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Analysis and solutions of heat radiation

and

conduction problems

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Declaration

I certify that thesis, submitted for the degree of Master, is the result of my own research except where otherwise acknowledged, and that thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ……

Qasem Mohammed Ibrahim Al-Heeh

Date: 13/ 1/ 2008

Dedication

To My Father, My Mother, My Wife And My sons

Omar, May and Mohammed

Acknowledgement

Thanks is given first to God.

I would like to express my thanks to my supervisor, Dr. Naji Qatanani for his help and support during all phases of my graduate study.

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Abstract

This work is concerned with the numerical solution of the heat radiation equation and the mathematical analysis of the problem involving coupling radiation with heat conduction. First , we present a systematic derivation of the equation of the heat radiation. This is proceded by thorough definitions of all quantities needed to derive this equation. As the governing equation of radiation is a Fredholm integral equation, the idea of using the boundary element method to the solution of this equation naturally arises. Hence only the boundary of the domain needs to be discretized. This is equivalent to a reduction of the dimensionality of the problem by one. This reduction leads to a substantial time economy in both data preparation and computing. The typical discretization used in boundary element method is the Galerkin-Bubnov scheme. This discretization process transforms the governing integral equation to a linear system of algebraic equations. As the linear system is symmetric and positive-definite we use the conjugate gradient method to solve it. To demonstrate the high efficiency of this iterative method, we construct a numerical experiment for two-dimensional convex enclosure geometries.

Moreover, we analyze a model for the radiative heat transfer in materials that are conductive, grey and semitransparent. The most important feature of this model is the non–local interaction due to exchange of radiation. This, together with the nonlinearity arising from the well–known Stefan–Boltzmann law, makes the resulting problem non–monotone. We will prove that the operator defining the problem is pseudomonotone. Hence we can prove the existence of weak solution for the cases where the coercivity can be obtained.

In the general case, we prove the solvability of the system using the technique of sub and supersolutions.

الملخص

دراسة وتحليل ظاهرتي الأشعاع والتوصيل الحراري بالطرق الرياضية المتقدمة, لذلك نقوم بداية باشتقاق معادلة الاشعاع الحراري والتى هي عبارة عن معادلة تكاملية من النوع الثاني. والتي تعرف ب "Fredholm integral equation of the second kind".

ومن ثم تحويل هذه المعادلة التكاملية باستخدام طريقة Boundary element method والذي يعتمد على Galerkin scheme الى نظام معادلات خطية. وبالتالي حل هذا النظام الخطي باستخدام طريقة Conjugate gradient method .

علاوة على ذلك نقوم ايضا بفحص وتحليل النظام الذي يتم فيه ربط الاشعاع الحراري مع التوصيل الحراري والمعطي على شكل Boundary value problem.

من أهم صفات هذا النظام هي صفة التفاعل غير المحلي نتيجة للاشعاع الحراري.

هذا وبالاضافة الى الصفة الغير خطية الناتجة من قانون ستيفان – بولتزمان يجعل المشكلة non-monotone. لقد قمنا ببرهنة ان Operator الذي يعرف هذة المشكلة هوPseudomonotone. نتيجة لذلك قمنا ايضا ببرهنة وجود Weak solution في الحالات التي يوجد فيها Coercivity . في الحالة العامة قمنا ايضا ببرهنة وجود حل للنظام باستخدام Sub and supersolutions .

Contents

Introduction

All bodies at high temperatures emit energy in a form of electromagnetic waves. A portion of this energy when impinging other bodies is absorbed. As a result, net energy flow occurs from a body of higher temperature to a body having lower temperature. This mode of energy transfer is called heat radiation. Heat radiation is, as each wave propagation phenomenon, of dual nature. It possesses the continuity properties of electromagnetic waves and the corpuscular properties characteristic of photons.

Radiation plays a very significant role in energy transfer at elevated temperatures and in the presence of rarefied gases. The amount of heat transported by radiation in industrial furnaces and combustion chambers typically reaches 90%. Heat exchange in space and solar heating devices is 100 % due to heat radiation. Even at relatively low temperatures characteristic of central heating systems, nearly half of the heat is transferred by radiation. Thus, radiative heat analysis constitutes the crucial portion of the calculation of temperature fields in various branches of science and technology.

Owing to the progress in computer technology, mathematical modeling has become a cheap and reliable tool of engineering design. Almost all phenomena that scientists deal with are governed by differential equations. There are many well established numerical techniques for solving differential equations of mathematical models. Radiation is one of the few phenomena that is governed by an integral equation. This feature is a source of both conceptual and computational difficulties to most scientists and engineers whose mathematical backgrounds are based on differential

equations. Additional complexities inherent in heat radiation computations result from the severe nonlinearity and very complex characteristics of the material properties appearing in the radiation transport equations. Another important energy transfer mode is heat conduction. The difference between heat radiation and heat conduction can be discussed briefly as follows. First the physical consequences arising from the nature of the integral equations will be pointed out. Consider a point laying on a boundary of an enclosure formed by solid walls. The temperature field within the solid walls is obtained upon solving a differential equation. The conductive heat flux is then obtained by differentiation of the temperature field at that point. Thus, the conductive heat flux depends mostly on temperatures laying in the close vicinity of the point under consideration. Radiative heat flux gained by a point laying on the concave surface of the solid is obtained upon solving an integral equation. This means that the radiative flux depends on all the temperatures of this surface. Contrary to the case of heat conduction, temperatures at points laying far from the considered point can significantly influence the heat flux at that point. Moreover, temperatures in the nearest vicinity of the point under consideration often do not exert any influence on the radiative flux at this point. Because radiation is transporterd via electromagnetic waves, it can be transferred even a vacuum. Other heat transfer modes, i.e. conduction and convection, require a physical medium for heat interchange to occur.

The simplest possible case of radiative heat exchange is two parallel isothermal black surfaces separated by a transparent medium. The Steffan–Boltzmann law states that in this case the heat flux is proportional to the difference of the fourth powers of the surface temperatures. This is in contrast with conductive heat transfer where the heat

flux is proportional to the temperature gradient. Typically, material properties entering the equations of radiative transfer depend strongly on the length of the electromagnetic wave. In the case of gases this dependence assumes a very complex form, arising from quantum mechanics.

Another characteristic feature of radiation is that it can be transferred directly from one location to another only when one point can be 'seen' when looking from another, i.e. it does not lay in a shadow zone. The presence of shadow zones should be taken into consideration in heat radiation calculations. This leads often to complex algorithms and long computing times, see for example [10, 11, 12]. Most of heat transfer problems are solved now a days using numerical method. The common feature of these numerical methods is the discretization of the problem, i.e. transforming the governing equations to a system of algebraic equations. There are three discretization methods among many other available techniques that are commonly used to solve heat transfer problems. These are Finite Differernce Method (FDM), Finite Element Method (FEM) and Boundary Element Method (BEM). The popularity of these methods is due to their simplicity and practicality in solving these types of problems. As the governing equation of radiation is an integral one, the idea of using the boundary element method to the solution of radiation problems naturally arises. Employing BEM for the solution to the heat radiation problems has been addressed in the literature [1, 4, 13, 14, 15, 16]. This technique is proved to be efficient and easy to implement. Because the integral equation of the BEM is formulated usually only on the boundary of the domain, hence only the boundary needs to be descretized. It is equivalent to the reduction of the dimensionality of the problem by one. This reduction leads to a substantial time economy in computing.

This thesis is organized in the following manner: The initial part of Chapter 1 contains general characteristics of heat radiation and the derivation of the radiation integral equation. Chapter 2 contains the details of the boundary element method discretization technique applied to the heat radiation integral equation. This discritization scheme will transform the integral equation into a set of algebraic equations. In Chapter 3 we present an efficient iterative method called Conjugate–Gradient method to solve this linear system including some numerical examples.

Chapter 4 examines the problem of coupling radiation with heat conduction. We analyze a model for the radiative heat transfer in bodies that are conductive, grey and semitransparent.

Index of Special Notation

R^n	Euclidean n–dimensional space
R	The set of all real numbers
Ω	Solid angle
$d\Omega$	Differential solid angle
θ	Azimuth angle
ϕ	Polar angle
T	The temperature

C_1 and C_2 constants in Planck's spectral energy distribution

$C_1 = 0.59544 \times 10^{-16}\ W.m^2$ and $C_1 = 1.4388 \times 10^{-2}\ m.K$

σ	Stefan–Boltzmann constant which has the value
	$5.669 \times 10^{-8}\ W/(m^2.K^4)$
ε	Emissivity
ρ	Reflexivity
λ	Wavelength
α	Absorptivity
γ	Scattering
Q^x	Radiative heat flux.
i , o	Incoming and Outgoing radiation respectively.
ϕ_y	The angle between the line of sight and the normal at point y
L_{xy}	Line of sight

T^m	Temperature of medium
q_{rad}	Heat flux due to radiation
h_{rad}	Heat source due to radiation
k	The coefficient of heat conductivity
$\frac{\partial}{\partial n}$	Differentiation along the outward normal
$\langle .,. \rangle$	Inner product
q_i^∞ *and* q_s^∞	Additional radiattive heat source/flux arriving from outside of the system

$$L_\mu^p = L^p(\Omega \cup \Gamma; \mu)$$

$$\| f \|_p = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}, \qquad 1 \le P < \infty$$

Chapter 1

Formulation of the problem

The heat flux of radiative energy emitted by an element blackbody surface known as the blackbody emissive power can be computed upon integrating the normal component of the intensity vectors over the hemisphere centered at that surface,

$$E_b = \int_{2\pi} J_b \cos\phi \, d\Omega \tag{1.1}$$

where J_b is the blackbody intensity of radiation. We can express the differential solid angle $d\Omega$ in terms of the polar angle ϕ and the azimuth angle θ by

$$d\Omega = \sin\phi \, d\phi \, d\theta \tag{1.2}$$

By virtue of equation (1.2) and performing an appropriate integration, equation (1.1) yields

$$E_b = J_b \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta = \pi J_b \tag{1.3}$$

Equation (1.3) links the intensity and the emissive power of the blackbody. The blackbody spectral emissive power can be expressed as a function of temperature and wavelength, that is,

$$E_{b\lambda} = \frac{2\pi c_1}{\lambda^5 \left[\exp\left(c_2 / \lambda T \right) - 1 \right]} \tag{1.4}$$

Equation (1.4) is known as the Plank's function, where λ is the wavelength, T is the temperature, c_1 and c_2 are constants which have the value $0.59544 \times 10^{-16}\ W.m^2$ and $1.4388 \times 10^{-2}\ m.K$ respectively.

The energy emitted by a unit blackbody surface in a unit time within the entire spectrum can be computed from the Stefan–Boltzmann law

$$E_b = \int_0^\infty E_{b\lambda}\, d\lambda = \sigma T^4 \tag{1.5}$$

where σ is the Boltzmann constant which has the value $5.669 \times 10^{-8}\ W/(m^2.K^4)$. Let J_ϕ denote the intensity of radiation emitted in a direction inclined by an angle ϕ to the surface normal. For surfaces that are diffusive and gray as emitters and reflectors the emissivity ε does not depend on the direction. It follows

$$J_{\phi} = \varepsilon J_b \tag{1.6}$$

Consider an elemental surface impinged by rays incoming from all directions. In this case only normal components of the incoming intensity vectors contribute to the energy absorbed by the surface. Let ϕ be the angle made by the incoming ray and the normal to the surface, the irradiation can be written as

$$i = \int_{2\pi} J_{\phi}^{i} \cos\phi \, d\Omega \tag{1.7}$$

where J_{ϕ}^{i} is the radiation intensity incident on the infinitesimal surface from the direction inclined by ϕ to the normal. For diffusively reflecting and emitting surfaces, radiosity and irradiance are related by a simple heat balance over the infinitesimal surface. That is, (see [20])

$$b = \varepsilon E_b + \rho i \tag{1.8}$$

where $\rho = (1-\varepsilon)$ is the reflectivity.

Let Q^x be the radiative heat flux defined by

$$Q^x = \varepsilon \int_{2\pi} \left(J_{\phi}^{i} - J_b \right) \cos\phi \, d\Omega \tag{1.9}$$

In view of equations (1.1) and (1.7) equation (1.9) can be written as

$$Q^x = \varepsilon i - \varepsilon E_b \tag{1.10}$$

With the help of equation (1.8), equation (1.10) takes the form

$$b = E_b + \frac{1-\varepsilon}{\varepsilon} Q^x \tag{1.11}$$

Let J_ϕ^o be the radiation intensity leaving the surface in the direction inclined by the angle ϕ to the normal. This outgoing radiation associated with an elemental solid angle can be expressed as

$$db = J_\phi^o \cos\phi \, d\Omega \tag{1.12}$$

Integrating equation (1.12) over the solid angle 2π yields

$$b = \pi J_\phi^o \tag{1.13}$$

To this end, the intensity of radiation leaving the surface can be written in terms of emissive power and radiative heat flux. Inserting (1.11) into equation (1.13) gives

$$J_{\phi}^{o} = \frac{1}{\pi}\left(E_b + \frac{1-\varepsilon}{\varepsilon} Q^{x} \right). \tag{1.14}$$

Consider a pencil of rays traveling from point x to point y a long a line of sight. The increase of radiation intensity taking place as the radiation passes an infinitesimal path along the line of sight can be described by a differential equation see [21].

$$\frac{dJ}{dL_{xy}} = -\left[J - J_b(T^m) \right] \tag{1.15}$$

where dL_{xy} is the infinitesimal path along the line of sight and T^m denotes the temperature of the medium. Let $J(x)$ denote the intensity of the point where the ray originates and $J_b\left(T^m(x) \right)$ denote the intensity of radiation of blackbody having a temperature at point x. Hence the integrated equation of radiation can be written as

$$J_{\phi}^{i}(y) = J_{\phi}^{o}(x) + \int_{L_{xy}} J_b \left[T^m(x) \right] dL_{xy}(x) \tag{1.16}$$

According to equation (1.14) the intensity outgoing from the surface in a direction of the line of sight can be expressed as a linear combination of radiative flux and blackbody emissive power. Taking into account, and making use of equation (1.3) to replace the blackbody intensity by its emissive power, equation (1.16) can be written into the form

$$J_{\phi}^{i}(y) = \frac{1}{\pi}\left[\ E_b(x) + \frac{1-\varepsilon(x)}{\varepsilon(x)}\, Q^{x}(x)\ \right] + \frac{1}{\pi}\int_{L_{xy}} E_b\left[\ T^{m}(x)\ \right] dL_{xy} \qquad (1.17)$$

An alternative form can be obtained by substituting equation (1.17) into the heat balance on the unit surface equation (1.9). The resulting equation reads as

$$Q^{x}(y) + \varepsilon(y)\, E_b\left[\ T(y)\ \right] = \varepsilon(y) \int_{2\pi} \frac{1}{\pi}\left\{\ E_b\left[\ T(x)\ \right] + \frac{1-\varepsilon(x)}{\varepsilon(x)}\, Q^{x}(x)\ \right\} \cos\phi_y\ d\Omega \qquad (1.18)$$

where ϕ_y denotes the angle between the line of sight and the normal at point y. The integral in equation (1.18) can be converted to a surface integral. This can be accomplished by noting that the infinitesimal solid angle can be expressed in terms of the infinitesimal area of the bounding surface

$$d\Omega = \frac{dS(x)\cos\phi_x}{\left|x-y\right|^{2}} \qquad (1.19)$$

where $|x-y|$ stands for the distance between the points x and y.

Setting equation (1.19) into equation (1.18) we obtain the boundary integral equation of radiation

$$Q^x(y) + \varepsilon(y)\, E_b[\, T(y)\,] = \varepsilon(y) \int_S \left\{ E_b[\, T(x)\,] + \frac{1-\varepsilon(x)}{\varepsilon(x)}\, Q^x(x) \right\} G(x,y)\, \beta(x,y)\, dS(x) \tag{1.20}$$

where the kernel $G(x,y)$ denotes the view factor between the points x and y of S. For two–dimensional enclosure geometries $G(x,y)$ has the representation [15], [16]

$$G(x,y) = \frac{[\, n(x).(x-y)\,].[\, n(y).(y-x)\,]}{2\,|x-y|^3} \tag{1.21}$$

Moreover, for three–dimensional enclosure geometries $G(x,y)$ has the representation [15], [16]

$$G(x,y) = \frac{[\, n(x).(x-y)\,].[\, n(y).(y-x)\,]}{\pi\,|x-y|^4} \tag{1.22}$$

The visibility function $\beta(x,y)$ appearing in equation (1.20) takes into account the shadow zones and is defined by [15], [16]

$$\beta(x,y) = \begin{cases} 1 \quad , \text{ if the point } x \text{ and } y \text{ can see each other} \\ \\ 0 \quad , \text{ otherwise.} \end{cases} \tag{1.23}$$

In fact equation (1.20) can also be written in the more simplified form (see for example [13]),

$$Q(x) = \varepsilon(x)\,\sigma T^4(x) + \big(\,1-\varepsilon(x)\,\big)\int_S G(x,y)\,\beta(x,y)\,Q(y)\,dS_y \tag{1.24}$$

This is a Fredholm integral equation of the second kind with the kernel $G(x,y)$ as defined in (1.21) – (1.22).

The properties of the radiosity equation (1.20) have been thoroughly investigated in [12] [16].

Chapter Two

Discretization and boundary element method

2.1. Boundary element method and Galerkin discretization

In a two–dimensional case we let S be a curve that is given by a regular parameter representation [19].

$$S : y = Z_j(t) \quad \text{for } t \in R, \quad j = 1,.....,L \tag{2.1.1}$$

We choose on R a family of 1–periodic interval partition:

$$0 = t_0 < t < ..... < t_N = 1,$$

$$\Pi_h = \{t_k\}_{-\infty}^{\infty}, \; t_{k+N} = t_k + 1 \quad \text{with} \quad h = \max\{t_{k+1} - t_k\} \to 0. \tag{2.1.2}$$

Let $S_h^{d,r}$ be a family of 1–periodic piecewise polynomials of degree (d – 1) with respect to the partition Π_h in the sense of Babuska an Aziz [2] which is (r –1) times continuous and differentiable. We denote with $\Phi_k(t)$ the basis trial functions with a smallest possible support (*B*–splines) (see Fig.1). The approximate solution has the general form

$$Q_h(t) = \sum_{k=1}^{n} Q_k \Phi_{k,n}(t) \qquad (2.1.3)$$

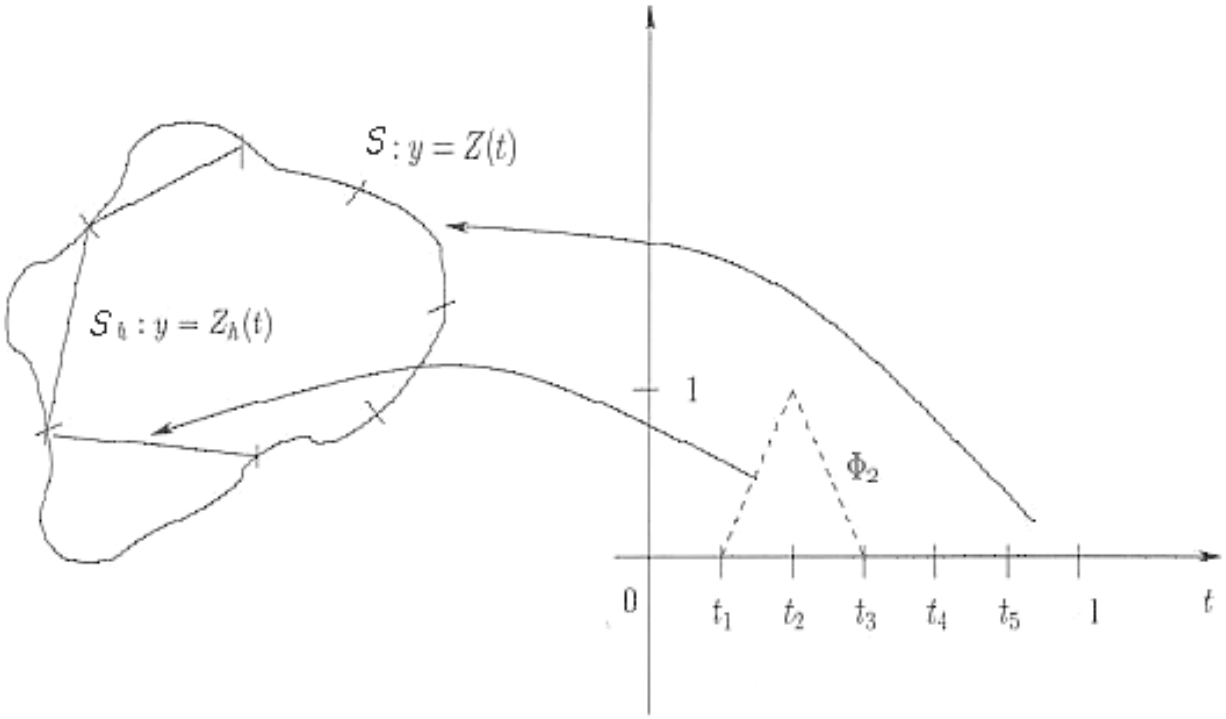
where n is the number of free grids and $Q_k \in R$, k = 1,……,n are the partition coefficients. On partition in the parameter domain we use $S_h^{m+1,1}$–Lagrange–System of finite elements. Then the local representation of S transplant these finite element function onto S_h. The ansatz function (2.1.3) on S_h will then be defined by

$$S_h : y = Z_{jh}(t) \qquad (2.1.4)$$

with

$$Z_{jh}(t) = Z_j(t_k).$$

Fig.1



2.2 Representation of System of Equations

The Fredholm integral equation (1.24) can be expressed as

$$Q \;=\; g + K\,Q \tag{2.2.1}$$

where

$$K\,Q = (1-\varepsilon)\,\widetilde{K}\,Q$$

and

$$\widetilde{K}\,Q(x) \;=\; \int_S G(x,y)\;\beta(x,y)\;Q(y)\;dS_y \qquad for\;\; x \in S \tag{2.2.2}$$

We let

$$\langle u\,,v\rangle_S \;:=\; \int_0^1 u(t)\;v(t)\,\left|\dot{x}(t)\right|\,dt\,.$$

The Galerkin discretization of the integral equation (1.24) with the ansatz function (2.1.3) is given by

$$\sum_{k=1}^{n} Q_k\,\langle \Phi_{k,n}, \Phi_{l,n}\rangle_S \;=\; \langle g, \Phi_{l,n}\rangle_S + \sum_{k=1}^{n} Q_k\,\langle K\,\Phi_{k,n}, \Phi_{l,n}\rangle_S \tag{2.2.3}$$

Equation (2.2.3) can be written in the following short form:

$$\left(A_n - B_n \right) a_n = b_n \tag{2.2.4}$$

using the abbreviation $A = (A_{l,k})\, l,k = 1,.....,n$ for the mass matrix, with

$$A_{l,k} = \langle \Phi_{k,n}, \Phi_{l,n} \rangle_S = \int_0^1 \Phi_{l,n}(t)\, \Phi_{k,n}(t) \left| \dot{x}(t) \right| dt, \tag{2.2.5}$$

$B = (B_{l,k})\, l,k = 1,.....,n$ for the view factor matrix with

$$B_{l,k} = \langle K \Phi_{k,n}, \Phi_{l,n} \rangle_S = \int_0^1 \int_0^1 \left(1 - \varepsilon(t) \right) \Phi_{l,n}(t)\, G(t,\tau)\, \Phi_{k,n}(\tau) \left| \dot{x}(t) \right| \left| \dot{x}(\tau) \right| dt\, d\tau \tag{2.2.6}$$

and the vectors $a = (Q_k)\, k = 1,.....,n$ and $b = \langle g, \Phi_{l,n} \rangle_S, l = 1,.....,n$.

Properties of the matrices

The mass matrix *A* in (2.2.4) is symmetric, positive definite and diagonal dominant hence it is invertible. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of the matrix *A*, respectively. Then follows the known estimations

$$\lambda_{\min} \|Q\|_{l^2}^2 \le (A_n Q, Q) \le \lambda_{\max} \|Q\|_{l^2}^2 \tag{2.2.7}$$

$$\frac{1}{\lambda_{\min}} \|Q\|_{l^2}^2 \le (A_n^{-1} Q, Q) \le \frac{1}{\lambda_{\max}} \|Q\|_{l^2}^2 \tag{2.2.8}$$

where $(\,.\,,\,.\,)$ denotes the Euclidean scalar product of R^n with $(Q,Q) = \|Q\|_{l^2}^2$. Furthermore

$$\|A_n\|_{l^2} = \lambda_{\max}\,,\ \frac{1}{\|A_n^{-1}\|_{l^2}} = \lambda_{\min} \quad . \tag{2.2.9}$$

Also the system of equations $(A_n - B_n)$ is symmetric and positive definite. Since the mass matrix *A* is invertible, equation (2.2.4) can then be expressed in the form

$$(I - A_n^{-1} B_n) a_n = A_n^{-1} b_n \tag{2.2.10}$$

Equation (2.2.10) can also be written as

$$Q_n = g_n + K_n Q_n\,, \tag{2.2.11}$$

where $Q_n = a_n$, $g_n = A_n^{-1} b_n$, $K_n = A_n^{-1} B_n$.

Chapter Three

Iterative method and numerical results

3.1 Conjugate Gradient Iteration

In this section we apply a general iteration method for solving simultaneous linear system, one called the conjugate gradient method, to the solution of discretizations of integral equations of the second kind. The conjugate gradient method is restricted to solving linear systems

$$C_l a_l = b_l \tag{3.1.1}$$

where

$$C_l = (A_l - B_l).$$

It is an effective method for symmetric and positive definite systems.

This CG–iteration is given by the following algorithm [8]:

1. Choose an initial vector a_l^0 and compute $r_0 = C_l a_l^0 - b_l$.

   Set $p_0 = r_0$ and $k = 0$

2. Compute

$$\alpha_k = \frac{r_k^T p_k}{p_k^T C_l p_k}$$

$$a_l^{k+1} = a_l^k + \alpha_k p_k$$

$$r_{k+1} = C_l a_l^{k+1}$$

3. Stop if $\frac{\| r_{k+1} \|_2}{\| r_k \|_2} < \varepsilon$

4. Compute

$$\beta_k = \frac{r_{k+1}^T C_l p_k}{p_k^T C_l p_k}$$

$$p_{k+1} = r_{k+1} + \beta_l p_k \ .$$

Convergence of the conjugate gradient method

From [13] follows that

$$\varepsilon . \langle Q,Q \rangle_{L^2(S)} \leq \langle AQ,Q \rangle_{L^2(S)} \leq (2-\varepsilon) . \langle Q,Q \rangle_{L^2(S)} \qquad (3.1.2)$$

where $A = (I - K)$.

Let $Q \in H_l \subset L^2(S)$ then we define

$$Q(t) = \sum_{k=1}^{n_l} Q_k \Phi_k(t) \tag{3.1.3}$$

Set equation (3.1.3) into (3.1.2) we get

$$\begin{aligned} \varepsilon . \left\| \sum_{k=1}^{n_l} Q_k \Phi_k \right\|^2_{L^2(S)} &\le \sum_{k,j}^{n_l} Q_k Q_j \langle A\Phi_k , \Phi_j \rangle_{L^2(S)} \\ &\le (2-\varepsilon) . \left\| \sum_{k=1}^{n_l} Q_k \Phi_k \right\|^2_{L^2(S)} \end{aligned} \tag{3.1.4}$$

Now

$$\begin{aligned} \left\| \sum_{k=1}^{n_l} Q_k \Phi_k \right\|^2_{L^2(S)} &= \int_0^1 \left| \sum_{k=1}^{n_l} Q_k \Phi_k(t) \right|^2 dt \\ &= \sum_{k,j}^{n_l} Q_k Q_j \int_0^1 \Phi_k(t) \Phi_j(t) dt \;=\; (A_l Q, Q) \end{aligned} \tag{3.1.5}$$

Substituting (3.1.5) into (3.1.4) yields

$$\varepsilon (A_l Q, Q) \le (C_l Q, Q) \le (2-\varepsilon) (A_l Q, Q) . \tag{3.1.6}$$

Theorem 3.1.1

For a positive matrix C_l converges the conjugate gradient iteration with the convergence estimation [9]

$$\left\| e^k \right\|_{C_l} \leq 2\left(\frac{\left(\kappa(C_l)-1 \right)^{\frac{1}{2}}}{\left(\kappa(C_l)+1 \right)^{\frac{1}{2}}} \right)^k \left\| e^0 \right\|_{C_l} \tag{3.1.7}$$

where

$$\left\| e^k \right\|_{C_l} = \left\| a_l^k - a_l \right\|_{C_l}$$

and

$$\left\| e^0 \right\|_{C_l} = \left\| a_l^0 - a_l \right\|_{C_l} .$$

3.2 Numerical results

3.2.1 Numerical examples for the solution of the system of equations

Since the convergence requirements of the CG–iteration method is satisfied [13], then we can apply now this method to solve the two–dimensional convex enclosure.

Convex Enclosure

Example

Let Ω be the domain of an ellipse. The boundary of this ellipse has the following parameterization

$$S = \left\{ x \in R^2 : x = \begin{pmatrix} a\cos 2\pi t \\ b\sin 2\pi t \end{pmatrix}, a = 4, b = 2, 0 \le t < 1 \right\} \qquad (3.2.1)$$

The computation of the coefficients

$$A_{l,k} = \langle \Phi_{k,n}, \Phi_{l,n} \rangle \ , \ b_n = \langle g, \Phi_{l,n} \rangle \ , \ B_{l,k} = \langle K\Phi_{k,n}, \Phi_{l,n} \rangle$$

have been carried out by Gaussian quadrature form.

Here we have $g(t) = \varepsilon(t)\sigma T^4(t)$ with

The emissivity coefficient $\varepsilon = 0.9$

The Boltzmann coefficient $\sigma = 5.6696 \times 10^{-8}$ and

The surface temperature $T(t) = \frac{1}{2}(T_1 + T_2) - \frac{1}{2}(T_2 - T_1)\cos 2\pi t$,where

$T_1 = 1000$ and $T_2 = 1800$.

Table (I) shows the numerical results for the solutions by using CG –iteration method for the ellipse. The number n_l denotes the dimension parameter of the solved problem and l is the level number.

Table I. conjugate gradient scheme

n_l	CG	
	Iter	Sec
32	16	<1
64	18	<1
128	19	<1
256	20	0.51
512	20	2.05
1024	20	8.16

Chapter Four

Coupling heat radiation with conduction

4.1 Introduction

Let us consider a three–dimensional connected domain Ω filled by conductive–semitransparent and gray material. We assume that Ω is surrounded by an opaque medium and we denote the boundary of Ω by Γ. If we assume that the system is at steady state, the temperature field is modeled by stationary heat equation that is augmented by the terms modeling radiative heat exchange. Locally, the role of radiation is that of an additional heat source. Hence we can write the balance equations for absolute temperature T as

$$-k\,\Delta T = f + h_{rad} \quad \text{in} \quad \Omega \tag{4.1.1}$$

$$k\,\frac{\partial T}{\partial n} = g + q_{rad} \quad \text{on} \quad \Gamma \tag{4.1.2}$$

where h_{rad} is the heat source due to radiation, q_{rad} is the heat flux due to radiation, k is the coefficient of heat conductivity and f and g are known data, namely the internal heat source and the heat flux coming to the surface from outside of the system.

We denote by $\tilde{Q}_i$ the amount of emitted and scattered radiation in the volume, and by $\tilde{Q}_s$ the corresponding quantity on the surface. Here we distinguish with " ~ " the physical variables from their scaled variants that are used in mathematical analysis. The radiative heat source can be written as

$$h_{rad} = (\alpha + \gamma)\left(\tilde{K}_{ii}\tilde{Q}_i + \tilde{K}_{is}\tilde{Q}_s \right) - \tilde{Q}_i \qquad (4.1.3)$$

Where $\tilde{K}_{ii}\tilde{Q}_i + \tilde{K}_{is}\tilde{Q}_s$ is the amount of incoming radiation, $\alpha \geq 0$ is emission$\big/$absorbtion coefficient and $\gamma \geq 0$ is the scattering coefficient. $\tilde{K}_{ii}$ and $\tilde{K}_{is}$ are integral operators with kernels defined on $\Omega \times \Omega$ and $\Omega \times \Gamma$, respectively. On the radiating boundary we have the radiative heat flux

$$q_{rad} = \tilde{K}_{si}\tilde{Q}_i + \tilde{K}_{ss}\tilde{Q}_s - \tilde{Q}_s \qquad (4.1.4)$$

with similar interpretation. The radiosities $\tilde{Q}_i$ and $\tilde{Q}_s$ depend on the temperature in a non–linear and non–local fashion. Namely, we have that radiosity at a point is a sum of Stefan–Boltzmann radiation emitted by the point and the scattered$\big/$reflected part of incoming radiation. Thus, we can write

$$\widetilde{Q}_i \;=\; 4\,\alpha\,\sigma\,T^4 + \gamma\left(\;\widetilde{K}_{ii}\widetilde{Q}_i + \widetilde{K}_{is}\widetilde{Q}_s\;\right) \qquad (4.1.5)$$

$$\widetilde{Q}_s \;=\; \varepsilon\,\sigma\,T^4 + (1-\varepsilon)\left(\;\widetilde{K}_{si}\widetilde{Q}_i + \widetilde{K}_{ss}\widetilde{Q}_s\;\right) \qquad (4.1.6)$$

where $0 \le \varepsilon \le 1$ is the emissivity coefficient of the surface. In the some cases the radiative coefficients α , γ and ε may depend on temperature. However, in this thesis we shall restrict ourselves to materials which do not have this property. To this end, the organization of this chapter is as follows: In section 2 we derive in detail the model that was sketched above, in section 3 we analyze the integral operators appearing in the model. Furthermore, we show that Q_i and Q_s can be solved from (4.1.5) and (4.1.6), so that (4.1.1) – (4.1.2) can be viewed as a non–linear and non–local problem for *T* alone. In section 4 we study the solvability of the problem. The main problems are that T^4 is not necessarily integrable on the boundary when *T* is in $H^1(\Omega),$ and that the problem is not monotone. First we consider situations where we have sufficient a priori information to deal with T^4 on the boundary. This means that either *T* is known on Γ or we can prove the coercivity in $H^1(\Omega) \cap L^5(\Gamma)$. The case lacking the proof of coercivity is more difficult. However, we can prove the solvability provided there exists a pair of sub–and supersolutions. Then we give some examples of cases where the super–and subsolutions can be constructed. Throughout this thesis we shall assume that Ω is connected and it has Lipschitz boundary, that is, the boundary Γ can be locally presented as a graph of a Lipschitz function.

We shall also use the standard Lebesque and Sobolev spaces, for their notation see for example [6]. We denote by $(f)^+$ the positive part of a function f: $(f)^+ = f$ when $f > 0$ and $(f)^+ = 0$ otherwise. We shall call an operator positive, if it maps nonnegative functions to nonnegative functions.

4.2 Heat radiation model

The basic theory for radiative heat transfer is well established in [20, 21]. However, the point of view and notation adopted in this thesis is some what different to those references. Hence we try to give brief introduction to the basic principles behind the radiative terms h_{rad} and q_{rad}.

Interior heat balance

Let x be an interior point of a three–dimensional semitransparent material Ω. The radiative heat source at x equals to the difference between absorbed and emitted energy. In addition, part of the radiation may be redirected by scattering. Moreover, let $J(w)$ be the intensity of radiation that is incident from direction w, and $\hat{Q}_i(w)$ be the intensity of radiation that leaves x to direction w due to emission and scattering. Then the radiative heat source is given by

$$h_{rad} = \int_S \left(\alpha(x) + \gamma(x) \right) J(w)\, dw - \tilde{Q}_i(x) \quad , \qquad (4.2.1)$$

where S is the surface of the unit ball , α is emission $\big/$ absorption coefficient and γ is the scattering coefficient. In fact these coefficients depend on both direction w and wavelength λ of the radiation. In this case, J and $\hat{Q}_i$ are also functions of λ

and one has to integrate the corresponding formula over λ. We assume that the material is gray this means, that α and γ do not depend on λ. Thus we need to model only the total intensities, not the spectral quantities. Moreover, we assume that the material is isotropic absorber, emitter and scatterer. This means that the coefficients and $\hat{Q}_i$ do not depend on w and we can deal with the volumetric radiosity

$$\tilde{Q}_i = \int_S \hat{Q}_i \, dw = 4\pi \hat{Q}_i .$$

J on the other hand depends always on w and, also, it depends on radiosity in all points that are visible to x. If we let $p = p(w)$ be the point where the direction w meets the boundary (for the first time) and by r and s the points on the line $x\,p$, then

$$J(w) = \hat{Q}_s\big(p(w)\big) \exp\Big[-\int_{p(w)}^{x} \beta(s)\, ds \Big] + \int_{p(w)}^{x} \exp\Big[-\int_{r}^{x} \beta(s)\, ds \Big] \frac{\tilde{Q}_i(r)}{4\pi}\, dr \quad (4.2.2)$$

where $\beta = \alpha + \gamma$ determines the rate of attenuation due to absorption and scattering. $\hat{Q}_s\big(p(w)\big)$ is the intensity of emitted and scattered surface radiation at p. Now, we assume that the surface is gray and diffuse as emitter and reflecter.

Then we can write $\hat{Q}_s\big(p(w)\big) = \frac{1}{\pi} \tilde{Q}_s(p)$, where $\tilde{Q}_s(p)$ is the radiosity at p.

Integration of (4.2.2) over S

gives

$$\int_S J(w) \;=\; \tilde{K}_{is}\,\tilde{Q}_s \;+\; \tilde{K}_{ii}\,\tilde{Q}_i \quad ,$$

where

$$\left(\;\tilde{K}_{is}\,\tilde{Q}_s\;\right)(x) \;=\frac{1}{\pi}\int_S \tilde{Q}_s\big(p(w)\big)\exp\Big[\;-\int_{p(w)}^{x}\beta(s)\,ds\;\Big]\,dw\;,$$

$$\left(\;\tilde{K}_{ii}\,\tilde{Q}_i\;\right)(x) \;=\int_S\int_{p(w)}^{x}\frac{\tilde{Q}_i(r)}{4\pi}\;\exp\Big[\;-\int_r^x\beta(s)\,ds\;\Big]dr\,dw\quad,$$

Hence we have

$$h_{rad}\;=\beta(x)\left(\;\tilde{K}_{is}\,\tilde{Q}_s\;\right)(x)\;+\;\beta(x)\left(\;\tilde{K}_{ii}\,\tilde{Q}_i\;\right)(x)-\tilde{Q}_i(x)\qquad\qquad(4.2.3)$$

Surface heat balance

As in the interior, the radiative flux on the surface consists of the difference of incoming and outgoing radiation. Let x be a point on the surface Γ that separates the semitransparent material from its opaque surrounding. We assume that Γ has a unique outer unit normal vector n at x. The surface can emit and receive radiation

only in directions pointing to the semitransparent material. Thus the radiative flux at x is

$$q_{rad}(x) \;=\; \int_S J(w)\,\big(\; w\cdot n(x)\;\big)^{+}\,dw \;-\; \tilde{Q}_s(x) \quad ,$$

where $J(w)$ is as in (4.2.2) . By denoting

$$\big(\; \tilde{K}_{si}\;\tilde{Q}_i \;\big)(x) \;=\; \int_S \int_{p(w)}^{x} \frac{\tilde{Q}_i(r)}{4\pi}\;\exp\Big[\; -\int_r^x \beta(s)\,ds \;\Big]\,dr\,\big(\; w\cdot \mathrm{n}(x)\;\big)^{+}\,dw \quad ,$$

$$\big(\; \tilde{K}_{ss}\;\tilde{Q}_s \;\big)(x) \;=\; \int_S \frac{\tilde{Q}_s(\,p(w)\,)}{\pi}\;\exp\Big[\; -\int_{p(w)}^{x} \beta(s)\,ds \;\Big]\big(\; w\cdot \mathrm{n}(x)\;\big)^{+}\,dw \quad ,$$

Thus we can write

$$q_{rad}(x) \;=\; \big(\; \tilde{K}_{si}\;\tilde{Q}_i \;\big)(x) \;+\; \big(\; \tilde{K}_{ss}\;\tilde{Q}_s \;\big)(x) - \tilde{Q}_s(x) \qquad (4.2.4)$$

Radiosities

It remains to couple the radiosities $\tilde{Q}_i$ and $\tilde{Q}_s$ with the absolute temperature. According to Stefan–Boltzmann law, emission over unit volume equal to $4\alpha n^2 \sigma T^4$, where n is the index of refraction and σ is the Stefan–Boltzmann constant. For simplicity we assume $n = 1$ throughout this thesis. Now the volumetric radiosity $\tilde{Q}_i$ consists of Stefan–Boltzmann radiation together with the scattered part of the incident radiation

$$\tilde{Q}_i = 4\alpha\sigma T^4 + \gamma\left(\tilde{K}_{is}\tilde{Q}_s + \tilde{K}_{ii}\tilde{Q}_i\right) \qquad (4.2.5)$$

Similarly, on the surface we have that the emission obeys the law $\varepsilon\sigma T^4$, where $0 \le \varepsilon \le 1$ is the emissivity $\big/$ absorptivity of the surface. The reflected part of the incoming intensity is $(1-\varepsilon)J$. Thus

$$\tilde{Q}_s = \varepsilon\sigma T^4 + (1-\varepsilon)\left(\tilde{K}_{si}\tilde{Q}_i + \tilde{K}_{ss}\tilde{Q}_s\right) \qquad (4.2.6)$$

Cartesian form of integral

Let $w = \dfrac{(z-x)}{\|z-x\|}$. If z is an interior point we have the coordinate transform $dr\ dw = \dfrac{dV}{\|z-x\|^2}$ and when z is on the surface we have $dw = \dfrac{w \,.\, n(z)\, dS}{\|z-x\|^2}$.

In a non–convex domain we consider the radiosities only in points which are visible on x. Hence we multiply the kernels by visibility factor: $v(x,z) = 1$ if x and z can see each other and $v(x,z) = 0$ otherwise.

If we denote $\tau(x,z) = \exp\left[-\int_x^z \beta(s)\, ds \right] v(x,z)$, then the integral formulas above can be expressed in Cartesian coordinates

$$\left(\tilde{K}_{ii}\, \tilde{Q}_i \right)(x) = \int_\Omega \frac{\tau(x,z)}{4\pi\|z-x\|^2}\, \tilde{Q}_i(z)\, dV(z) \qquad (4.2.7)$$

$$\left(\tilde{K}_{is}\, \tilde{Q}_s \right)(x) = \int_\Gamma \frac{\tau(x,z)\, n_z \cdot (z-x)}{\pi\|z-x\|^3}\, \tilde{Q}_s(z)\, dS(z) \qquad (4.2.8)$$

$$\left(\tilde{K}_{si}\, \tilde{Q}_i \right)(x) = \int_\Omega \frac{\tau(x,z)\, n_x \cdot (z-x)}{4\pi\|z-x\|^3}\, \tilde{Q}_i(z)\, dV(z) \qquad (4.2.9)$$

$$\left(\tilde{K}_{ss}\, \tilde{Q}_s \right)(x) = \int_\Gamma \frac{\tau(x,z)\, n_x \cdot (z-x)\, n_z \cdot (z-x)}{\pi\|z-x\|^4}\, \tilde{Q}_s(z)\, dS(z) \qquad (4.2.10)$$

Systems where part of the radiation can escape

In some practical cases part of the boundary allows the radiation to escape the system. Think of an oven with a hatch, for example. In this case we denote $\partial\Omega = \Gamma \cup \Gamma_0$, where Γ is radiating part of the boundary as earlier and Γ_0 is the transparent part of the boundary. Γ separates Ω from opaque surroundings whereas Γ_0 is an interface between Ω and transparent surroundings. This does not change the structure of the model derived above. However, additional data terms appear in (4.2.3) – (4.2.6), namely the radiation coming from outside of the system. Moreover, the integral operators become contractive in the sense specified in the next section.

4.3 Operator form of the radiative heat sources

In this section we introduce simplifying notations and derive tools for the existence result. In particular, we show h_{rad} and q_{rad} can be expressed by means of the temperature T only. Unless otherwise stated we consider the problem where Γ is an enclosure ($\partial\Omega = \Gamma$). Now, we assume that the radiating body is absorbing and emitting at every point of Ω and Γ, so that there exist a constant c, such that $\alpha \geq c > 0$ and $\varepsilon \geq c > 0$. Moreover, we assume that $\alpha, \gamma, \beta \in L^{\infty}(\Omega)$. To simplify the notation , let us define the scaled radiosity Q as

$$Q = \begin{cases} \widetilde{Q}_i(x)/4\beta(x) & \text{if} \quad x \in \Omega \\ \widetilde{Q}_s(x) & \text{if} \quad x \in \Gamma \end{cases} \tag{4.3.1}$$

Next we define ε also in the interior Ω: we set $\varepsilon(x) = \alpha(x)/\beta(x)$ for $x \in \Omega$. In this way we have $\varepsilon \leq 1$ in both Ω and Γ and $\gamma/\beta = 1 - \varepsilon$. Now, we rewrite The system (4.2.3) – (4.2.6) as

$$h_{rad} = -4\beta\left(I - K\right)Q \qquad x \in \Omega \qquad , \tag{4.3.2}$$

$$q_{rad} = -\left(I - K \right) Q \qquad x \in \Gamma \qquad , \qquad (4.3.3)$$

$$\varepsilon \sigma T^4 = \left(I - (I - E)\, K \right) Q \qquad x \in \Omega \cup \Gamma \qquad , \qquad (4.3.4)$$

where E is the operator induced by multiplication with ε. The operator K is defined as

$$\left(K Q\right)(x) = \begin{cases} \left(K_{ii} Q\right)(x) + \left(K_{is} Q\right)(x) & \text{if} \quad x \in \Omega \\ \left(K_{si} Q\right)(x) + \left(K_{ss} Q\right)(x) & \text{if} \quad x \in \Gamma \end{cases} \qquad (4.3.5)$$

where

$$K_{ii} = \tilde{K}_{ii}\, \beta$$

$$K_{is} = \frac{1}{4} \tilde{K}_{is}$$

$$K_{si} = 4\, \tilde{K}_{si}\, \beta$$

$$K_{ss} = \tilde{K}_{ss}$$

Next, we introduce function spaces for Q and K. The standard $L^P(\Omega)$ will not do, as we want to measure Q also on Γ. Hence, we define a measure μ such that

$$\int Q\, d\mu = \int_{\Omega} 4\,\beta(x)\, Q\, dx + \int_{\Gamma} Q\, dS \tag{4.3.6}$$

Let us denote by $L^P_\mu = L^p(\Omega \cup \Gamma\,;\mu)$ the class of functions $f : \Omega \cup \Gamma \to R$ whose p–th powers are integrable with respect to μ. The corresponding norms are

$$\| f \|_p = \left(\int | f |^p d\mu \right)^{1/p} = \left(\int_{\Omega} 4\,\beta(x) | f |^p dx + \int_{\Gamma} | f |^p dS \right)^{1/p} \tag{4.3.7}$$

when $1 \le p < \infty$, and $\| f \|_\infty = \inf \{ c : | f | \le c \}$ when $p = \infty$. Note that $f \in L^p_\mu$ if and only if $f \in L^p(\Omega) \cap L^p(\Gamma)$.

Moreover,

$$\| f \|_\infty = \max \left\{ \| f \|_{L^\infty(\Omega)}, \| f \|_{L^\infty(\Gamma)} \right\}. \tag{4.3.8}$$

We shall also use the dual systems $\langle L_\mu^p , L_\mu^q \rangle$ defined as

$$\langle f, g \rangle_\mu = \int_\Omega 4\beta f g + \int_\Gamma f g \qquad (4.3.9)$$

when $f \in L_\mu^p$ and $g \in L_\mu^q$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We extend the notation of self–adjointness to dual systems: We shall call the operator K self–adjoint, if $\langle K f , g \rangle_\mu = \langle f , Kg \rangle_\mu$ for every $p \in [1, \infty)$. Of course, this makes sense if K maps L_μ^p to itself for every p. To begin with, we consider the integrability of the kernels of K. In this we have to note that the terms $(K_{si} Q)(x)$ and $(K_{ss} Q_s)(x)$ are not defined for non–smooth $x \in \Gamma$. However, for Lipschitz boundary the set of non–smooth points has zero surface measure and zero μ–measure. In what follows we denote the kernels of K_{ii} , K_{is} , K_{si} and K_{ss} by corresponding lowercase letters.

Lemma 4.3.1

Let Ω be bounded with a Lipschitz boundary Γ.

Then the integrals $\int_\Omega K_{ii}(x,z)\, dz$ and $\int_\Gamma K_{is}(x,z)\, dz$ exist for all $x \in \Omega$.

The integrals

$\int_\Omega K_{si}(x,z)\, dz$ and $\int_\Gamma K_{ss}(x,z)\, dz$ exist for all $x \in \Gamma$

for which the surface normal is defined. Moreover, every non zero constant is an eigenfunction of K with eigenvalue $\lambda = 1$.

Proof:

Clearly we have

$$\int_S \exp\Big[-\int_{p(w)}^{x} \beta(s)\, ds \Big] dw < \infty,$$

which shows integrability of k_{is} and k_{ss}. Moreover

$$K_{ii}(1) = \frac{1}{4\pi} \int_S \int_{p(w)}^{x} \exp[-\int_r^x \beta(s)\, ds\,] \beta(r)\, dr\, dw$$

$$= \frac{1}{4\pi} \int_S \exp[-\int_r^x \beta(s)\, ds\,] \Big|_{r=p(w)}^{r=x} dw$$

$$= 1 - \frac{1}{4\pi} \int_S \exp\Big[-\int_{p(w)}^{x} \beta(s)\, ds \Big] dw$$

$$= 1 - K_{is}(1) .$$

Using similar arguments, we observe that $K_{si}(1) = 1 - K_{ss}(1)$. □

The following two Lemmas show that the formulation (4.3.2) – (4.3.4) and the measure μ are, in the some sense, natural for the problem.

Remark 4.3.1

If Γ is not an enclosure, K does not have constant eigenfunctions associated with eigenvalue $\lambda = 1$. If x sees Γ_0 we have $(K(1))(x) < 1$ and if x does not see Γ_0, we have $(K(1))(x) = 1$.

Lemma 4.3.2

The operator K is self–adjiont.

Proof:

Let $f \in L^p_\mu$ and $g \in L^q_\mu$. Then

$$\langle Kf\ ,\ g \rangle_\mu \ = \int_\Omega 4\beta\big(\ K_{ii}\ f + K_{is}\ f\ \big)\, g\ dV + \int_\Gamma \big(\ K_{si}\ f + K_{ss}\ f\ \big)\, g\ dS\ .$$

Now $4\,\beta(x)\,k_{ii}(x,z) \ = \ 4\,\beta(z)\,k_{ii}(z,x)$ and $k_{ss}(x,z) \ = \ k_{ss}(z,x)$.

This implies that

$$\int_\Omega 4\beta\big(\ K_{ii}\ f\ \big)\, g + \int_\Gamma \big(\ K_{ss}\ f\ \big)\, g \ = \int_\Omega 4\beta\big(\ K_{ii}\ g\ \big)\, f + \int_\Gamma \big(\ K_{ss}\ g\ \big)\, f\ .$$

Thus it remains to show that

$$\int_\Omega 4\beta\big(\ K_{is}\ f\ \big)\, g\ dV \ = \ \int_\Gamma \big(\ K_{si}\ g\ \big)\, f\ dS\ .$$

This follows from the fact that $4\,\beta(x)\,k_{is}(x,z) = k_{si}(z,x)$.

□

Lemma 4.3.3

Let $1 \le p \le \infty$. Then the operator K maps L^p_μ into itself compactly, and, in addition, $\| K \|_p \le 1$.

Proof:

Let first $p, q \in (1,\infty)$, such that $\frac{1}{p} + \frac{1}{q} = 1$. From Lemma 4.3.1 and Hölder's inequality it follows that,

$$\left| K_{ii}\, f + K_{is}\, f \right| = \left| \int_\Omega (k_{ii})^{1/p+1/q}\, f + \int_\Gamma (k_{is})^{1/p+1/q}\, f \right|$$

$$\le \left(\int_\Omega k_{ii} + \int_\Gamma k_{is} \right)^{1/q} \left(\int_\Omega k_{ii} \left| f \right|^p + \int_\Gamma k_{is} \left| f \right|^p \right)^{1/p}$$

$$= \left(K_{ii} \left| f \right|^p + K_{is} \left| f \right|^p \right)^{1/p},$$

and similarly

$$\left| K_{si}\, f + K_{ss}\, f \right| \le \left(K_{si} \left| f \right|^p + K_{ss} \left| f \right|^p \right)^{1/p}.$$

Therefore,

$$\| K f \|_p^p = \int_\Omega 4\beta \left| K_{ii}\, f + K_{is}\, f \right|^p dV + \int_\Gamma \left| K_{si}\, f + K_{ss}\, f \right|^p dS$$

$$\le \int_\Omega 4\beta \left(K_{ii} \left| f \right|^p + K_{is} \left| f \right|^p \right) dV + \int_\Gamma K_{si} \left| f \right|^p + K_{ss} \left| f \right|^p dS$$

$$= \langle\ K \mid f\mid^{p}, 1\ \rangle_{\mu}\ = \langle\ \mid f\mid^{p}, K(1)\rangle_{\mu}\ = \|\ f\ \|_{p}^{p}\ .$$

The cases with $p = 1$ and $p = \infty$ are straight–forward.

To prove the compactness, we show that there is a sequence $\{K^{\varepsilon}\}$ of compact operators, that is, uniformly convergent to *K*.

Let $\varepsilon\ > 0$. We define K^{ε} as in (4.3.5) except that we make the kernels of K_{ii}^{ε} , K_{is}^{ε} , K_{si}^{ε} and K_{ss}^{ε} bounded. We define

$$k_{ss}^{\varepsilon}(x,y) = \begin{cases} 0 & \text{if} \quad \|x-y\| \le \varepsilon \\ k_{ss}(x,y) & \text{if} \quad \|x-y\| > \varepsilon \end{cases}$$

and treat other kernels of K^{ε} similarly. Then K^{ε} is compact operator. Moreover, $\|\ K - K^{\varepsilon}\ \|_{p} \to 0$ when $\varepsilon \to 0$, as the kernel of K is integrable . □

As we assumed that $\varepsilon\ >\ \varepsilon_{0}\ > 0$ we have that $\|\ (\ I - E\ )\ K\ \|_{p}\ <\ 1$, and hence,

$(\ I - (\ I - E\ )\ K\ )$ is invertible. Thus, *Q* can be eliminated from (4.3.2), and the radiation heat source and flux can be expressed by means of T alone

$$h_{rad}\ = -4\beta G(\sigma T^{4}) \quad ,\quad x \in \Omega \qquad (4.3.10)$$

$$q_{rad}\ = -G(\sigma T^{4}) \quad ,\quad x \in \Gamma \qquad (4.3.11)$$

where the operator G is defined by

$$G = (I - K) \left(\ I - (I - E) \mathrm{K} \ \right)^{-1} E . \qquad (4.3.12)$$

The operator G maps L_μ^p to itself, and it can be written also as

$$G = E - EK \left(\ I - (I - E) K \ \right)^{-1} E = E - E \left(\ I - K (\mathrm{I} - \mathrm{E}) \ \right)^{-1} K E. \qquad (4.3.13)$$

In the following Lemmas we formulate some properties of G. An important argument here is Riesz–Thorin theorem, see [3] for example.

Lemma 4.3.4

The operator G is self–adjiont. As a mapping from L_μ^2 into itself, G is positive semidefinite with respect to $\langle . , . \rangle_\mu$ inner product.

Proof:

The self–adjiont is a consequence of (4.3.13).

Let $v \in L_\mu^2$ be arbitrary and denote by u the solution of

$$\left(\ I - (I - E) K \ \right) u \ = \ E v .$$

Then

$$\langle v , G v \rangle_\mu = \langle E^{-1} \left(\ I - (I - E) K \ \right) u \ , \ (I - K) u \rangle_\mu$$

$$= \langle u , (I - K) (E^{-1} - I) (I - K) u \rangle_{\mu} + \langle u , (I - K) u \rangle_{\mu} \geq 0$$

as $\| K \|_2 \leq 1$ and $\varepsilon \leq 1$. □

Lemma 4.3.5

The operator G can be written as $G = I - H$, where H is self–adjoint positive and $\| H \|_p \leq 1$. Moreover, every nonzero constant is an eigenfunction of H with eigenvalue $\lambda = 1$.

Proof:

Indeed we can write

$$G = I - H = I - \left[(I - E) + E K \left(I - (I - E) K \right)^{-1} E \right] \quad (4.3.14)$$

where H is self–adjoint. We can write the inverse term in H as

$$\left(I - (I - E) K \right)^{-1} = \sum_{i=0}^{\infty} \left((I - E) K \right)^{i}.$$

All terms in the series are positive, as K is positive. Consequently, H is positive. As G is self–adjoint, we can write

$$H = I - G = I - E\left(I - K\,(I - E) \right)^{-1} (I - K) \qquad (4.3.15)$$

From Lemma 4.3.1 it follows that $H(c) = c$ for every constant c.

Next we show that $\| H \|_1 \le 1$ and $\| H \|_\infty \le 1$. Then, from Riesz–Thorin theorem it follows that $\| H \|_p \le 1$ for $1 < p < \infty$. As H is positive we have that

$H\left(1 - f / \| f \|_\infty \right) \ge 0$ for all $f \in L_\mu^\infty$, $f \ne 0$. Hence

$$\| H \|_\infty = \sup \frac{\| H f \|_\infty}{\| f \|_\infty} \le \| H(1) \|_\infty = \| 1 \|_\infty = 1.$$

Moreover,

from self–adjointness it follows that

$$\| H \|_1 = \| H^* \|_\infty = \| H \|_\infty \le 1 . \qquad \square$$

Remark 4.3.2

If Γ is not an enclosure, *H* does not have constant eigenfunctions associated with eigenvalue $\lambda = 1$. However, from (4.3.15) we see that $H(1) \leq 1$, as $\left(I - K \right)(1) \geq 0$ and $E\left(I - K\left(I - E \right) \right)^{-1}$ is a positive operator.

Let us now consider the case where part of the radiation can escape the system. Then, With the exception of the facts mentioned in remarks after Lemma 4.3.1 and 4.3.5, all the previous Lemmas hold. In addition, the operators K and H become constractive.

Lemma 4.3.6

Assume that $\partial\Omega \setminus \Gamma$ has positive surface measure. The $\| K \|_P < 1$ and $\| H \|_P < 1$ for $1 < p < \infty$.

Proof:

We shall prove the norm estimates in the case $p = 2$. When $p \in (1, 2)$ or $p \in (2, \infty)$, we can apply the Riesz–Thorin theorem. As *K* is self–adjoint the spectral radius $r(K)$ equals to $\| K \|_2$, see [17]. Now, we prove that $r(K) < 1$ by comparing *K* with operators defined in $\Omega \cup \partial\Omega$. For this reason, let $\hat{u}$ be defined in $\Omega \cup \partial\Omega$ and extend *K* by zero to Γ_0: We define $\left(\hat{K}\,\hat{u} \right)(x) = \left(K\,\hat{u}\big|_{\Omega \cup \Gamma} \right)(x)$ when $x \in \Omega \cup \Gamma$ and $\left(\hat{K}\,\hat{u} \right)(x) = 0$ when $x \in \Gamma_0$. Now clearly $r(K) = r(\hat{K})$.

Let us now consider a system where also Γ_0 is radiating and consider the radiation operator $\hat{K}$ associated to enclosure $\Omega \cup \Gamma \cup \Gamma_0$. Now $r(\tilde{K}) = 1$. To conclude the proof, we have to show that $r(\hat{K}) < r(\tilde{K})$. As $\hat{K}$ is positive and compact, $r(\hat{K})$ is an eigenvalue with non–negative eigenfunction $\hat{u}$; see [22]. Moreover, if $\hat{u}$ were zero in some region $\tilde{\Omega} \subset \Omega \cup \Gamma$, $\hat{u}$ had to be zero in every point that sees $\tilde{\Omega}$. Hence $\hat{u} > 0$ in $\Omega \cup \Gamma$. Now let $\tilde{u} = 1$, which is eigenfunction of $\hat{K}$. Then $\langle \hat{u}, \tilde{u} \rangle_{\mu^0} = \langle \hat{u}, (\tilde{K} - \hat{K})\tilde{u} \rangle_{\mu^0} + r(\hat{K}) \langle \hat{u}, \tilde{u} \rangle_{\mu^0}$, where μ^0 is the measure associated to the enclosure $\Omega \cup \Gamma \cup \Gamma_0$. Now, $(\tilde{K} - \hat{K})\tilde{u} > 0$ in those regions of $\Omega \cup \Gamma$ that sees Γ_0. This together with $\langle \hat{u}, \tilde{u} \rangle_{\mu^0} > 0$ implies that $r(\hat{K}) < 1$.

Let us now consider the L^2_μ norm of H. As H is self–adjoint and positive, $r(H) = \| H \|_2$ an eigenvalue of H; see [18]. Hence $\delta = 1 - r(H)$ is an eigenvalue of G with real eigenfunction μ. From the proof of Lemma 4.3.4 we see that G is positive definite if $\| K \|_2 < 1$. Thus, $\delta \langle u, u \rangle_\mu = \langle u, Gu \rangle_\mu > 0$, which concludes the proof. □

4.4 Conductive – radiative problem

Let us now analyze the problem of determining temperature in a system where conduction and radiation are present. We shall analyze situations that is in general enough to give ideas of the difficulties related to radiation terms and of results that can be obtained. We consider a system that consist of conductive materials occupying $\Omega = \Omega \cup \Omega_0 \cup \Gamma$. In Ω the material is assumed to be gray and semitransparent, where as Ω_0 is opaque. By Γ we denote the common boundary of Ω and Ω_0. The exterior boundary of Ω_0 is denoted by Γ_1 and possible exterior boundary of Ω by Γ_0. First, we assume the situation in which part of the wall Γ_0 is assumed semi– transparent allowing part of the radiation to escape .

Coercive cases

Let us consider the situation analyzed above. The heat balance is governed by the system

$$-k\,\Delta T = f \qquad \text{in} \quad \Omega_0 \tag{4.4.1}$$

$$-k\,\Delta T = f - 4\beta G(\sigma T^4) + q_i^{\infty} \qquad \text{in} \quad \Omega \tag{4.4.2}$$

$$[\,k\,\partial T / \partial n\,] = -G\,(\sigma T^4) + q_s^{\infty} \qquad \text{on} \quad \Gamma \tag{4.4.3}$$

where [.] denotes the jump across $\Gamma : [\ f\ ] = \lim_{x \to \Gamma^+} f(x) - \lim_{x \to \Gamma^-} f(x)$. Here q_i^∞ and q_s^∞ denote the additional radiative heat source / flux arriving from outside of the system. On Γ_1 we need some appropriate boundary condition. Here, we choose the condition

$$k \frac{\partial T}{\partial n} + c\left(T - T_0 \right) = 0 \qquad \text{on} \quad \Gamma_1 \tag{4.4.4}$$

On Γ_0 we have the condition

$$k \frac{\partial T}{\partial n} = 0 \qquad \text{on} \quad \Gamma_0 \tag{4.4.5}$$

as Γ_0 does not conduct or radiate outside of the system. As the Stefan–Boltzmann law makes physical only for positive values of *T*, we can alter it freely for mathematical convenience, if *T* is negative. Thus, we make the Stefan–Boltzmann law Monotone by replacing σT^4 with $\sigma \left| T \right|^3 T$. From now on, we write simply T^4 even when we actually mean $\left| T \right|^3 T$. The next step is to write the system in variational form. This is not entirely trivial as the non–linear terms on the boundary

Γ are not necessarily integrable if T in $H^1(\Omega)$. In fact, as G maps L^p_μ to itself, the natural space to work is

$$V = \left\{ v \in H^1(\Omega) : v\Big|_\Gamma \in L^5(\Gamma) \right\} \tag{4.4.6}$$

Then G is defined for all $v \in V$, $(G v^4)\Big|_\Omega \in L^{5/4}(\Omega)$ and $(G v^4)\Big|_\Gamma \in L^{5/4}(\Gamma)$.

If V is reflexive Banach space and we can write (4.4.1) – (4.4.5) in weak form as

$$A(T,v) = a(T,v) + b(T,v) = \langle \tilde{f}, v \rangle \quad \forall v \in V \tag{4.4.7}$$

where

$$a(T,v) = \int_{\Omega \cup \Omega_\circ} k \nabla T \nabla v + \int_\Gamma c T v \tag{4.4.8}$$

$$b(T,v) = \int_\Omega 4\beta G(\sigma T^4) v + \int_\Gamma G(\sigma T^4) v = \langle G(\sigma T^4), v \rangle_\mu \tag{4.4.9}$$

$$\langle \tilde{f}, v \rangle = \int_{\Omega \cup \Omega_\circ} (f + q_i^\infty) v + \int_\Gamma c T_\circ v + \int_\Gamma q_s^\infty v \tag{4.4.10}$$

As the radiative term $G(\sigma T^4)$ is not monotone, we shall apply the theory of pseudomonotone operators; see [23].

Now, if we can prove coercivity in V, the existence result follows.

Theorem 4.4.1

Assume that Ω is bounded, connected and that all the boundaries are Lipschitz. Let $\tilde{f} \in V'$ and assume that $\partial\Omega \setminus \Gamma$ has positive surface measure. Then there exists a solution of (4.4.7).

Proof:

By using Lemma 4.3.6 we get

$$b(T,T) \geq \sigma\left(\|T\|_5^5 - \|T\|_5^4 \left\| H^* T \right\|_5 \right) \|T\|_5^5$$

$$\geq \sigma\left(1 - \|H\|_{5/4} \right) \|T\|_5^5 \geq c \|T\|_5^5 ,$$

as $\|H\| < 1$. Moreover,

$$a(T,T) \geq c_\circ \|\nabla T\|_{L^2(\Omega)}^2 .$$

The norm

$\|\|u\|\| = \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^5(\Gamma)}$, is an equivalent norm in V, see [5].

Hence we have that

$A(T,T) / \|\|T\|\| \to \infty$, as $\|\|T\|\| \to \infty$. This means that A is coercieve in V. As A is also pseudomonotone the problem (4.4.7) has at least one solution ; see [23]. □

Another important situation where the coercivity can be obtained is that of two–dimensional models. If $\Omega \subset R^2$ the coercivity with respect to $L^5(\Gamma)$ norm is not needed as $H^1(\Omega) \subset L^5(\Gamma)$, i.e. $V = H^1(\Omega)$. Heat transfer can be modeled in 2D, if the radiating body has adequate symmetry; see [20]. Atypical example of this is a cross–section of a long cylindrical body. Of course, in 2D the definition of *K* must be changed to keep the physical properties.

General case

Let us consider the situation in which Γ is an enclosure. This system is modelled with equations (4.4.1) – (4.4.4) and it has variational formulation (4.4.7), except that now $q_i^\infty = 0$ and $q_s^\infty = 0$. Now the form A is not necessarily coercieve in *V* (at least we have no proof for coercivity). Hence we shall prove the existence by assuming the existence of sub–and supersolutions. For this reason, let us denote by V^+ the cone of non–negative elements $V^+ = \{ v \in V : v \geq 0 \}$.

Theorem 4.4.2

Assume that Ω is bounded, connected and that all boundaries are Lipschitz. Let $\tilde{f} \in V'$ and assume that there exist functions φ and ψ in *V*, such that $\varphi \leq \psi$ and

$$A(\varphi, w) \leq \langle \tilde{f}, w \rangle \qquad \forall\, w \in V^+$$

$$A(\psi, w) \leq \langle \tilde{f}, w \rangle \qquad \forall\, w \in V^+$$

Then there exists a solution T for (4.4.7). Moreover, $\varphi \leq T \leq \psi$ in Ω.

Proof:

By using Lemma 4.3.5 we can rewrite the problem as

$$a(T,v)+c(T,v)-d(T,v) = \langle \tilde{f},v\rangle \quad ,\forall v\in V\ , \qquad (4.4.11)$$

where

$$c(T,v) = \langle \sigma T^4 ,v\rangle_\mu \text{ and } d(T,v)=\langle H(\sigma T^4),v\rangle_\mu .$$

We choose now $T_1=\psi$ and construct a sequence $\{T_n\}$ as follows:

$$a(T_{n+1},w)+c(T_{n+1},w) = d(T_n,w)+\langle \tilde{f},w\rangle \quad ,\forall w\in V .$$

We claim that T_n is a decreasing sequence of supersolutions that is bounded from below. To prove this, assume that T_n is a supersolution. Then

$$a(T_{n+1},w)+c(T_{n+1},w) - a(T_n,w)-c(T_n,w) =$$

$$\langle \tilde{f},w\rangle - a(T_n,w)-c(T_n,w)+d(T_n,w)\le 0 \quad \forall w\in V^+ ,$$

as T_n was a supersolution. In particular, for $w=(T_{n+1}-T_n)^+$ we set

$a(T_{n+1}-T_n,w)=a(w,w)\ge 0$, and

$c(T_{n+1},w)-c(T_n,w) = \int_\Omega 4\beta\cdot 4\sigma\left|T_\xi\right|^3 w^2 + \int_\Gamma 4\sigma\left|T_\xi\right|^3 w^2 \ge 0$, where T_ξ is

between T_n and T_{n+1}. Consequently, $a(w, w) \equiv 0$. As the form a is coercieve in $H^1(\Omega)$, it follows that $w \equiv 0$ and $T_{n+1} \leq T_n$. On the other hand, $d(T_{n+1}, w) \leq d(T_n, w)$, as $H^* w \geq 0$. Hence, it is easy to see that T_{n+1} is also a supersolution. Moreover, if $T_n \geq \varphi$, then $T_{n+1} \geq \varphi$. The case with subsolutions is treated analogously. This means that the sequence $\{T_n\}$ converges monotonically to a limit which clearly solves the problem. □

Now we give some examples of cases, where the super and subsolutions can be easily constructed. Here we use the fact that $G(\sigma T^4) = 0$, whenever T is a constant.

Example 4.4.1

Assume that there is no internal heating source, so that $f = 0$ in Ω, and let $T_\circ \in L^\infty(\Gamma)$. Then we can choose φ and ψ as constant functions which satisfy $\varphi \leq T_\circ \leq \psi$. In the case of internal heating the construction is not as simple.

Example 4.4.2

Assume that $f \equiv 0$ in Ω. Now, we try to construct ψ as follows:

(i) in Ω we set $\psi = \xi$, where ξ is a constant to be determined;

(ii) ψ solves the problem

$$-k\,\Delta\psi = f \qquad \text{in } \Omega_0$$

$$\psi = \xi \qquad \text{on } \Gamma$$

$$k\frac{\partial\psi}{\partial n} + c(\psi - T_0) = 0 \qquad \text{on } \Gamma;$$

(iii) $$\frac{\partial\psi}{\partial n} \geq 0 \qquad \text{on } \Gamma.$$

These conditions can be satisfied, for example, if Γ has C^1 smoothness and Ω_0 satisfies the interior ball condition on Γ: for every $x \in \Gamma$ there is a ball $B \subset \Omega_0$ with $x \in \partial B$.

To construct ψ in Ω_0, let T_1 and T_2 be solutions to

$$-k\,\Delta T_1 = f \quad , \quad -k\,\Delta T_2 = 0 \quad \text{in } \Omega_0,$$

$$T_1 = 0, \; T_2 = 1 \quad \text{on} \quad \Gamma$$

$$k\frac{\partial T_1}{\partial n} + c(T_1 - T_0) = 0 \quad , \quad k\frac{\partial T_2}{\partial n} + c(T_2 - T_0) = 0 \quad \text{on} \quad \Gamma.$$

Due to the regularity of Γ and Ω_0 we have that $\frac{\partial T_1}{\partial n}$ is bounded and $\frac{\partial T_2}{\partial n} > c_1 > 0$ on Γ, see [7].

If we choose ξ such that $\xi\, \partial T_2 / \partial n \geq \partial T_1 / \partial n$, we can take

$$\psi \Big|_{\Omega_0} = T_1 + \xi\, T_2 \ .$$

A subsolution φ of can be constructed similarly .

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