Deanship of Graduate Studies

Al-Quds University

MATRIX POLYNOMIALS

By

Ayed Mohamed Ahmed Abed Al-Ghani

M. Sc. Thesis

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Declaration:

I Certify that this thesis submitted for the degree of Master is the result of my own research except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed ....

Ayed Mohamed Ahmed Abed Al-Ghani

Date: 21/8/2002

Dedication

To the memory of the late my grandfather.

To my parents.

Acknowledgement

The words are unable to express how I am beholden to those memorable people who helped me to prepare and complete this study. Their names are graven in my memory.

I owe the success of this work to Dr. Amina Afaneh, to her constructive and helpful suggestions regarding the appropriate resources and references of this study. Her on going efforts and final checking of my work helped me a lot to overcome all the difficulties I faced while I was working on this dissertation.

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CHAPTER FOUR

JORDAN CHAIN AND DIFFERENTIAL EQUATIONS

Introduction

The matrix polynomial or a λ-matrix is a matrix-valued function of a complex variable of the form $A(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_0$ where $A_l, A_{l-1}, \cdots, A_0$ are $m \times n$ matrices of complex numbers. Matrix polynomial is a generalization of the matrix polynomial $\lambda I - A$ of degree 1, $\lambda I - A$ is very important in finding the eigenvalues and eigenvectors of the constant $n \times n$ matrix A.

We can also write the matrix polynomial $A(\lambda)$ in the form $A(\lambda) = [a_{ij}(\lambda)]_{i,j=1}^{n}$, so that when the entries of $A(\lambda)$ are evaluated for a particular value of λ, say $\lambda = \lambda_0$, then $A(\lambda_0) \in \mathbf{C}^{n\times n}$ and if we take $n = 1$, we get a scalar polynomial $a(\lambda)$.

In chapter one we study the notion and the kinds of a matrix polynomial, the condition which is necessary for the matrix polynomial to be invertible and the operations on the matrix polynomial.

In chapter two we turn our attention to the invariant polynomials. The importance of the invariant polynomials developed in this chapter allows us to obtain a canonical form of a matrix polynomial without using the elementary row (column) operations to obtain a canonical form. We also study the generally invertible matrix polynomial, generalized inverse, right inverse and left inverse of a matrix polynomial.

In chapter three, we study a factorization of selfadjoint matrix polynomial of the form $A(\lambda) = (M(\overline{\lambda}))^* D(\lambda) M(\lambda)$, where $D(\lambda)$ is a constant matrix or a matrix polynomial and $M(\lambda)$ is a matrix polynomial.

The study of factorization of a selfadjoint matrix polynomial is very important in several applied problems, such as filtering, see [1], chapter 9.

Factorization of matrix polynomials was developed by many researchers as: V. A. Jakubovic, see [9], I. Gohberg, P. Lancaster, and L. Rodman, see [6] and A. C. M. Ran and L. Rodman, see [15].

In chapter four, standard triple and Jordan chain for a matrix polynomial are used to solve the differential equation of the form $\sum_{i=0}^{l} L_i \vec{x}(t)^{(i)} = \vec{f}(t)$, where $L_i \in \mathbf{C}^{n\times n}$, $i = 0,1,\cdots,l$, $\vec{f}(t)$ is a vector-valued function.

In appendix A, we study the Jordan canonical form for a constant matrix and the exponential of a square matrix.

Remarks:

(i) We will use the following system of notion:

Equation j of chapter i in section k is denoted by $(i.k.j)$, similarly definition j of chapter i in section k is denoted by definition $(i.k.j)$ and similar conventions apply to theorems, corollaries and lemmas.

(ii) Here, we mean by a scalar polynomial a polynomial with scalar coefficients not a polynomial as $p(\lambda) = a$, $a \in \mathbf{C}$.

CHAPTER ONE

MATRIX POLYNOMIALS

This chapter contains definitions, theorems and ideas that we shall need in the following chapters. It consists of three sections, section one is about the notion of a matrix polynomial, in section two addition and multiplication of matrix polynomials are introduced and section three about division of matrix polynomials.

1.1 The Notion of a Matrix Polynomial

Definition 1.1.1: A matrix polynomial is a matrix-valued function of a complex variable of the form

$$A(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_1\lambda + A_0$$

where $A_0, A_1, \cdots, A_l$ are $m \times n$ matrices of complex numbers.

Example 1.1.1: The followings are examples of matrix polynomials

(*i*) $A(\lambda) = \begin{bmatrix} \lambda^2+\lambda+1 & \lambda^2-\lambda+2 \\ 2\lambda & \lambda^2-3\lambda-1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}\lambda + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

(*ii*) $B(\lambda) = I_n\lambda - B$, where I_n is the identity matrix and $B \in \mathbf{C}^{n\times n}$.

(*iii*) $C(\lambda) = C$, where $C \in \mathbf{C}^{n\times n}$, *i.e.* matrix with constant entries.

Definition 1.1.2: The **degree** of a matrix polynomial $A(\lambda)$ (denoted by $\deg A(\lambda)$) is the greatest degree of the scalar polynomials appearing as entries of $A(\lambda)$.

In example 1.1.1 $\deg A(\lambda) = 2$, $\deg B(\lambda) = 1$ and $\deg C(\lambda) = 0$.

We call a matrix polynomial $A(\lambda)$

(i) **monic** if the leading coefficient $A_l = I_n$,

(ii) **comonic** if $A_0 = I_n$,

(iii) **regular** if $\det A(\lambda) \not\equiv 0$,

(iv) **unimodular** if $\det A(\lambda)$ is a nonzero constant independent of λ.

Example 1.1.2:

(i) If $\det A_l \neq 0$ then A_l^{-1} exists

$$A_l^{-1}A(\lambda) = A_l^{-1}[A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_1\lambda + A_0]$$

$$= I_n\lambda^l + A_l^{-1}A_{l-1}\lambda^{l-1} + \cdots + A_l^{-1}A_1\lambda + A_l^{-1}A_0.$$

Therefore $A_l^{-1}A(\lambda)$ is a monic matrix polynomial.

(ii) If $\det A_0 \neq 0$ then A_0^{-1} exists

$$A_0^{-1}A(\lambda) = A_0^{-1}[A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_1\lambda + A_0]$$

$$= A_0^{-1}A_l\lambda^l + A_0^{-1}A_{l-1}\lambda^{l-1} + \cdots + A_0^{-1}A_1\lambda + I_n.$$

Therefore $A_0^{-1}A(\lambda)$ is a comonic matrix polynomial.

(iii) Let $A(\lambda) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$.

Since $\det A(\lambda) = \lambda^2 + 1 \not\equiv 0$, then $A(\lambda)$ is a regular matrix polynomial.

$$(iv) \quad A(\lambda) = \begin{bmatrix} 1 & \lambda & -2\lambda^2 \\ 0 & 1 & \lambda^4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det A(\lambda) = 1$, then $A(\lambda)$ is unimodular matrix polynomial.

We can transform a comonic matrix polynomial into a monic form, as follows:

Let $A(\lambda)$ be a comonic matrix polynomial, that is $A_0 = I_n$

$$\begin{aligned} B(\lambda) &= \lambda^l A(\lambda^{-1}) \\ &= \lambda^l [A_l(\lambda^{-1})^l + A_{l-1}(\lambda^{-1})^{l-1} + \cdots + A_1\lambda^{-1} + I_n] \\ &= \lambda^l [A_l\lambda^{-l} + A_{l-1}\lambda^{-l+1} + \cdots + A_1\lambda^{-1} + I_n] \\ &= A_l + A_{l-1}\lambda + \cdots + A_1\lambda^{l-1} + I_n\lambda^l \\ &= I_n\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l . \end{aligned}$$

Therefore $B(\lambda)$ is a monic matrix polynomial.

Definition 1.1.3: An $n \times n$ matrix polynomial $A(\lambda)$ is **invertible** if there exists an $n \times n$ matrix polynomial $B(\lambda)$ such that $A(\lambda)B(\lambda) = I_n$.

We denote $B(\lambda)$ by $(A(\lambda))^{-1}$.

Theorem 1.1.1: A matrix polynomial $A(\lambda)$ is invertible if and only if it is unimodular

Proof:

If $A(\lambda)$ is invertible then there exists $B(\lambda)$ such that, $A(\lambda)B(\lambda) = I_n$.

By taking the determinant of both sides, we obtain $\det A(\lambda) \det B(\lambda) = 1$.

The product of the scalar polynomials $\det A(\lambda)$ and $\det B(\lambda)$ is a nonzero constant.

This is possible only if they are both nonzero constants.

Therefore $A(\lambda)$ is unimodular.

Conversely, if $A(\lambda)$ is unimodular then $\det A(\lambda) = const. \neq 0$.

The entries of the inverse matrix are equal to the cofactors of $A(\lambda)$ (*i.e.* the minors of $A(\lambda)$ of order $n-1$ multiplied by 1 or -1) divided by $const. \neq 0$.

Therefore the inverse matrix is a matrix polynomial in λ. So $B(\lambda) = [A(\lambda)]^{-1}$ is a matrix polynomial. *i.e.* $A(\lambda)$ is invertible. □

1.2 Addition and Multiplication of Matrix Polynomials

Definition 1.2.1: Let $A(\lambda) = A_l\lambda^l + A_{l-1}\lambda^{l-1} + \cdots + A_1\lambda + A_0$ and $B(\lambda) = B_m\lambda^m + B_{m-1}\lambda^{m-1} + \cdots + B_1\lambda + B_0$ be two matrix polynomials of the same order n, then

$$A(\lambda) + B(\lambda) := \begin{cases} \sum_{i=0}^{l}(A_i + B_i)\lambda^i + \sum_{i=l+1}^{m} B_i\lambda^i & \text{if} \quad l < m \\ \sum_{i=0}^{l=m}(A_i + B_i)\lambda^i & \text{if} \quad l = m \\ \sum_{i=0}^{m}(A_i + B_i)\lambda^i + \sum_{i=m+1}^{l} A_i\lambda^i & \text{if} \quad l > m \end{cases}$$