_{s-k}) > \gamma \sum_{s=n_2}^{n-1} q_s x_{s-k} \end{aligned} \tag{4.2.3}$$

hence

$$\sum_{s=n_2}^{\infty} q_s x_{s-k} < \infty \quad , \tag{4.2.4}$$

Now, from theorem 1.4.1 $\Delta z_n > 0$ for $n \geq n_2$. This means that for $n \geq n_2$

$$z_n - z_{n-m} = x_n - x_{n-2m} - (h_n - h_{n-m}) \quad , \tag{4.2.5}$$

but h_n is m periodic, this yields

$$z_n - z_{n-m} = x_n - x_{n-2m} > 0 \tag{4.2.6}$$

or $x_n > x_{n-2m}$ for $n \geq n_2$. Therefore

$$\liminf_{n\to\infty} x_n > 0 \quad \text{and so} \quad \sum_{s=n_2}^{\infty} q_s < \infty$$

Which contradicts (C_4). □

Consider the second-order NDE

$$\Delta^2 \left(x_n + p x_{\tau_n}\right) + q_n G\left(n, x_n, x_{\sigma_n}\right) = f_n \quad , \quad n \geq 0 \tag{4.2.7}$$

where $\sigma, \tau : \mathbf{N} \to \mathbf{N}$, $\sigma_n \leq n$, $\tau_n \leq n$ and $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \tau_n = \infty$, $f : \mathbf{N} \to \mathbf{R}$. $G : \mathbf{N} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that $yG(n, x, y) > 0$ for $xy > 0$ and $n \in \mathbf{N}(s)$, $q_n = 1$ and $0 \leq p < 1$.

Theorem 4.2.2:

Suppose that :

I. ϕ_n is nonnegative function on I and $w(x) > 0$ for $x > 0$ is continuous and nondecreasing on $\mathbf{R}_+$ such that

$$\left|G(n, x, y)\right| \geq \phi_n w\left(\frac{|y|}{\sigma_n}\right) ,$$

and

$$\int_0^{\pm a} \frac{dx}{w(x)} < \infty \quad , \text{ for any } a > 0$$

II. h_n is an oscillatory function such that $\Delta^2 h_n = f_n$ and $\lim_{n\to\infty} h_n = 0$

III. $\sum_{n=0}^{\infty} \phi_n = \infty$,

Then every solution of equation (4.2.7) is oscillatory.

Proof: Let x_n be an eventually positive solution of equation (4.2.7), define $z_n = x_n + p x_{\tau_n}$ and $y_n = z_n - h_n$. It follows that z_n, y_n and $-\Delta^2 y_n$ are eventually positive, by Theorem 1.4.1 , we have that Δy_n is eventually positive and , since $\lim_{n\to\infty} \sigma_n = \infty$, it is clear that we can find a natural number n_0 such that $x_{\sigma_n} > 0$ on $\mathbf{N}(n_0)$, substituting y_n in equation (4.2.7) we have

$$\Delta^2 y_n = -G\left(n, x_n, x_{\sigma_n}\right) < 0 \ , \tag{4.2.8}$$

for $n \in \mathbf{N}(n_0)$, let $\{n_k\}$ be an increasing sequence in $\mathbf{N}$ so that $\lim_{k\to\infty} n_k = \infty$ and $\sigma_{n_k} \in \mathbf{N}(n_0)$ for $k \geq 1$. In view of corollary 1.4.1

$$y_{\sigma_{n_k}} > c_1\sigma_{n_k}\Delta y_{\sigma_{n_k}} \quad , \tag{4.2.9}$$

where c_1 is an appropriate constant, since $\sigma_n \le n$ and Δy_n is decreasing , it follows from inequality (4.2.9) that

$$y_{\sigma_{n_k}} > c_1\sigma_{n_k}\Delta y_{n_k} \quad , \tag{4.2.10}$$

on the other hand, since Δh_n is oscillatory function, then Δz_n must be eventually positive. In view of the fact that $x_n \le z_n$ and $\tau_n \le n$ using the increasing nature of z_n it follows that

$$z_n = x_n + px_{\tau_n} \le x_n + pz_n \quad ,$$

from which we obtain

$$x_n \ge (1-p)z_n \tag{4.2.11}$$

Since $\lim_{n\to\infty} h_n = 0$, there exists a $c_2 \in (0,1)$ such that for n sufficiently large

$$z_n \ge c_2 y_n \tag{4.2.12}$$

by inequality (4.2.11) we get

$$x_{\sigma_n} \ge c_2(1-p)y_{\sigma_n} \tag{4.2.13}$$

combining inequalities (4.2.10) and (4.2.13), we obtain

$$x_{\sigma_{n_k}} > c_1c_2(1-p)(\sigma_{n_k})\Delta y_{n_k}$$

let $c_1c_2(1-p) = c$ by condition (I) of this theorem we get

$$G(n_k, x_{n_k}, x_{\sigma_{n_k}}) \ge \phi_{n_k} w(c\Delta y_{n_k}) \tag{4.2.14}$$

If we set $v_n = c\Delta y_n$ it follows from inequalities (4.2.8) and (4.2.14) that

$$\frac{\Delta v_{n_k}}{w(v_{n_k})} + c\phi_{n_k} \le 0 \tag{4.2.15}$$

note that $v_n > 0$ is decreasing and by using the nondecreasing nature of $w(x)$, then

$$w(s) \le w(v_{n_k}) \quad \text{for} \quad v_{n_{k+1}} \le s \le v_{n_k}$$

In view of the last observation, we find

$$\int_{v_{n_{k+1}}}^{v_{n_k}} \frac{ds}{w(s)} \ge \frac{v_{n_k} - v_{n_{k+1}}}{w(v_{n_k})} \ge \frac{v_{n_k} - v_{n_k+1}}{w(v_{n_k})} = \frac{-\Delta v_{n_k}}{w(v_{n_k})} \tag{4.2.16}$$

Using (4.2.16) and (4.2.15) and summing both sides of the resulting inequality from $k=1$ to $k = K$ we have

$$-\int_{v_{n_1}}^{v_{n_{K+1}}} \frac{ds}{w(s)} + c\sum_{k=1}^{K}\phi_{n_k} \le 0 \tag{4.2.17}$$

let $\lim_{n\to\infty} v_n = L$. It is clear that the limit L exists and is nonnegative. If $L \ne 0$ then inequality (4.2.17) implies

$$\sum_{k=1}^{\infty}\phi_{n_k} < \infty$$

We get a contradiction with condition (III) , when $L = 0$ making use of the condition that $\int_0^{\pm a}\frac{dx}{w(x)} < \infty$ for any $a > 0$, we again get a contradiction with condition (III) , the proof is complete. □

Consider the second-order NDE of the form

$$\Delta^2\left(x_n + px_{n-m}\right) + q_n G\left(x_{k_n}\right) = f_n \quad , \tag{4.2.18}$$

where p is a nonnegative real number, $k \geq 1$, the sequence $\{k_n\}$ is a sequence of nonnegative integers with $\lim_{n\to\infty} k_n = \infty$ and $\{f_n\}$ and $\{q_n\}$ are also sequences of real numbers with $q_n \geq 0$ eventually, the function G is a real valued function satisfying $xG(x) > 0$ for $x \neq 0$.

We need the following assumptions:

(A_1) $G(x) \geq Mx$, $x \neq 0$ M is a positive number

(A_2) There exists a sequence $\{h_n\}$ of real numbers such that

$$\Delta^2 h_n = f_n \quad \text{and } \{h_n\} \text{ is oscillatory}$$

(A_3) The sequence $\{h_n\}$ is periodic of period m

Theorem 4.2.3: In addition to the assumptions (A_1)-(A_3), assume that there exists a sequence $\{l_n\}$ of positive integers such that $\{n - l_n\}$ is increasing, $\lim_{n\to\infty}(n - l_n) = \infty$ and $k_n \geq n - l_n$ for $n \in \mathbf{N}$. If

$$\liminf_{n\to\infty} \frac{1}{l_n} \sum_{i=n-l_n}^{n-1} (i - l_i) q_i > \gamma^* \limsup_{n\to\infty} \frac{l_n^{l_n}}{(l_n + 1)^{l_n+1}} \quad , \tag{4.2.19}$$

then equation (4.2.18) is oscillatory provided that $0 \leq p < 1$, $\gamma^* = \dfrac{2}{M(1-p)}$.

Proof: Assume that equation (4.2.18) has an eventually positive bounded solution x_n, then there exists a positive integer n_0 such that

$$x_n > 0 \ , \ x_{n-m} > 0 \text{ and } x_{k_n} > 0 \text{ for } n \geq n_0 \tag{4.2.20}$$

define

$$y_n = x_n + px_{n-m} \text{ and } z_n = y_n - h_n \tag{4.2.21}$$

Then

$$\Delta^2 z_n = -q_n G\left(x_{k_n}\right) \leq 0 \ , \text{ for } n \geq n_0 \tag{4.2.22}$$

Which implies that $\{\Delta z_n\}$ is an eventually nonincreasing sequences, also $z_n > 0$ for $n \geq n_0$, we claim that $\Delta z_n > 0$ for $n \geq n_1$ for some $n_1 \geq n_0$. If not, then there exists an integer $n_2 \geq n_1$ such that $\Delta z_n \leq 0$ for $n \geq n_2$. Since q_n is not eventually zero there exists an integer $n_3 \geq n_2$ such that $\Delta^2 z_n < 0$ for $n \geq n_3$. It follows that

$$\Delta z_{n_3+1} < \Delta z_{n_3} \leq \Delta z_{n_2} \leq 0$$

Hence

$$\Delta z_n \leq \Delta z_{n_3+1} < 0$$

Summing both sides of the above inequality from $n_3 + 1$ to $n \geq n_3 + 1$, we have

$$z_{n+1} - z_{n_3+1} \leq (n - n_3)\Delta z_{n_3+1} \to -\infty \text{ as } n \to \infty,$$

which is a contradiction. Thus

$$\Delta^2 z_n \leq 0 \ , \ \Delta z_n > 0 \ , \ z_n > 0 \text{ and } x_{k_n} > 0 \ , \text{ for } n \geq n_3 \tag{4.2.23}$$

now using (A_1) in (4.2.22) we get

$$\Delta^2 z_n \le -Mq_n x_{k_n} \text{ , for } n \ge n_1 \tag{4.2.24}$$

in view of inequality (4.2.23) there exists an integer $N_0 \ge n_1$ such that

$$x_n + px_{n-m} = z_n + h_n$$
$$x_n = z_n + h_n - px_{n-m}$$
$$= z_n + h_n - p(z_{n-m} + h_{n-m} - px_{n-2m})$$

so

$$x_n \ge (1-p)(z_n + h_n) \text{ , for } n \ge n_1$$

and

$$x_n \ge (1-p)(z_n + h_{N_0}) = \xi_n \text{ , for } N_0 \ge n_2$$

It is clear that

$$\Delta z_n = \frac{1}{1-p}\Delta\xi_n \text{ , for } n \ge n_2$$

and

$$\xi_n = (1-p)(z_n + h_{N_0})$$
$$\ge (1-p)(z_{N_0} + h_{N_0}) \text{ , for } N_0 \ge n_2$$

by inequality (4.2.24) we obtain

$$\Delta^2 \xi_n \le -M(1-p)q_n \xi_{k_n} \text{ , for } n \ge N_0 \tag{4.2.25}$$

clearly that ξ_n satisfies (4.2.23) with z replaced by ξ. Hence there exists an integer $N_1 \ge N_0$ such that

$$\Delta^2 \xi_n \le -M(1-p)q_n \xi_{n-l_n} \text{ , for } n \ge N_1 \tag{4.2.26}$$

by corollary 1.4.1 there exists an integer $N_2 \ge N_1$ such that

$$\xi_n \ge \frac{n}{2}\Delta\xi_n \text{ , for } n \ge N_2 \tag{4.2.27}$$

and

$$\xi_{n-l_n} \ge \left(\frac{n-l_n}{2}\right)\Delta\xi_{n-l_n} \text{ , for } n \ge N_2 \tag{4.2.28}$$

set $\Delta\xi_n = \theta_n$ we get

$$\Delta\theta_n \le -M(1-p)\left(\frac{n-l_n}{2}\right)q_n\theta_{n-l_n} \text{ , for } n \ge N_2 \tag{4.2.29}$$

By lemma 1.4.3 , condition (4.2.19) implies that inequality (4.2.29) has no eventually positive solution which is a contradiction. □

The following discussion is for equation (4.2.18) when G is superlinear.

Theorem 4.2.4: Let conditions (A_2) and (A_3) hold, and suppose that $\sigma_n = \min\{n, k_n\}$ such that

$$\Delta\sigma_n > 0 \text{ and } \Delta^2\sigma_n \le 0 \text{ for } n \in \mathbf{N} \tag{4.2.30}$$

If

$$\sum_{n=n_0}^{\infty} \sigma_n q_n = \infty \quad , \tag{4.2.31}$$

then equation (4.2.18) is oscillatory provided that $0 \le p < 1$.

Proof: Proceeding as in the proof of theorem 4.2.3 we get (4.2.20), also we get (4.2.22) and (4.2.23) hold for $n \ge N_0$, and that ξ_n satisfies (4.2.23) with z replaced by ξ, so

$$\Delta^2 \xi_n \le -M(1-p)q_n G(\xi_{k_n}) \text{ , for } n \ge N_0$$

since G is superlinear and using conditions (4.2.30) and (4.2.23) in the above inequality we get

$$\Delta^2 \xi_n + M(1-p)q_n G(\xi_{\sigma_n}) \le 0 \text{ , for } n \ge N_1 \ge N_0 \tag{4.2.32}$$

set

$$w_n = \sigma_n \frac{\Delta \xi_n}{G(\xi_{\sigma_n})} \quad , \quad n \ge N_1$$

then

$$\Delta w_n = \sigma_{n+1} \frac{\Delta \xi_{n+1}}{G(\xi_{\sigma_{n+1}})} - \sigma_n \frac{\Delta \xi_n}{G(\xi_{\sigma_n})}$$

$$= \sigma_n \frac{\Delta^2 \xi_n}{G(\xi_{\sigma_n})} + \Delta \xi_{n+1} \left(\frac{\sigma_{n+1}}{G(\xi_{\sigma_{n+1}})} - \frac{\sigma_n}{G(\xi_{\sigma_n})} \right)$$

Using (4.2.23) and (4.2.30) along with the fact that the sequences $\{\xi_n\}$ and $\{\sigma_n\}$ are increasing, we get

$$\Delta w_n \le -\sigma_n q_n + \Delta \sigma_n \left(\frac{\Delta \xi_{\sigma_{n-1}}}{G(\xi_{\sigma_n})} \right) \text{ , for } n \ge N_2 \ge N_1 \tag{4.2.33}$$

Summing both sides of inequality from N_2 to $s \ge N_2$, we have

$$w_{n+1} - w_{N_2} \le -\sum_{n=N_2}^{s} \sigma_n q_n + \Delta \sigma_{N_2} \sum_{n=N_2}^{s} \frac{\Delta \xi_{\sigma_{n-1}}}{G(\xi_{\sigma_n})} \quad ,$$

by condition (4.2.31) , we have

$$w_{s+1} \to -\infty \text{ as } s \to \infty \text{ ,}$$

which is a contradiction. □

Theorem 4.2.5: Suppose that the function G is nondecreasing and that conditions (A_2) and (A_3) hold. If

$$\sum_{n=n_0}^{\infty} q_n = \infty \text{ ,} \tag{4.2.34}$$

holds, then equation (4.2.18) is oscillatory provided that $0 \le p < 1$.

Proof: We proceed as in the proof of theorem 4.2.3 to get (4.2.20), following the proof of theorem 4.2.4 to arrive at inequality (4.2.23). Choose an integer $N_1 \ge N_2$ and a positive number c such that

$$\xi_{k_n} > c \text{ , for } n \ge N_1 \tag{4.2.35}$$

using inequality (4.2.35) in inequality (4.2.32) to get

$$\Delta^2 \xi_n + M(1-p)q_n G(c) \le 0 \text{ , for } n \ge N_1$$

summing both sides of the above inequality from N_1 to $s > N_1$ and using condition (4.2.34) we have

$$0 < \Delta \xi_{s+1} \to -\infty \text{ as } s \to \infty$$

which is a contradiction. □

Remark 4.2.1: The case when ($p>1$) are considered to NDE

$$\Delta^2(x_n + px_{n+m}) + q_n G(x_{k_n}) = f_n \quad ,$$

in [10] using similar arguments used in the proof of theorems 4.2.3 , 4.2.4 and 4.2.5.

Example 4.2.1: Consider the NDE

$$\Delta^2(x_n + x_{n-2}) + 8x_{n-4} = 16(-1)^n \quad , \tag{4.2.36}$$

for $h_n = 4(-1)^n$ which is periodic, $\sum_{n=N_0}^{\infty} q_n = \infty$, and there exists $\gamma = 1 > 0$ such that

$$\frac{G(x)}{x} = \frac{x}{x} = 1 \geq \gamma > 0 \quad , \text{for } x \neq 0$$

by theorem 4.2.1 every solution of equation (4.2.36) is oscillatory.

Example 4.2.2: Consider the NDE

$$\Delta^2\left(x_n + \frac{1}{2}x_{\tau_n}\right) + q_n\left|x_{\sigma_n}\right|^{\alpha}\operatorname{sgn}\left(x_{\sigma_n}\right) = \frac{2(-1)^n}{n(n+2)} \quad , \tag{4.2.37}$$

$0 < \alpha < 1$, q_n is nonnegative and not identically zero

we have

$$G(n,x,y) = q_n|y|^{\alpha}\operatorname{sgn}(y) \quad ,$$

$$\phi_n = (1-p)^{\alpha}(\sigma_n)^{\alpha} q_n \quad \text{with } p = \frac{1}{2}$$

$$w(x) = x^{\alpha} \quad , \quad h_n = \frac{(-1)^n}{n}$$

Using theorem 4.2.2 every solution of equation (4.2.37) is oscillatory.

Example 4.2.3: Consider the NDE

$$\Delta^2\left(x_n + \frac{2}{3}x_{n-2}\right) + 2\left(\frac{16n-1}{n-3}\right)\left|x_{n-3}\right|\operatorname{sgn}\left(x_{n-3}\right) = 4(-1)^n \quad , \tag{4.2.38}$$

where $n > 3$, $m = 2$, $k_n = n-3$

hence $f_n = 4(-1)^n$ and $h_n = (-1)^n$ which is periodic of period 2, let $l_n = 3$,

then by theorem 4.2.3 or theorem 4.2.4 every solution of equation (4.2.38) is oscillatory.

It is clear that (A_1)-(A_3) are satisfied with $M = 1$, with

$$\liminf_{n\to\infty} \frac{1}{3}\sum_{i=n-3}^{n-1}(i-3)\left(2\frac{6i-1}{i-3}\right) > \gamma^{*}\limsup_{n\to\infty}\left(\frac{3^3}{4^4}\right)$$

Remark 4.2.2: The results of this section are mainly referred to [10], [25] and [28].

4.3 Oscillation criteria of second-order NDE's with variable coefficients

Consider the second-order NDE of the form

$$\Delta^2(x_n + p_n x_{n-m}) + q_n G(x_{n-k}) = 0 \quad , \quad n \geq 0 \tag{4.3.1}$$

where p_n, q_n are sequences of real numbers with $q_n \ge 0$, $G \in C(\mathbf{R},\mathbf{R})$ is nondecreasing and satisfies $uG(u) > 0$ for $u \ne 0$, $m > 0$ and $k \ge 0$ are integers.

Many authors studied the oscillation of solutions of equation (4.3.1) considering various ranges of p_n. For example, in [19] Parhi and Tripathy established many sufficient conditions for the oscillation of equation (4.3.1). However, a different approach was achieved by Agarwal, Manuel and Thandapani depending on dividing the set S of all nontrivial solutions into four classes and trying to find under what conditions those classes are empty, as we will see later.

Lemma 4.3.1: Let x_n be a real-valued function on $\mathbf{N}(n_0)$ such that $x_n > 0$ and $\Delta^m x_n \le 0$ on $\mathbf{N}(n_0)$ and $\Delta^m x_n \not\equiv 0$, where $m \ge 2$. If $\Delta x_n > 0$ and $\Delta^{m-1} x_n > 0$, then there exists $n_2 > 2n_1$, where n_1 is a large number in $\mathbf{N}(n_0)$, such that $x_n > \dfrac{\Delta^{m-1} x_n}{(m-1)!}$ for $n \ge n_3$, where $n_3 > \max\{2^{m-1} n_2, N\}$ and $N > 0$ is a large integer such that $\left(2^{1-m} n\right)^{(m-1)} > 1$ for $n \ge N$.

Proof: From corollary 1.4.1

$$x_n \ge \frac{1}{(m-1)!}\left(\frac{n}{2}\right)^{(m-1)} \Delta^{m-1} x_{2^{m-2} n}$$

for $n \ge 2n_1$, where n_1 is a large number in $\mathbf{N}(n_0)$. Since x_n is increasing, then, for $n \ge 2^{m-1} n_1$

$$x_n \ge x_{2^{2-m} n} \ge \frac{1}{(m-1)!}\left(2^{1-m} n\right)^{(m-1)} \Delta^{m-1} x_n$$

Hence, for $n \ge n_3$

$$x_n > \frac{1}{(m-1)!} \Delta^{m-1} x_n$$

Thus the lemma is proved. □

We need the following conditions:

(C_1) For $u > 0$ there exists $\lambda > 0$ such that $G(u) \ge \lambda u$.

For $u < 0$, there exists $\lambda > 0$ such that $G(u) \le \lambda u$

(C_2) For $u > 0$ and $v > 0$, there exists $\lambda > 0$ such that $G(u+v) \le \lambda(G(u) + G(v))$

(C_3) For $u < 0$ and $v < 0$, there exists $\lambda > 0$ such that $G(u+v) \ge \lambda(G(u) + G(v))$

(C_4) $\displaystyle\int_0^{\pm c} \frac{du}{G(u)} < \infty$ for every $c > 0$

(C_5) $\displaystyle\sum_{n=m}^{\infty} q_n^* = \infty$, where $q_n^* = \min\{q_n, q_{n-m}\}$

(C_6) For $u > 0$ and $v > 0$, $G(uv) \le G(u)G(v)$

(C_7) $G(-u) = -G(u)$, $u \in \mathbf{R}$

(C_8) For $u, v \in \mathbf{R}$, $G(uv) = G(u)G(v)$

Theorem 4.3.1: Let $-\infty < p_1 \le p_n \le 0$ and $k \ge m$. If (C_1) holds and if

$$\liminf_{n\to\infty}\sum_{i=n-k+m}^{n-1} q_i > \max\left\{\lambda^{-1}\left(\frac{k}{k+1}\right)^{k+1}, -p_1\lambda^{-1}\left(\frac{k-m}{k-m+1}\right)^{k-m+1}\right\},$$

then every solution of equation (4.3.1) is oscillatory.

Proof: Let x_n be an eventually positive solution of equation (4.3.1), $x_n > 0$ for $n > n_0$. Setting

$$z_n = x_n + p_n x_{n-m} \tag{4.3.2}$$

we get

$$\Delta^2 z_n + q_n G(x_{n-k}) = 0 \tag{4.3.3}$$

let $r = \max\{m,k\}$, $z_n \le x_n$ and $\Delta^2 z_n \le 0$ for $n \ge n_0 + r$, either $z_n > 0$ or $z_n < 0$.
If $z_n > 0$, the inequalities in theorem 1.4.1 hold. Then $z_n > \Delta z_n$ for $n \ge n_2 > n_1$ by lemma 4.3.1. Hence, using (C_1) equation (4.3.3) gives

$$\Delta^2 z_n + \lambda q_n \Delta z_{n-k} \le 0,$$

that is the last inequality admits a positive solution $\{\Delta z_n\}$, a contradiction in view of lemma 1.4.3. Hence $z_n < 0$ for $n \ge n_1$, setting $y_n = -z_n$ for $n \ge n_1$, we have $y_n > 0$, $y_n < -p_1 x_{n-m}$, and

$$\Delta^2 y_n - q_n G(x_{n-k}) = 0 \tag{4.3.4}$$

Hence $\Delta^2 y_n \ge 0$ for $n \ge n_1$.
By theorem 1.4.1

$$-\Delta y_n > 0$$

so equation (4.3.4) yields that y_n is a positive solution of

$$\Delta y_n - \frac{\lambda}{p_1} q_n y_{n+m-k} \le 0,$$

which is a contradiction by lemma 1.4.3, so every solution of equation (4.3.1) is oscillatory. □

Theorem 4.3.2: Let $-\infty < p_1 \le p_n \le -1$. If

$$\sum_{n=0}^{\infty} q_n = \infty, \tag{4.3.5}$$

then every solution of equation (4.3.1) is oscillatory.

Proof: Let x_n be an eventually positive solution of equation (4.3.1), $x_n > 0$ for $n \ge n_0 > 0$, let z_n as in (4.3.2) we get (4.3.3), for $r = \max\{m,k\}$ we have

$$z_n < x_n \text{ and } \Delta^2 z_n \le 0 \text{ for } n \ge n_0 + r,$$

let $z_n > 0$ for $n \ge n_1 \ge n_0 + r$, then theorem 1.4.1, $\Delta z_n > 0$ for $n \ge n_2 \ge n_1$. Since z_n is increasing, $z_n > M > 0$ for $n \ge n_2$, from equation (4.3.3) we obtain

$$\Delta^2 z_n + q_n G(M) \le 0, \text{ for } n \ge n_3 > n_2 + r$$

hence $\sum_{n=n_3}^{\infty} q_n < \infty$, because $\Delta z_n > 0$, which is a contradiction.

Let $z_n < 0$ for $n \geq n_1$ .Set $y_n = -z_n$ we get $y_n > 0$, $y_n < -p_1 x_{n-m}$, and $\Delta^2 y_n - q_n G(x_{n-k}) = 0$ for $n \geq n_1$, by lemma 1.4.1 we have $-\Delta y_n > 0$ for $n \geq n_2 > n_1$. As in the proof of theorem 4.3.1 we proceed until we reach to

$$\Delta y_n + q_n G(x_{n-k}) \leq 0$$

Thus

$$\liminf_{n\to\infty} x_n = 0 \quad ,$$

Which is a contradiction, so equation (4.3.1) must have an oscillating solution. □

Theorem 4.3.3: Let $-\infty < p_1 \leq p_n \leq p_3 \leq -1$. If condition (4.3.5) holds, then every solution of equation (4.3.1) is oscillatory.

Proof: Let x_n be an eventually positive solution of equation (4.3.1) then $x_n > 0$ for $n \geq n_0 > 0$. Proceeding as in the proof of theorem 4.3.2 we obtain $\liminf_{n\to\infty} x_n = 0$.

We consider the case $y_n > 0$, and $-\Delta y_n > 0$, hence $\lim_{n\to\infty} y_n = \alpha$, $0 \leq \alpha < \infty$. If $\alpha > 0$ then $y_n > \beta > 0$ for $n \geq n_4 > n_3$.

Then

$$\Delta y_n + q_n G(x_{n-k}) = 0 \text{ and } x_{n-k} > -\frac{y_{n+m-k}}{p_1} \quad ,$$

this yields $\sum_{n=n_4}^{\infty} q_n < \infty$, a contradiction, hence $\alpha = 0$ and $\lim_{n\to\infty} z_n = 0$

but

$$p_3 x_{n-m} \geq p_n x_{n-m} = z_n - x_n \quad ,$$

then

$$\liminf_{n\to\infty}(p_3 x_{n-m}) \geq \liminf_{n\to\infty}(z_n - x_n) \geq \lim_{n\to\infty} z_n - \limsup_{n\to\infty} x_n \quad ,$$

that is

$$(1 + p_3)\limsup_{n\to\infty} x_n \geq 0 \quad ,$$

that is

$$\limsup_{n\to\infty} x_n = 0 \quad ,$$

hence

$$\lim_{n\to\infty} x_n = 0 \quad ,$$

which is a contradiction, equation (4.3.1) is oscillatory. □

Theorem 4.3.4: Let $0 \leq p_n \leq 1$ and $k \geq m$. If (C_2)-(C_5) hold, then every solution of equation (4.3.1) oscillates.

Proof: Let x_n be an eventually positive solution of equation (4.3.1), then $x_n > 0$ for $n \geq n_0 > 0$, setting z_n as in equation (4.3.2) we get equation (4.3.3) and $0 < z_n \leq x_n + x_{n-m}$ for $n \geq n_0 + r$ ($r = \max\{m, k\}$). Hence $\Delta^2 z_n \leq 0$ for $n \geq n_0 + r$, using theorem 1.4.1 we have $\Delta z_n > 0$ for $n \geq n_1 \geq n_0 + r$. Hence $z_n > \Delta z_n$ for $n \geq n_2 > n_1$ by lemma 4.3.1. From equation (4.3.3) and using (C_2) we have, for $n \geq n_3 > n_2 + r$

$$\begin{aligned} 0 &= \Delta^2 z_n + q_n G(x_{n-k}) + \Delta^2 z_{n-m} + q_{n-m} G(x_{n-m-k}) \\ &\geq \Delta^2 z_n + \Delta^2 z_{n-m} + q_n^* [G(x_{n-k}) + G(x_{n-m-k})] \\ &\geq \Delta^2 z_n + \Delta^2 z_{n-m} + \lambda^{-1} q_n^* G(x_{n-k} + x_{n-m-k}) \\ &\geq \Delta^2 z_n + \Delta^2 z_{n-m} + \lambda^{-1} q_n^* G(z_{n-k}) \\ &\geq \Delta^2 z_n + \Delta^2 z_{n-m} + \lambda^{-1} q_n^* G(\Delta z_{n-k}) \end{aligned}$$

As $\Delta z_n > 0$, then the above inequality yields

$$\lambda^{-1} q_n^* \leq -\frac{\Delta^2 z_n}{G(\Delta z_{n-k})} - \frac{\Delta^2 z_{n-m}}{G(\Delta z_{n-k})}$$

Since Δz_n is decreasing and $k \geq m$, then

$$\begin{aligned} \lambda^{-1} q_n^* &\leq -\frac{\Delta^2 z_n}{G(\Delta z_n)} - \frac{\Delta^2 z_{n-m}}{G(\Delta z_{n-m})} \\ &\leq \int_{\Delta z_{n+1}}^{\Delta z_n} \frac{du}{G(u)} + \int_{\Delta z_{n+1-m}}^{\Delta z_{n-m}} \frac{du}{G(u)} \end{aligned}$$

Hence

$$\lambda^{-1} \sum_{j=n_3}^{t-1} q_j^* \leq \int_{\Delta z_t}^{\Delta z_{n_3}} \frac{du}{G(u)} + \int_{\Delta z_{t-m}}^{\Delta z_{n_3-m}} \frac{du}{G(u)} \quad ,$$

that is $\sum_{j=n_3}^{t-1} q_j^* < \infty$, since $\lim_{t\to\infty} \Delta z_t$ exists and greater or equals zero. This contradicts (C_5), so every solution of equation (4.3.1) is oscillatory. □

Theorem 4.3.5: Let $0 \leq p_n \leq p_2 < \infty$ and $k \geq m$. If (C_2), (C_4), (C_6) and (C_7) hold, and if

$$\sum_{n=m}^{\infty} Q_n = \infty \,, \tag{4.3.6}$$

where $Q_n = \min\left\{ q_n, \frac{q_{n-m}}{G(p_2)} \right\}$,

then every solution of equation (4.3.1) oscillates.

Proof: The proof is similar to that of theorem (4.3.4), using (C_2) and (C_6) we get

$$0 \geq \Delta^2 z_n + \Delta^2 z_{n-m} + \lambda^{-1} Q_n G(\Delta z_{n-k})$$

using (C_4) and proceeding as in theorem 4.3.4 yields $\sum_{n=n_3}^{\infty} Q_n < \infty$, a contradiction, and the proof is complete. □

Theorem 4.3.6: Let $-\infty < p_1 \leq p_n \leq 0$, and $k \geq m$. If (C_4) and (C_5) hold, then every solution of equation (4.3.1) oscillates.

Proof: Let x_n be an eventually positive solution of equation (4.3.1), then $x_n > 0$ for $n \geq n_0$. Setting z_n as in equation (4.3.2) we get equation (4.3.3) and $z_n \leq x_n$ for $n \geq n_0 + r$ ($r = \max\{m, k\}$), either $z_n > 0$ or $z_n < 0$ for $n \geq n_1 > n_0 + r$.

Let $z_n > 0$ for $n \ge n_1$, then by theorem 1.4.1 $\Delta z_n > 0$, for $n \ge n_2 > n_1$, hence $z_n > \Delta z_n$ for $n \ge n_3 > n_2$, and we get

$$\Delta^2 z_n + q_n G(\Delta z_{n-k}) \le 0 \quad ,$$

since $\Delta z_n > 0$ and decreasing, then

$$q_n \le -\frac{\Delta^2 z_n}{G(\Delta z_{n-k})} \le \int_{\Delta z_{n+1}}^{\Delta z_n} \frac{du}{G(u)}$$

Hence

$$\sum_{n=n_3}^{t-1} q_n \le \int_{\Delta z_t}^{\Delta z_{n3}} \frac{du}{G(u)}$$

This leads to a contradiction due to (C_4) and (C_5) so $z_n < 0$ for $n \ge n_1$. Set $y_n = -z_n$, we have $y_n > 0$, $y_n < -p_1 x_{n-m}$, we get also equation (4.3.4), by theorem 1.4.1 $-\Delta y_n > 0$ and equation (4.3.4) yields

$$\Delta y_n + q_n G(x_{n-k}) \le 0 \quad ,$$

that is

$$\Delta y_n + q_n G\left(-\frac{y_n}{p_1}\right) \le 0 \quad ,$$

using (C_4) and (C_5) we get a contradiction which completes the proof. □

Theorem 4.3.7: Let p_n be allowed to change sign with $-\infty < p_1 \le p_n \le p_2 < \infty$, where $p_2 > 0$ and $p_1 < 0$ are constants. Let $k \ge m$ and (C_2), (C_4), (C_6) and (C_7) hold. Suppose that equation (4.3.6) holds if $G(p_2) \ge 1$ or (C_5) holds if $G(p_2) < 1$. Then every solution of equation (4.3.1) oscillates.

Proof: Let $x_n > 0$ for $n \ge n_0$ an eventually positive solution of equation (4.3.1), setting z_n as in equation (4.3.2) to get equation (4.3.3).
Let $G(p_2) \ge 1$, then equation (4.3.6) holds and hence (C_5) holds because $Q_n \le q_n^*$, since $\Delta z_n \le 0$ for $n \ge n_0 + r$, then $z_n > 0$ or $z_n < 0$ for $n \ge n_0 + r$. Let $z_n > 0$ for $n \ge n_1$, we have

$$z_n \le x_n + p_2 x_{n-m}$$

Proceeding as in the proof of theorem 4.3.5 we obtain a contradiction. Hence $z_n < 0$ for $n \ge n_1$. Putting $y_n = -z_n$ we obtain $y_n > 0$, $y_n < -p_1 x_{n-m}$ and equation (4.3.4) holds, and we obtain a contradiction as in the proof of the second half of theorem 4.3.6 .
Let $G(p_2) < 1$, then (C_5) holds and hence equation (4.3.6) holds because $Q_n \ge q_n^*$. If $z_n > 0$ for $n \ge n_1$ then continue as in the proof of theorem 4.3.5 to get a contradiction. If $z_n < 0$ for $n \ge n_1$ then continue as in the proof of the second half of theorem 4.3.6 to arrive a contradiction and the proof is complete. □

Remark 4.3.1: Results for the oscillation when $0 \le p_n < 1$ and $p_n \ge 0$ positive real numbers was achieved by Zafer in [27], those results were applicable to the form

$$\Delta^2\left(x_n + p_n x_{\tau_n}\right) + G\left(n, x_{\sigma_n}\right) = 0 \quad ,$$

provided that

$$\sum_{n=0}^{\infty} G\left(n,\left(1-p(\sigma_n)c\right)\right)=\infty \ , \ c>0 .$$

Consider the second order NDE

$$\Delta\left(c_n\Delta\left(x_n+p_n x_{n-m}\right)\right)+q_{n+1}G\left(x_{n+1-k}\right)=0 \quad , \quad n\ge 0 \tag{4.3.7}$$

where m,k are fixed nonnegative integers, $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are sequences of real numbers, and the following three conditions hold:

(i) $c_n>0$, for all $n\in\mathbf{Z}$ and q_n is not identically zero for large n

(ii) $G\in(\mathbf{R},\mathbf{R})$ with $uG(u)>0$ for $u\ne 0$

(iii) there exists a nonnegative function g such that

$$G(u)-G(v)=g(u,v)(u-v) \ , \quad \text{for all } u\ne v$$

Let S denote the set of all nontrivial solutions of equation (4.3.7) , S can be divided into:

$M^+=\{\{x_n\}\in S:$ there exists an integer $N\in\mathbf{Z}$ such that $x_n\Delta x_n\ge 0$ for all $n\ge N$ $\}$

$M^-=\{\{x_n\}\in S:\{x_n\}$ is nonoscillatory and there exists an integer $N\in\mathbf{Z}$ such that $x_n\Delta x_n\le 0$ for all $n\ge N$ $\}$

$OS=\{\{x_n\}\in S:$ for every integer $N\in\mathbf{Z}$, there exists $n\ge N$ such that $x_n x_{n+1}\le 0$ for all $n\ge N$ $\}$

$WOS=\{\{x_n\}\in S:$ $\{x_n\}$ is nonoscillatory and for every $N\in\mathbf{Z}$ there exists $n\ge N$, such that $\Delta x_n\Delta x_{n+1}\le 0$ $\}$

Theorem 4.3.8: Assume that with respect to equation (4.3.7) the following conditions hold:

(1) $m\ge 1$ and $-1\le p_n\le 0$,

(2) $q_n\ge 0$ for all $n\ge n_0\in\mathbf{Z}$,

(3) $\lim_{n\to\infty}\sum_{s=n_0}^{n-1} q_{s+1}=\infty$,

(4) $\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)=\infty$, then $M^+=\phi$

Proof : Let equation (4.3.7) has a solution $\{x_n\}\in M^+$, such that $x_n>0$, $\Delta x_n\ge 0$, $x_{n-r}>0$, $\Delta x_{n-r}>0$ for all $n\ge n_1>n_0+r$, $r=\max\{k,m\}$, let $z_n=x_n+p_n x_{n-m}$, then by condition (3) and the fact that $\{x_n\}\in M^+$, we have

$$z_n>x_{n-m}+p_n x_{n-m}\ge 0 \text{ , for all } n\ge n_1$$

but equation (4.3.7) becomes

$$\Delta\left(c_n\Delta z_n\right)=-q_{n+1}G\left(x_{n+1-k}\right) \ , \ n\ge n_1$$

the $\{c_n\Delta z_n\}$ is nonincreasing for all $n\ge n_1$, to prove that $c_n\Delta z_n>0$, suppose it is not, that is, $c_n\Delta z_n<0$ for all $n\ge n_1$, then there exists an integer $n_2>n_1$ such that

$$c_n\Delta z_n < c_{n_2}\Delta z_{n_2} < 0 \ , \ \text{for all } n \ge n_2$$

taking the summation from n_2 to $n-1 \ge n_2$ we have

$$z_n - z_{n_2} \le \sum_{s=n_2}^{n-1} \frac{c_{n_2}\Delta z_{n_2}}{c_s} \ ,$$

which implies that that $z_n \to -\infty$ as $n \to \infty$, contradiction, thus $c_n\Delta z_n \ge 0$, now by equation (4.3.7)

$$\frac{c_n\Delta z_n}{G(x_{n-k})} - \frac{c_{n_1}\Delta z_{n_1}}{G(x_{n_1-k})} + \sum_{s=n_1}^{n-1} \frac{c_s\Delta z_s\, g(x_{s+1-k}, x_{s-k})\Delta x_{s-k}}{G(x_{s+1-k})G(x_{s-k})} = -\sum_{s=n_1}^{n-1} q_{s+1}$$

and hence

$$\frac{c_n\Delta z_n}{G(x_{n-k})} - \frac{c_{n_1}\Delta z_{n_1}}{G(x_{n_1-k})} \le -\sum_{s=n_1}^{n-1} q_{s+1} \ , \ n \ge n_1$$

using condition (3), we find

$$\lim_{n\to\infty} \frac{c_n\Delta z_n}{G(x_{n-k})} = -\infty \ ,$$

which is a contradiction to the assumption that $c_n\Delta z_n \ge 0$. □

Theorem 4.3.9: With respect to equation (4.3.7), suppose in addition to condition (3) and (4) of theorem 4.3.8, suppose that

$$q_n > 0 \ , \ \text{for all large values of } n \tag{4.3.8}$$

and

$$-1 < p_1 \le p_n \le 0 \ \text{for all } n \ge n_0 \in \mathbf{Z} \tag{4.3.9}$$

then $M^- = \phi$

Proof: Suppose that equation (4.3.7) has a solution $\{x_n\} \in M^-$ so there exists an integer $n \ge n_0 \in \mathbf{Z}$ such that $x_n > 0$, $\Delta x_n \le 0$, $x_{n-r} > 0$ and $\Delta x_{n-r} \le 0$ for all $n \ge n_1$, let $z_n = x_n + p_n x_{n-m}$, since we have that $z_n > 0$, (otherwise x_n vanishes as $n \to \infty$) then equation (4.3.7) becomes

$$\Delta(c_n\Delta z_n) = -q_{n+1}G(x_{n+1-k}) \ , \ n \ge n_1$$

with $\{c_n\Delta z_n\}$ is decreasing for $n \ge n_1$ as in theorem 4.3.8 we can prove that $c_n\Delta z_n > 0$ for $n \ge n_1$

Define

$$w_n = \frac{c_n\Delta z_n}{G(z_n)} \ , \quad n \ge n_1$$

we obtain

$$\Delta w_n = -\frac{q_{n+1}G(x_{n+1-k})}{G(z_{n+1})} - \frac{c_n(\Delta z_n)^2 g(z_n, z_{n+1})}{G(z_n)G(z_{n+1})}$$

$$\Delta w_n \le -\frac{q_{n+1}G(x_{n+1-k})}{G(z_{n+1})} \ , \quad n \ge n_1 \tag{4.3.10}$$

using inequality (4.3.9) we have that $z_{n+1} \le x_{n+1}$, since $\{x_n\} \in M^-$, then

$$z_{n+1} \le x_{n+1} \le x_{n+1-k} \ ,$$

and

$$G(z_{n+1}) \le G(x_{n+1-k}) \text{ , for } n \ge n_1$$

using this inequality in inequality (4.3.10) and taking the summation from n_1 to $n-1 \ge n_1$, to the resulting inequality we get

$$w_n \le w_{n_1} - \sum_{s=n_1}^{n-1} q_{s+1} \quad ,$$

and hence $w_n \to -\infty$ as $n \to \infty$, which is a contradiction so $M^- = \phi$. □

Theorem 4.3.10: With respect to equation (4.3.7), assume in addition to condition (2) of theorem 4.3.8 that

$$m \text{ is odd positive integer ,} \tag{4.3.11}$$

and

$$p_n = p \le 0 \text{ , for all } n \ge n_0 \in \mathbf{Z} \tag{4.3.12}$$

then $WOS = \phi$

Proof: Let $\{x_n\} \in WOS$, suppose that there exists an integer $n_1 \ge n_0 \in \mathbf{Z}$ such that $x_n > 0$, $x_{n-r} > 0$ for all $n \ge n_1$, let $z_n = x_n + p_n x_{n-m}$, then

$$\Delta z_n \Delta z_{n+1} = \Delta x_n \Delta x_{n+1} + p[\Delta x_n \Delta x_{n+1-m} + \Delta x_{n+1} \Delta x_{n-m}] + p^2 \Delta x_{n-m} \Delta x_{n+1-m}$$

Using condition (4.3.11) and (4.3.12), we find that

$$\Delta z_n \Delta z_{n+1} \le 0 \text{ ,}$$

hence $\{\Delta z_n\}$ is oscillatory

Define

$$w_n = c_n \Delta z_n \quad ,$$

then w_n is oscillatory , but

$$\Delta w_n = -q_{n+1} G(x_{n+1-k}) \text{ ,} \quad n \ge n_1$$

using condition (2) of theorem 4.3.8

$$\Delta w_n \le 0 \text{ ,}$$

and so w_n is nonincreasing which is a contradiction. □

Theorem 4.3.11: Assume in addition to the hypothesis of theorem 4.3.9 that $m \ge 1$, then every solution of equation (4.3.7) is either oscillatory or weakly oscillatory.

Proof: It is an immediate consequence of theorems 4.3.8 and 4.3.9. □

Theorem 4.3.12: Assume that conditions (2), (3) and (4) of theorem 4.3.8 hold, and condition (4.3.11) holds, also the inequality $-1 \le p_n = p < 0$ holds. Then every solution of equation (4.3.7) is oscillatory.

Proof: It follows from theorems 4.3.8, 4.3.9 and 4.3.10 immediately. □

Example 4.3.1: Consider the NDE

$$\Delta^2 \left(x_n - \left(4 + (-1)^n\right) x_{n-1} \right) + 2x_{n-3}^3 = 0 \text{ ,} \quad n \ge 0 \tag{4.3.13}$$

it clear that $-\infty < p_1 \le p_n = -\left(4+(-1)^n\right) \le p_3 \le -1$ and $\sum_{n=0}^{\infty} q_n = \infty$, $G = x_{n-3}^3$ is continuous on $\mathbf{R}$ and nondecreasing satisfies $uG(u) > 0$ for $u \ne 0$,
so by theorem 4.3.3 every solution of equation (4.3.13) is oscillatory.

Example 4.3.2: Consider the NDE

$$\Delta^2\left(x_n - 5(-1)^n x_{n-3}\right) + 4x_{n-4}^{\frac{1}{3}} = 0 \ , \quad n \ge 0 \tag{4.3.14}$$

using theorem 4.3.7 , equation (4.3.14) is oscillatory since $p_n = \begin{cases} 5 & , \ n \ even \\ -5 & , \ n \ odd \end{cases}$, and taking $G(p_2) = 5^{\frac{1}{3}} \ge 1$, also $\sum_{n=0}^{\infty} Q_n = \sum_{n=0}^{\infty} \frac{q_n}{G(p_2)} = \sum_{n=0}^{\infty} \frac{4}{5^{\frac{1}{3}}} = \infty$ is satisfied in addition to $uG(u) > 0$ for $u \ne 0$

Example 4.3.3: Consider the NDE

$$\Delta\left(\frac{1}{n}\Delta\left(x_n + \sqrt[3]{-0.5}\,x_{n-1}\right)\right) + \frac{1}{n}x_{n-2} = 0 \ , \quad n \ge 0 \tag{4.3.15}$$

applying theorem 4.3.12 it is clear that

$$-1 < p_n = p = (-0.5)^{\frac{1}{3}} \le 0 \, , \quad q_n = \frac{1}{n} \ge 0 \qquad \text{for all } n \ge n_0 \in \mathbf{Z}$$

$$\lim_{n\to\infty} \sum_{s=0}^{n} \frac{1}{(s+1)} = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} n = \infty$$

so every solution of equation (4.3.15) is oscillatory.

Remark 4.3.2: The results of this section are mainly referred to [3] and [19].

4.4 Oscillation criteria of second-order forced NDE's with variable coefficients

Consider the second order forced NDE

$$\Delta^2\left(x_n + p_n x_{n-m}\right) + q_n G\left(x_{n-k}\right) = f_n \ , \tag{4.4.1}$$

where $\{p_n\}$, $\{q_n\}$ and $\{f_n\}$ are sequences of real numbers with $q_n \ge 0$, $G \in C(\mathbf{R},\mathbf{R})$ is nondecreasing and $uG(u) > 0$ for $u \ne 0$, $m > 0$ and $k \ge 0$ are integers.

This section studies theorems that establish sufficient conditions for the oscillation of equation (4.4.1). In [19] Parhi and Tripathy proved the oscillation using sufficient conditions used in section 4.3 in addition to conditions concerning the forced term. However, in [25] the authors proved the oscillation of equation (4.4.1) depending on the technique used in the proof of the unforced related equations.

Assuming in the first three theorems that G satisfies the following two conditions
(i) There exists $K > 0$ such that

$$|G(uv)| \ge K|G(u)||G(v)| \quad \text{for all } u, v \in \mathbf{R}$$

(ii) $\int_0^{\pm c} \frac{ds}{G(s)} < \infty$ for all $c > 0$

We need the following conditions:
(H_1) $0 \le p_n < p_1 < 1$, where p_1 is a constant.
(H_2) there exists a real sequence $\{h_n\}$ such that $\Delta^2 h_n = f_n$.
(H_3) $\sum_{n=N_0}^{\infty} q_n G\left(\frac{n-k}{2}\right) = \infty$.
(H_4) $\{h_n\}$ is oscillatory and $\lim_{n\to\infty} h_n = 0$.
(H_5) $\{h_n\}$ is m periodic .
(H_6) $\sum_{n=N_0}^{\infty} q_n = \infty$.
(H_7) there exists $\gamma > 0$ such $\frac{G(u)}{u} > \gamma > 0$ for $u \ne 0$.
The following result that concerns with the oscillation if all solution of equation (4.4.1) with $f_n = 0$ is needed to establish the oscillation of all solutions of equation (4.4.1).

Theorem 4.4.1: Let $f_n = 0$ for all $n \in \mathbf{N}$, and let (H_1) and (H_3) hold. Then all solutions of equation (4.4.1) are oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.4.1) , then $x_n > 0$, $x_{n-m} > 0$ and $x_{n-k} > 0$ for $n \ge N_1 \ge N_0$. Setting

$$z_n = x_n + p_n x_{n-m} \tag{4.4.2}$$

we obtain that $z_n \ge x_n > 0$ and

$$\Delta^2 z_n = -q_n G(x_{n-k}) \le 0 \quad \text{, for } n \ge N_1 \tag{4.4.3}$$

by theorem 1.4.1 then $\Delta z_n > 0$ for $n \ge N_2$, from equation (4.4.2) we have

$$x_n = z_n - p_n x_{n-m} \tag{4.4.4}$$

so $z_n \ge x_n$ and $\{z_n\}$ is increasing, so

$$0 < (1-p_1) z_n \le (1-p_n) z_n \le x_n \tag{4.4.5}$$

by corollary 1.4.1 there exists $N_3 \ge N_2$

$$x_n \ge (1-p_1) z_n \ge \frac{(1-p_1)n}{2} \Delta z_n \quad \text{, for } n \ge 2N_3 \tag{4.4.6}$$

applying (i) and (ii) to inequality (4.4.6) gives

$$G(x_{n-k}) \ge K^2 G(1-p_1) G\left(\frac{n-k}{2}\right) G(\Delta z_{n-k}) \tag{4.4.7}$$

$$\ge K_1 G\left(\frac{n-k}{2}\right) G(\Delta z_n) \quad \text{, for } n \ge N_4 \ge 2N_3$$

where $K_1 = K^2 G(1-p_1) > 0$. Combining equation (4.4.3) and inequality (4.4.7), we get

$$\Delta^2 z_n + K_1 q_n G\left(\frac{n-k}{2}\right) G(\Delta z_n) \le 0 \quad \text{, for } n \ge N_4 \tag{4.4.8}$$

Summing the last inequality, we get

$$K_1 \sum_{s=N_4}^{n-1} q_s G\left(\frac{s-k}{2}\right) \le -\sum_{s=N_4}^{n-1} \frac{\Delta^2 z_s}{G(\Delta z_s)} \le \int_{\Delta z_n}^{\Delta z_{N_4}} \frac{du}{G(u)} \tag{4.4.9}$$

letting $n \to \infty$ and using (i) and (ii), we get

$$\sum_{n=N_4}^{\infty} q_n G\left(\frac{n-k}{2}\right) < \infty \tag{4.4.10}$$

which contradicts (H_3). □

Theorem 4.4.2: If (H_1) and (H_2)-(H_4) hold, then all solutions of equation (4.4.1) are oscillatory.

Proof: Let x_n be an eventually positive solution of equation (4.4.1) then $x_n > 0$, $x_{n-m} > 0$ and $x_{n-k} > 0$ for all $n \ge N_1 \ge N_0$. Let

$$y_n = x_n + p_n x_{n-m} - h_n \tag{4.4.11}$$

from equation (4.4.11) and (C_2), we have

$$\Delta^2 y_n = -q_n G(x_{n-k}) \le 0 \tag{4.4.12}$$

It is clear that y_n must be greater than zero, thus $y_n > 0$ for $n \ge N_2 \ge N_1$, by theorem 1.4.1 we have

$$\Delta y_n > 0 \quad \text{for } n \ge N_3 \ge N_2 \ ,$$

now for $0 < \varepsilon < (1 - p_1) y_n$, ($H_4$) implies that there exists an integer $N_4 > N_3$ such that

$$|h_n| < \frac{\varepsilon}{2} \ . \ \text{ for } n \ge N_4$$

From equation (4.4.11), we have $x_n \le y_n + h_n$. So

$$\begin{aligned} y_n - p_n y_{n-m} &\le x_n - h_n + p_n h_{n-m} \\ &< x_n + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} p_n \end{aligned} \tag{4.4.13}$$

Hence

$$0 < (1 - p_1) y_{N_3} - \varepsilon < (1 - p_1) y_n - \varepsilon < x_n \quad , \ \text{for } \ n \ge N_4 \tag{4.4.14}$$

Set $r_n = (1 - p_1) y_n - \varepsilon$ for $n \ge N_4$, we get

$$0 < r_n < x_n \ , \ \Delta r_n > 0$$

and

$$\Delta^2 r_n = -(1 - p_1) q_n G(x_{n-k}) \le 0$$

We see that this step is similar to equation (4.4.3) in theorem 4.4.1, so we proceed as in the proof of theorem 4.4.1 to get a contradiction which completes the proof. □

Remark 4.4.1: The conclusion of theorem 4.4.2 will be that the solution either oscillate or converge to zero if (H_4) is replaced by $\lim_{n \to \infty} h_n = 0$

Theorem 4.4.3: If (H_1)-(H_3) and (H_5) hold, then every solution of equation (4.4.1) is oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.4.1) , then $x_n > 0$, $x_{n-m} > 0$ and $x_{n-k} > 0$ for $n \geq N_1 \geq N_0$. Define y_n as in equation (4.4.11) , we have that equation (4.4.12) holds and so either $y_n > 0$ or $y_n < 0$ for $n \geq N_2$ for some $N_2 \geq N_1$.
We claim that $\{x_n\}$ is bounded. If not, then $\{x_n\}$ is unbounded and since $x_n < y_n + h_n$ and $\{h_n\}$ bounded, $\{y_n\}$ must be unbounded and eventually positive. Clearly that $\Delta y_n > 0$ for large n since $\Delta y_n < 0$ implies that $\{y_n\}$ is bounded. From equation (4.4.11) we have

$$y_n - p_n y_{n-m} = x_n - h_n + p_n p_{n-m} x_{n-2m} + p_n h_{n-m} \ , \quad \text{for } n \geq N_3 \qquad (4.4.15)$$

for some $N_3 \geq N_2$. That is

$$(1 - p_n) y_n \leq x_n - (1 - p_n) h_n \quad , \qquad (4.4.16)$$

or

$$0 < (1 - p_1)(y_n + h_n) \leq x_n \qquad (4.4.17)$$

since $\{h_n\}$ is periodic, there exist real numbers a_1 and a_2 and two increasing sequences $\{n_i'\}$ and $\{n_i''\}$ of natural numbers such that

$$\lim_{i\to\infty} n_i' = \lim_{i\to\infty} n_i'' = \infty \ ,$$

and

$$h_{n_i'} = a_1 \ , \ h_{n_i''} = a_2 \ , \ \ a_1 < h_n < a_2 \ \text{ for all } \ n > N_0$$

hence

$$y_n + a_1 \geq y_{n_i'} + a_1 = y_{n_i'} + h_{n_i'} \geq x_{n_i'} > 0 \ \ , \text{for } n > n_i' \ , \ i > 1 \qquad (4.4.18)$$

Thus

$$0 < (1 - p_1)(y_n + a_1) \leq (1 - p_1)(y_n + h_n) \leq x_n \ \ , \ \text{for } n \geq n_i' \qquad (4.4.19)$$

Setting

$$r_n = (1 - p_1)(y_n + a_1) \ \ , \text{for } n \geq n_i' \ \text{ and } i \geq 1$$

We obtain

$$0 < r_n \leq x_n \ \text{ and } \Delta r_n > 0$$

and

$$\Delta^2 r_n = -(1 - p_1) q_n G(x_{n-k}) \leq 0 \qquad (4.4.20)$$

Proceeding as in the proof of theorem 4.4.1 to arrive a contradiction, thus $\{x_n\}$ is bounded as we claim $\{x_n\}$ is bounded

The boundedness of $\{x_n\}$ implies that $\{y_n\}$ is bounded and $\Delta y_n > 0$ for $n > N_2$ using theorem 1.4.1. Again we proceed as the proof of theorem 4.4.1 we arrive a contradiction. Hence $\{x_n\}$ oscillates. □

In fact, more general ranges of p_n are considered in [19], we need some conditions used in section 4.3, so we will mention those conditions as they were in section 4.3 in addition to new conditions (C_9)-(C_{13})

(C_2) For $u > 0$ and $v > 0$, there exists $\lambda > 0$ such that $G(u+v) \leq \lambda(G(u) + G(v))$
(C_3) For $u < 0$ and $v < 0$, there exists $\lambda > 0$ such that $G(u+v) \geq \lambda(G(u) + G(v))$
(C_6) For $u > 0$ and $v > 0$, $G(uv) \leq G(u)G(v)$

(C_7) $G(-u)=-G(u)$, $u\in\mathbf{R}$

(C_8) For $u,v\in\mathbf{R}$, $G(uv)=G(u)G(v)$

(C_9) There exists a real valued function $\{h_n\}$ defined on N which changes sign and satisfies that $\Delta^2 h_n = f_n$

(C_{10}) $\sum_{n=r}^{\infty} q_n^* G(h_{n-k}^+)=\infty$ and $\sum_{n=r}^{\infty} q_n^* G(-h_{n-k}^-)=-\infty$

where $h_n^+ = \max\{h_n, 0\}$, $r=\max\{m,k\}$

$h_n^- = \max\{-h_n, 0\}$, $q_n^* = \min\{q_n, q_{n-m}\}$

(C_{10}^1) $\sum_{n=r}^{\infty} q_n^* G(h_{n-k}^+)=\infty$ and $\sum_{n=r}^{\infty} q_n^* G(h_{n-k}^-)=\infty$

(C_{11}) $\sum_{n=k}^{\infty} q_n G(h_{n-k}^+)=\infty$ and $\sum_{n=k}^{\infty} q_n G(h_{n+m-k}^-)=\infty$

(C_{12}) $\sum_{n=k}^{\infty} q_n G(h_{n-k}^-)=\infty$ and $\sum_{n=k}^{\infty} q_n G(h_{n+m-k}^+)=\infty$

(C_{13}) $\sum_{n=k}^{\infty} q_n G(h_{n+m-k}^+)=\infty$ and $\sum_{n=k}^{\infty} q_n G(h_{n+m-k}^-)=\infty$

Theorem 4.4.4: Let $0\le p_n \le 1$. If (C_2) , (C_3), (C_9) and (C_{10}) hold, then every solution of equation (4.4.1) is oscillatory.

Proof:: Let $\{x_n\}$ be an eventually positive solution of equation (4.4.1) , then $x_n>0$ for $n\ge n_0>0$. Setting

$$z_n = x_n + p_n x_{n-m} \quad , \tag{4.4.21}$$

and

$$w_n = z_n - h_n \quad , \tag{4.4.22}$$

we obtain for $n\ge n_0+r$

$$\Delta^2 w_n + q_n G(x_{n-k}) = 0 \quad , \tag{4.4.23}$$

now

$$0 < z_n \le x_n + x_{n-m} \quad ,$$

and $\Delta^2 w_n \le 0$. Hence either $w_n>0$ or $w_n<0$. It is clear that $w_n>0$, otherwise we have a contradiction with (C_9). Hence $w_n>0$ for $n\ge n_1$.

By theorem 1.4.1 , $\Delta w_n>0$, for $n\ge n_2>n_1$. We have

$$x_n + x_{n-m} \ge h_n^+$$

using (C_2) from (4.4.23) we get

$$\begin{aligned} 0 &= \Delta^2 w_n + q_n G(x_{n-k}) + \Delta^2 w_{n-m} + q_{n-m} G(x_{n+m-k}) \\ &\ge \Delta^2 w_n + \Delta^2 w_{n-m} + q_n^*\left[G(x_{n-k}) + G(x_{n+m-k})\right] \\ &\ge \Delta^2 w_n + \Delta^2 w_{n-m} + \lambda^{-1} q_n^* G(x_{n-k} + x_{n+m-k}) \\ &\ge \Delta^2 w_n + \Delta^2 w_{n-m} + \lambda^{-1} q_n^* G(h_{n-k}^+) \quad , \text{ for } n\ge n_3 > n_2 + r \end{aligned}$$

Hence

$$\sum_{n=n_3}^{\infty} q_n^* G\left(h_{n-k}^+\right) < \infty \quad ,$$

which is a contradiction to (C_{10}), then $\{x_n\}$ is oscillatory. □

Theorem 4.4.5: Let $0 \le p_n < p_2 < \infty$. If (C_2) , (C_6), (C_7), (C_9) and (C_{10}) hold, then every solution of equation (4.4.1) is oscillatory.

Proof: Proceeding as in the proof of theorem (4.4.4) we reach to

$$0 < z_n < x_n + p_2 x_{n-m}$$

to conclude that $w_n > 0$ and $x_n + p_2 x_{n-m} \ge h_n^+$

Using (C_2) and (C_6) yield

$$\begin{aligned} 0 &= \Delta^2 w_n + q_n G(x_{n-k}) + G(p_2)\Delta^2 w_{n-m} + G(p_2) q_{n-m} G(x_{n-m-k}) \\ &\ge \Delta^2 w_n + G(p_2)\Delta^2 w_{n-m} + \lambda^{-1} q_n^* G\left(h_{n-k}^+\right) \end{aligned}$$

Hence

$$\sum_{n=n_3}^{\infty} q_n^* G\left(h_{n-k}^+\right) < \infty \quad ,$$

which contradicts (C_{10}^1), hence $\{x_n\}$ oscillates. □

Theorem 4.4.6: Let $-\infty < p_1 \le p_n \le 0$. If ($C_8$), ($C_9$), ($C_{11}$) and ($C_{12}$) hold, then every solution of equation (4.4.1) is oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.4.1), let w_n as in (4.4.22) we obtain equation (4.4.23) since $\Delta^2 w_n \le 0$ for $n \ge n_0 + r$, either $w_n > 0$ or $w_n < 0$ for $n \ge n_1 > n_0 + r$

<u>If</u> $w_n > 0$ for $n \ge n_1$, then $x_n \ge h_n^+$ and $\Delta w_n > 0$, by theorem 1.4.1 from equation (4.4.23) we obtain

$$\sum_{n=n_2}^{\infty} q_n G\left(h_{n-k}^+\right) < \infty \quad , \quad n_2 \ge n_1 + r$$

which contradicts (C_{11}).

<u>If</u> $w_n < 0$ for $n \ge n_1$, then $x_n > -p_1^{-1} h_{n+m}^-$, set $y_n = -w_n$, then $y_n > 0$ and

$$\Delta^2 y_n - q_n G(x_{n-k}) = 0 \quad ,$$

since $\Delta^2 y_n \ge 0$, then $\Delta y_n < 0$ for $n \ge n_3 > n_2$ by theorem 1.4.1 , using (C_8) we obtain

$$\sum_{n=n_2}^{\infty} q_n G\left(h_{n+m-k}^-\right) < \infty \quad ,$$

which again contradicts (C_{11}). Thus $\{x_n\}$ is oscillatory. □

Theorem 4.4.7: Let p_n changes sign such that $-\infty < p_1 \le p_n \le p_2 < \infty$, where $p_1 < 0$ and $p_2 > 0$ are constants. If (C_2), (C_8), (C_9), (C_{10}^1) and (C_{13}) hold, then every solution of equation (4.4.1) is oscillatory.

Proof: Let x_n be an eventually positive solution of equation (4.4.1) then $x_n > 0$ for $n \geq n_0 > 0$. Set w_n as in (4.4.22), we get (4.4.23). For $n \geq n_0 + r$, $\Delta^2 w_n \leq 0$ and hence either $w_n > 0$ or $w_n < 0$ for $n \geq n_1$.

If $w_n > 0$ for $n \geq n_1$ we proceed as in the proof of theorem 4.4.5 we get a contradiction.

If $w_n < 0$ for $n \geq n_1$ we proceed as in the proof of theorem 4.4.6 to arrive a contradiction. Thus x_n is oscillatory. □

Theorem 4.4.8: Let $p_n \geq 0$. If

$$\liminf_{n\to\infty} n^{-1} h_n = -\infty \quad \text{and} \quad \limsup_{n\to\infty} n^{-1} h_n = \infty \ ,$$

then every solution of equation (4.4.1) is oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.4.1), then $x_n > 0$ for $n \geq n_0 > 0$. Setting z_n as in (4.4.21) and w_n as in (4.4.22), we obtain $z_n > 0$ for $n \geq n_0 + r$ and $\Delta^2 w_n \leq 0$ for $n \geq n_0 + r$ from equation (4.4.23). Hence for $n \geq n_1 \geq n_0 + r$, we have

$$\Delta w_n \leq \Delta w_{n_1} \ ,$$

and

$$w_n \leq w_{n_1} + (n - n_1)\Delta w_{n_1} \ ,$$

hence

$$0 < \frac{z_n}{n} \leq \frac{h_n}{n} + \frac{w_{n_1}}{n} + \frac{(n-n_1)}{n}\Delta w_{n_1}$$

Thus

$$\begin{aligned}
0 \leq \liminf_{n\to\infty} \frac{z_n}{n} &\leq \liminf_{n\to\infty}\left(\frac{h_n}{n} + \frac{w_n}{n} + \frac{(n-n_1)}{n}\Delta w_{n_1}\right) \\
&= \liminf_{n\to\infty} \frac{h_n}{n} + \limsup_{n\to\infty}\left(\frac{w_n}{n} + \frac{(n-n_1)}{n}\Delta w_{n_1}\right) \\
&\leq \liminf_{n\to\infty} \frac{h_n}{n} + \lim_{n\to\infty}\left(\frac{w_n}{n} + \frac{(n-n_1)}{n}\Delta w_{n_1}\right) \\
&\leq \liminf_{n\to\infty} \frac{h_n}{n} + \Delta w_{n_1} = -\infty \ ,
\end{aligned}$$

which is a contradiction so $\{x_n\}$ is oscillatory. □

Example 4.4.1:

$$\Delta^2\left(x_n + \frac{1}{3}x_{n-2}\right) + 3\left(2 - \frac{3}{2^n}\right)x_{n-5}^{\frac{3}{5}} = \frac{9(-1)^n}{2^{n+2}} \tag{4.4.24}$$

It is clear that the condition of theorem 4.4.2 are satisfied since

(H_1) $0 \leq p_n = \dfrac{1}{3} < p_1 < 1$

(H_2) Take $h_n = \dfrac{(-1)^n}{2^n}$ with $\Delta h_n = \dfrac{3(-1)^{n+1}}{2^{n+1}}$ and $\Delta^2 h_n = \dfrac{9(-1)^n}{2^{n+2}}$

(H3) $\sum_{n=N_0}^{\infty} 3\left(2-\frac{3}{2^n}\right)\left(\frac{n-5}{2}\right)^{\frac{3}{5}} = \infty$

(H4) $\{h_n\}$ is oscillatory $\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$

Then by theorem 4.4.2 every solution of equation (4.4.24) is oscillatory.

Example 4.4.2:

$$\Delta^2\left(x_n + \frac{1}{8}x_{n-2}\right) + 3x_{n-6}^{\frac{1}{3}} = 6(-1)^n \qquad (4.4.25)$$

It is clear that equation (4.4.25) satisfies the conditions of theorem 4.4.3 since

(C1) $0 \le p_n = \frac{1}{8} < p_1 < 1$, where p_1 is a constant

(C2) Taking $h_n = \frac{3}{2}(-1)^{-n}$ with $\Delta h_n = 3(-1)^{-n+1}$ and $\Delta^2 h_n = 6(-1)^{-n} = 6(-1)^n$

(C3) $\sum_{n=N_0}^{\infty} 3\left(\frac{n-6}{2}\right)^{\frac{1}{3}} = \infty$

(C4) $\{h_n\}$ is periodic of period 2

By theorem 4.4.3 every solution of equation (4.4.25) is oscillatory.

Example 4.4.3:

$$\Delta^2\left(x_n + (-1)^n x_{n-2}\right) + 2x_{n-1}^5 = (-1)^n \qquad (4.4.26)$$

we have $h_n = \frac{1}{4}(-1)^n$, so

$$h_n^+ = \begin{cases} \frac{1}{4} & n \ even \\ 0 & n \ odd \end{cases} \quad , \quad \text{and} \quad h_n^- = \begin{cases} 0 & n \ even \\ \frac{1}{4} & n \ odd \end{cases}$$

also we have

(C_{10}^1) $\sum_{n=2}^{\infty} q_n^* G\left(h_{n-1}^+\right) = 2\sum_{n=2}^{\infty}\left(h_{n-1}^+\right)^5 = \infty$

(C8) is satisfied

(C13) $\sum_{n=1}^{\infty} q_n G\left(h_{n+1}^+\right) = 2\sum_{n=1}^{\infty}\left(h_{n+1}^+\right)^2 = \infty$

So by theorem 4.4.7 every solution of equation (4.4.26) is oscillatory.

Example 4.4.4:

$$\Delta^2\left(x_n + \left(5 + (-1)^n\right)x_{n-1}\right) + x_{n-2}^5 = 2(-1)^n\left(4n^2 + 8n + 6\right) \ , \ n \ge 0 \qquad (4.4.27)$$

we have $h_n = 2n^2(-1)^n$ which changes sign with

$$\Delta h_n = 2(-1)^n\left(2n^2 + 2n + 1\right)$$

$$\Delta^2 h_n = 2(-1)^n\left(4n^2 + 8n + 6\right)$$

and

$$\liminf_{n\to\infty} \frac{h_n}{n} = -\infty$$
$$\limsup_{n\to\infty} \frac{h_n}{n} = \infty$$
By theorem 4.4.8 we have that every solution of equation (4.4.27) is oscillatory.

Remark 4.4.2: The results of this section are mainly referred to [19] and [25].

4.5 Oscillation criteria of sublinear, linear and superlinear NDE's

Consider the second-order NDE
$$\Delta\left(c_n \Delta\left(x_n + \delta\, p_n x_{n-m}\right)\right) + \delta\, q_n x_{n+1-k}^{\beta} = 0 \quad , \; n \ge 0 \tag{4.5.1}$$

Of course the value of β determines the type of the NDE whether it is sublinear or superlinear. In [24] the authors studied the oscillation of equation (4.5.1) where β is in general the ratio of odd positive integers, $0 \le p_n \le p < 1$. However in [17] Lin studied the superlinear NDE, but this time the results were for bounded oscillation.

The following conditions are needed in our discussion

(C_1) $\{p_n\}$ is nondecreasing such that $0 \le p_n \le p < 1$ and $q_n \ge 0$ and $\{q_n\}$ is not identically zero for infinitely many values of n

(C_2) $c_n > 0$ and $\sum_{n=n_0}^{\infty} \frac{1}{c_n} < \infty$,

Let $r = \max\{m, k\}$, $Q_n = \sum_{s=n}^{\infty} \frac{1}{c_s}$, $\delta = 1$

We need the following lemmas

Lemma 4.5.1: Let $\{x_n\}$ be an eventually positive solution of equation (4.5.1), then for $z_n = x_n + p_n x_{n-m}$, one of the following two cases holds for all sufficiently large n

(I) $z_n > 0$, $c_n \Delta z_n > 0$,

(II) $z_n > 0$, $c_n \Delta z_n < 0$

Proof: Assume that $x_{n-k-m} > 0$ for $n \ge N_0 \in \mathbf{N}(n_0)$. Then by the condition (C_1), we have $z_n > 0$ and $\Delta(c_n \Delta z_n) \le 0$ for $n \ge N_0$. Hence $\{c_n \Delta z_n\}$ is eventually of one sign. □

Lemma 4.5.2: Let $\{x_n\}$ be an eventually positive solution of equation (4.5.1) and suppose case (I) of lemma 4.5.1 holds. Then there exists an integer $N \in \mathbf{N}(n_0)$ such that
$$(1-p) z_n \le x_n \le z_n \tag{4.5.3}$$

Proof: Cleary that (C_1) implies that $z_n \ge x_n$. Moreover we have
$$x_n = z_n - p_n x_{n-k} \ge z_n - p_n z_{n-k} \ge z_n (1-p)$$
Since $\{z_n\}$ is nondecreasing. This completes the proof. □

Lemma 4.5.3: Let $\{x_n\}$ be an eventually positive solution of equation (4.5.1) and suppose case (II) of lemma 4.5.1 holds. Then there exists an integer $N \in \mathbf{N}(n_0)$ such that

$$x_{n-k} \geq \frac{z_n}{1+p} \geq (1-p)z_n \ , \ \text{for } n \geq N$$

Proof: Since $\{z_n\}$ is nonincreasing we may assume that $\{x_n\}$ is also nonincreasing. Hence

$$z_n \leq x_{n-k} + px_{n-k} = (1+p)x_{n-k} \tag{4.5.4}$$

Since $1 \geq 1-p^2$, we have $\dfrac{1}{1+p} \geq 1-p$, therefore

$$\frac{z_n}{1+p} \geq (1-p)z_n \ . \quad \square$$

Lemma 4.5.4: Let $\{x_n\}$ be an eventually positive solution of equation (4.5.1). Then $\{z_n\}$ is bounded above and satisfies

$$z_n \geq -Q_n c_n \Delta z_n \ , \tag{4.5.5}$$

for all sufficiently large n.

Proof: Proceeding as in the proof of lemma 4.5.1 we obtain $\Delta(c_n \Delta z_n) \leq 0$ for $n \geq N \in \mathbf{N}(n_0)$. Then $c_n \Delta z_n \leq c_N \Delta z_N$ for $n \geq N$. Dividing the last inequality by c_n and summing we obtain

$$z_n - z_N \leq c_N \Delta z_N \sum_{s=N}^{n-1} \frac{1}{c_s} < \infty$$

Hence $\{z_n\}$ is bounded above. Letting $n \to \infty$, we obtain $z_N \geq -c_N \Delta z_N Q_N$ for sufficiently large n.

Theorem 4.5.1: Assume that (C_1) and (C_2) hold and $\beta = 1$ and $k > m$ in equation (4.5.1). If

$$\sum_{n=n_0}^{\infty} q_n (1 - p_{n+1-k}) = \infty \ , \tag{4.5.6}$$

and

$$\sum_{s=n}^{n+k-m-1} q_s Q_s > 1 + p \quad , \tag{4.5.7}$$

are satisfied, then all solutions of equation (4.5.1) are oscillatory.

Proof: Let $\{x_n\}$ be an eventually positive solution of equation (4.5.1) , then $x_{n-r} > 0$ for $n \geq N \in \mathbf{N}(n_0)$, using lemma (4.5.1) we have two cases:

(I) $z_n > 0$, $c_n \Delta z_n > 0$,

by lemma 4.5.2 we have

$$\Delta(c_n \Delta z_n) + q_n (1 - p_{n+1-k}) z_{n+1-k} \leq 0 \ , \ \ n \geq N \tag{4.5.8}$$

Let $w_n = \dfrac{c_n \Delta z_n}{z_{n-k}}$, for $n \geq N \geq n_0 + k$

then we have

$$\Delta w_n \le -q_n(1-p_{n+1-k}) - \frac{c_n \Delta z_n \Delta z_{n-k}}{z_{n-m} z_{n+1-k}} \le q_n(1-p_{n+1-k})$$

Summing the last inequality, we get

$$\sum_{s=N}^{n} q_s(1-p_{s+1-k}) < w_N \quad ,$$

as $n \to \infty$ we have

$$\sum_{s=N}^{\infty} q_s(1-p_{s+1-k}) < \infty \quad ,$$

which is a contradiction.

(II) $z_n > 0$, $c_n \Delta z_n < 0$

Taking the summation to equation (4.5.1) from N to $n-1$, we obtain

$$c_n \Delta z_n - c_N \Delta z_N + \sum_{s=N}^{n-1} q_s x_{s+1-k} = 0$$

or

$$\Delta z_n + \frac{1}{c_s} \sum_{s=N}^{n-1} q_s x_{s+1-k} \le 0$$

Summing again from n to $j-1$ and rearranging, we obtain

$$\sum_{s=n}^{\infty} q_s Q_s x_{s+1-k} < z_n \ , \tag{4.5.9}$$

using inequality (4.5.4) in (4.5.9), and using the fact that $\{z_n\}$ is nonincreasing, we obtain

$$\sum_{s=n}^{n+k-m+1} q_s Q_s \le (1+p) \ ,$$

which contradicts condition (4.5.7), $\{x_n\}$ must be oscillatory. □

Theorem 4.5.2: Assume that (C_1) and (C_2) hold and $\beta > 1$ and $k \ge m$ in equation (4.5.1). If

$$\sum_{n=n_0}^{\infty} q_n Q_{n+1}^{\beta} = \infty \quad , \tag{4.5.10}$$

then all solutions of equation (4.5.1) are oscillatory.

Proof: Proceeding as in the proof of theorem 4.5.1 we see that lemma 4.5.1 holds for $n \ge N \in \mathbf{N}(n_0)$

(I) $z_n > 0$, $c_n \Delta z_n > 0$

Summing equation (4.5.1) from $N \in \mathbf{N}(n_0)$ to $(n-1)$, yields

$$c_n \Delta z_n - c_N \Delta z_N + \sum_{s=N}^{n-1} q_s x_{s+1-k}^{\beta} = 0 \quad , \quad n \ge N \tag{4.5.11}$$

then

$$\sum_{s=N}^{n-1} q_s x_{s+1-k}^{\beta} < c_N \Delta z_N$$

letting $n \to \infty$, we obtain

$$\sum_{s=N}^{\infty} q_s x_{s+1-k}^{\beta} < \infty \tag{4.5.12}$$

but $\{z_n\}$ is increasing, so there exists a positive constant a such that $z_n > a$ for $n \geq N$, together with equation (4.5.2) yields

$$x_n \geq a(1-p) \ , \ \text{for } n \geq N$$

Thus there exists an integer $N_1 \geq N$ such that

$$x_n \geq a(1-p)Q_n \ , \ \ n \geq N_1$$

since $Q_n \to 0$ as $n \to \infty$. Because $\{Q_n\}$ is decreasing the last inequality implies that

$$x_{n+1-k} \geq a(1-p)Q_{n+1-k} \geq a(1-p)Q_{n+1} \tag{4.5.13}$$

Combining (4.5.12) and (4.5.13) we get

$$\sum_{s=N_1}^{\infty} q_s Q_{s+1}^{\beta} < \infty \tag{4.5.14}$$

which contradicts (4.5.10)

(II) $z_n > 0$, $c_n \Delta z_n < 0$

From equation (4.5.4) and (4.5.5), the facts that $k \geq m$ and z_n is decreasing we have

$$\begin{aligned} x_{n+1-k} &= x_{n+1-k+m-m} \\ &\geq (1-p)z_{n+1-k+m} \\ &\geq (1-p)z_{n+1} \\ &\geq -(1-p)c_{n+1}\Delta z_{n+1} Q_{n+1} \ , \ \ n \geq N \end{aligned} \tag{4.5.15}$$

Consider the difference

$$\begin{aligned} \Delta\left((c_n \Delta z_n)^{-\beta+1}\right) &= (-\beta+1)\, t^{-\beta} \Delta(c_n \Delta z_n) \\ &= (\beta-1)\left(q_n x_{n+1-k}^{\beta}\right) t^{-\beta} \end{aligned} \tag{4.5.16}$$

where $c_{n+1}\Delta z_{n+1} < t < c_n \Delta z_n$

using (4.5.15), equation (4.5.16) becomes

$$\begin{aligned} \Delta\left((c_n \Delta z_n)^{\beta-1}\right) &\leq (-\beta+1)\left(-q_n x_{n+1-k}^{\beta}\right)(c_{n+1}\Delta z_{n+1})^{-\beta} \\ &\leq (-\beta+1) q_n (1-p)^{\beta} Q_{n+1}^{\beta} (c_{n+1}\Delta z_{n+1})^{\beta} (c_{n+1}\Delta z_{n+1})^{-\beta} \end{aligned}$$

Hence

$$\Delta\left((c_n \Delta z_n)^{1-\beta}\right) \leq -(\beta-1)(1-p)^{\beta} q_n Q_{n+1}^{\beta} \ , \ \ n \geq N \tag{4.5.17}$$

Summing (4.5.17) from N to n, we have

$$(c_{n+1}\Delta z_{n+1})^{1-\beta} - (c_N \Delta z_N)^{1-\beta} \leq -(1-p)^{\beta}(\beta-1)\sum_{s=N}^{n} q_s Q_{s+1}^{\beta}$$

and so letting $n \to \infty$, we obtain

$$\sum_{s=N}^{n} q_s Q_{s+1}^{\beta} < \infty$$

which contradicts (4.5.10). □

Theorem 4.5.3: Assume that (C_1) and (C_2) hold and $0 < \beta < 1$ and $k \geq m$ in equation (4.5.1). If

$$\sum_{n=n_0}^{\infty} q_n Q_{n+1} = \infty \ , \tag{4.5.18}$$

Then all solutions of equation (4.5.1) are oscillatory.

Proof: Again proceeding as in the proof of theorem 4.5.1, we see that lemma 4.5.1 holds so

(I) $z_n > 0$, $c_n \Delta z_n > 0$

then (4.5.13) and (4.5.14) hold, we have

$$Q_n \le 1 \quad \text{and} \quad Q_n^{\beta} > Q_N \quad \text{for large } n$$

from (4.5.14) we get

$$\sum_{n=N}^{\infty} q_n Q_{n+1} \le \sum_{n=N}^{\infty} q_n Q_{n+1-k}^{\beta} < \infty$$

a contradiction to (4.5.18)

(II) $z_n > 0$, $c_n \Delta z_n < 0$

from (4.5.11) we have

$$-\Delta z_{n+1} \ge \left(\frac{1}{c_{n+1}}\right) \sum_{s=n}^{\infty} q_s x_{s+1-k}^{\beta} \quad , \text{ for } n \ge N$$

If we consider the difference $\Delta\left(z_n^{2\varepsilon}\right), \varepsilon > 0$ is such that $2\varepsilon < 1 - \beta$ and

$$-\Delta\left(z_{n+1}^{2\varepsilon}\right) = -2\varepsilon t^{2\varepsilon - 1} \Delta z_{n+1}$$

$$\ge 2\varepsilon t^{2\varepsilon-1} \left(\frac{1}{c_{n+1}}\right) \sum_{s=N}^{\infty} q_s x_{s+1-k}^{\beta}$$

$$\ge 2\varepsilon z_{n+1}^{2\varepsilon-1} \left(\frac{1}{c_{n+1}}\right) \sum_{s=N}^{\infty} q_s x_{s+1-k}^{\beta} \quad , \quad t \in \left(z_{n+2}, z_{n+1}\right)$$

and $\{z_n\}$ is decreasing, then

$$-\Delta\left(z_{n+1}^{2\varepsilon}\right) \ge \frac{2\varepsilon}{c_{n+1}} \sum_{s=N}^{\infty} q_s x_{s+1-k}^{\beta} z_{s+1}^{2\varepsilon-1} \quad , \tag{4.5.19}$$

since $a \ge z_n > 0$ and $a \ge x_n > 0$, where a is a constant , from (4.5.4),we obtain

$$x_{n+1-k} \ge (1-p) z_{n+1} \quad , \; n \ge N \tag{4.5.20}$$

From (4.5.19) and (4.5.20) we obtain

$$-\Delta\left(z_{n+1}^{2\varepsilon}\right) \ge \frac{2\varepsilon(1-p)^{\beta}}{c_{n+1}} \sum_{s=N}^{n} q_s z_{s+1}^{\beta+2\varepsilon-1}$$

$$\ge \frac{K}{c_{n+1}} \sum_{s=N}^{n} q_s$$

where $K = 2\varepsilon(1-p)^{\beta} a^{\beta} + 2\varepsilon - 1$

Summing the last inequality we get

$$z_{N+1}^{2\varepsilon} - z_{n+2}^{2\varepsilon} \ge K \sum_{i=N}^{n} q_i \sum_{s=i}^{n} \frac{1}{c_{s+1}}$$

as $n \to \infty$, we obtain

$$\sum_{i=N}^{n} q_i Q_{i+1} < \infty \quad ,$$

which is a contradiction. □

In the following two theorems, we suppose that equation (4.5.1) satisfies

(C4) $\sum_{n=i}^{j} p_n = 0$ whenever $j \le i - 1$

Theorem 4.5.4: Assume that (C_4) holds and there exists a constant $p \in (0,1)$, such that

$$p \le p_n \le 1 \text{ , for large } n \tag{4.5.21}$$

If there exists $\lambda > 0$ such that

$$\liminf_{n\to\infty} \left[q_n \exp\left(-e^{\lambda n}\right)\right] > 0 \quad , \tag{4.5.22}$$

then all bounded solutions of equation (4.5.1) are oscillatory provided that $\delta = -1$ and $c_n = 1$

Proof: Suppose that equation (4.5.1) has a bounded eventually solution $\{x_n\}$, let $z_n = x_n - p_n x_{n-m}$, then $\{z_n\}$ is bounded and $\Delta^2 z_n = q_n x_{n-k}^{\beta} \ge 0$ for large n , thus Δz_n is monotone and does not change sign eventually, by theorem 1.4.1, we obtain that there exists an integer $n_1 > n_0$, such that

$$z_n > 0 \ , \quad -\Delta z_n > 0 \text{ , for } n \ge n_1 \tag{4.5.23}$$

Choosing N, such that

$$\lambda > \frac{1}{k + Nm} \ln \beta \quad ,$$

then for $n \ge n_1 + k + Nm$

$$\begin{aligned} \Delta^2 z_n &= q_n x_{n-k}^{\beta} \\ &\ge q_n \left(z_{n-k} + p z_{n-k-\tau} + \ldots + p^N z_{n-k-Nm}\right)^{\beta} \\ &\ge p^{\beta N} q_n z_{n-k-Nm}^{\beta} \end{aligned}$$

Thus

$$\Delta^2 z_n \ge p^{\beta N} q_n z_{n-k-Nm} \quad , \quad n \ge n_1 + k + Nm$$

Summing the last inequality from n to $m-1$ we have

$$\Delta z_n + \sum_{i=n}^{\infty} p^{\beta N} q_i z_{n-k-Nm}^{\beta} \le 0$$

Thus

$$\Delta z_n + p^{\beta N} q_n z_{n-k-Nm}^{\beta} \le 0 \quad , \quad n > n_1 + k + Nm$$

Note that $\lambda > \dfrac{1}{(k + Nm)\ln \beta}$ and condition (4.5.22), then by lemma 1.4.2 the last inequality has no eventually positive solution, which is a contradiction. □

Theorem 4.5.5: Assume that (C_4) holds and there exists a $p^* \in (1, \infty)$, such that

$$1 \le p_n \le p^* \text{ , for large } n \tag{4.5.24}$$

and

$$\sum_{n=n_0}^{\infty} n q_n = \infty \ , \tag{4.5.25}$$

then all bounded solutions of equation (4.5.1) are oscillatory provided that $\delta = -1$ and $c_n = 1$.

Proof: Let $\{x_n\}$ be an eventually bounded positive solution of equation (4.5.1), again set $z_n = x_n - p_n x_{n-m}$, proceeding as in the proof of theorem (4.5.4), we have

$$z_n > 0 \ , \quad -\Delta z_n > 0 \ \text{ eventually} \tag{4.5.26}$$

Hence, $x_n > x_{n-m}$ eventually. Choosing $M > 0$ and $n_1 > n_0$, such that $x_n > M$, $n \ge n_1$, so

$$\Delta^2 z_n > M^{\beta} q_n \ , \ n \ge n_1 \tag{4.5.27}$$

Summing (4.5.27) and using (4.5.26), we have

$$z_n > M^{\beta} \sum_{i=n}^{\infty} (i-n+1) q_i \ ,$$

hence

$$\sum_{i=n}^{\infty} (i-n+1) q_i < \infty \ ,$$

and so

$$\sum_{n=n_0}^{\infty} n q_n < \infty \ ,$$

which is a contradiction. □

Example 4.5.1:

$$\Delta\left(2^n \Delta\left(x_n + \frac{1}{4} x_{n-2}\right)\right) + 3(5^n) x_{n-4} = 0 \ , \ \ n \ge 4 \tag{4.5.28}$$

To check the conditions of theorem 4.5.1

$$\sum_{n=n_0}^{\infty} 3(5^n)\left(1 - \frac{1}{4}\right) = \sum_{n=n_0}^{\infty} \frac{9}{4}(5^n) = \infty$$

$$Q_n = \sum_{s=n}^{\infty} \frac{1}{2^n} = 2^{-n+1}$$

now

$$\sum_{s=n}^{n+1} 3(5^s) 2^{-s+1} = 6\left(\frac{5}{2}\right)^n \left[1 + \frac{5}{2}\right] = 21\left(\frac{5}{2}\right)^n > \frac{5}{4}$$

It follows by theorem 4.5.1 that all solutions of the above equation are oscillatory.

Example 4.5.2:

$$\Delta\left(2^n \Delta\left(x_n + \frac{1}{4} x_{n-2}\right)\right) + 2^5 x_{n-5}^3 = 0 \ , \ \ n \ge 5 \tag{4.5.29}$$

To check the conditions of theorem 4.5.2

$$\sum_{n=n_0}^{\infty} q_n Q_{n+1}^{\beta} = \sum_{n=n_0}^{\infty} 2^5 \left(2^{-n}\right)^3 = \sum_{n=n_0}^{\infty} 2^5 \left(\frac{1}{8}\right)^n = \infty$$

So by theorem 4.5.2 we found that equation (4.5.29) is oscillatory.

Remark 4.5.1: The results of this section are referred to [17] and [24].

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